Combinatorial geometries and determinantal measures on graphs

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Determinantal and permanental point processes, quantum physics, and signal processing

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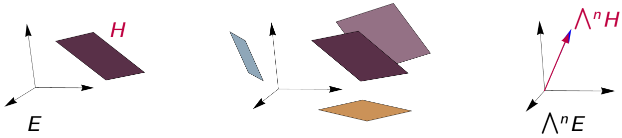
Outline

Determinantal measures

Definition

Setup: Euclidean vector space E, subspace $H \subset E$, orthonormal basis $B = \{e_1, \ldots, e_d\}$ of E.



The Plücker coordinates of H in the basis B are scalar coefficients associated with the subsets of B of cardinality $n = \dim H$.

By Pythagoras' theorem in the n-th exterior power of E, the sum of the squared modulus of these coefficients is equal to 1.

This defines a probability measure on $\binom{S}{n}$ where $S = \{1, \ldots, d\}$.

Let X be the corresponding random n-subset of S.

Incidence measure

For each $I \in \binom{S}{n}$, the probability $\mathbb{P}(\mathsf{X} = I)$ is the square of a determinant; the squared volume of the projected shadow of a unit cube in H on the coordinate subspace $\mathrm{Vect}(e_i : i \in I)$:

$$\mathbb{P}(\mathsf{X} = I) = \det(\Pi^H)_I^I = \cos^2(H, \mathrm{Vect}(e_i : i \in I))$$

One can view X as a point process. It is called determinantal[1] in view of the following equalities for its incidence measure:

$$\mathbb{P}(J \subset \mathsf{X}) = \det(\Pi^H)_J^J$$

for all $J \subset S$.

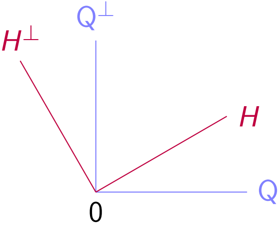
[1]Terminology due to Alexei Borodin ($\sim$ 2000), the original name given by Odile Macchi in her foundational work (1975) being *fermionic process*

Mean projection theorem

Let $\mathsf{Q} = \bigoplus_{i \in \mathsf{X}} \mathbb{C} e_i$ be the random coordinate subspace generated by X.
We have $E = \mathsf{Q} \oplus H^{\perp}$.

Lyons (2003): $$\Pi^H = \mathbb{E}[\mathsf{P}^{\mathsf{Q}}_{\| H^{\perp}}]$$

where $\mathsf{P}^{Q}_{\| H^{\perp}}$ is the projection on Q parallel to $H^{\perp}$



Bases and independent sets

The n-subsets I for which $\mathbb{P}(\mathsf{X} = I) > 0$ are precisely those for which the family $\{\Pi^H(e_i), i \in I\}$ is linearly independent and thus forms a basis of H.

In particular, for any n-subset I in the support, and $J \subset I$, the family $\{\Pi^H(e_i), i \in J\}$ is linearly independent.

This introduces a notion of combinatorial geometry on the finite set S.

This combinatorial geometry is measured: the incidence measure assigns a weight to each independent set, whose restriction to the collection of bases is the determinantal probability measure.

This point of view on determinantal measures was introduced by Russell Lyons (2003).

Two complementary viewpoints

combinatorial geometry
$(S, \mathcal{I})$

intrinsic combinatorial weight

$\updownarrow$ normalisation by partition function

determinantal probability measure

Euclidean space and o.n. basis
$H \subset E$, $(e_1, \ldots, e_d)$

Examples[2]:

- uniform measure on spanning trees
- weighted measure $\prod_{c:\text{cycle}} |1 - \mathrm{hol}_h(c)|^2$ on forests of unicycles

To put these examples into a larger family, I will now present

- combinatorial geometries on the set of edges of a graph,
- determinantal probability measures associated with particular Euclidean spaces attached to a graph.

[2] In both cases, the partition function is given by $\det' \Delta$

Combinatorial geometries on graphs

Combinatorial geometries

A matroid or a combinatorial geometry[3] on a finite set S is a non-empty collection $\mathcal{I}$ of subsets of S, called independent subsets, such that

- if $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$
- if $I, J \in \mathcal{I}$ and $|J| < |I|$, then $\exists i \in I \setminus J$, such that $J \cup \{i\} \in \mathcal{I}$

The maximal independent sets are called bases.

The minimal dependent sets are called circuits.

Examples:

- Linear algebra: S family of vectors of a vector space, $\mathcal{I}$ the linearly independent subsets
- Graph theory: S the set of edges of a finite graph, $\mathcal{I}$ the acyclic subsets

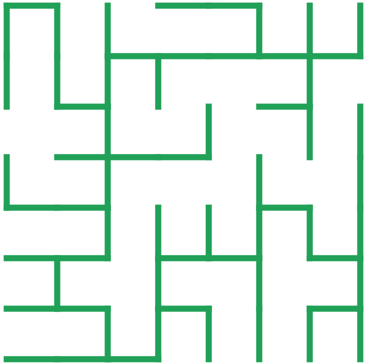
[3] Terminology proposed by Gian-Carlo Rota to replace the first introduced by Hassler Whitney (1935) in his founding study (independently carried out by Takeo Nakasawa)

The circular matroid

S set of edges of a finite connected graph, $\mathcal{I}$ collection of spanning forests.

Bases: spanning trees

Circuits: simple cycles



Number of sp. trees: $|\mathcal{F}_1(\mathsf{G})| = |\mathsf{V}(\mathsf{G})|^{-1} \det' \Delta$, where $\Delta = d^* d$ is the Laplacian (*Kirchhoff, 1847*)

The circular matroid and associates

Two simple operations to construct new matroids on a given set S:

Union: $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$
Dual: $\mathcal{B}^* = \{S \setminus B : B \in \mathcal{B}\}$

From the circular matroid, by taking unions and duals, one can generate new matroids, the bases of which are classes of subgraphs with specific topological properties.

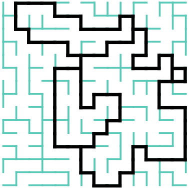
The circular matroid and associates

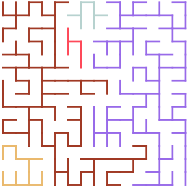
The case of $\mathcal{M}_{k,0}$:

- Bases: spanning forests with $k+1$ connected components
- Circuits: simple cycles and spanning forests with k connected components

The case of $\mathcal{M}_{0,\ell}$:

- Bases: connected subgraphs with ℓ cycles
- Circuits: minimal subgraphs X with $\ell+1$ cycles





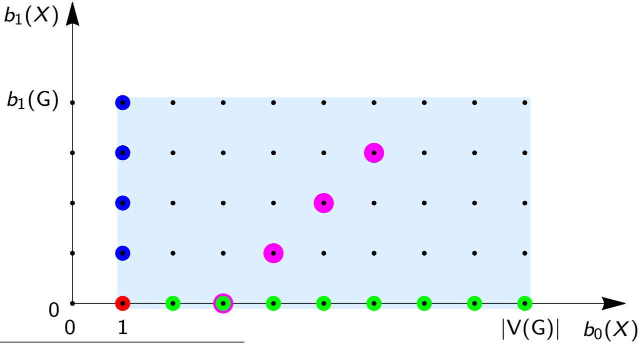
Liu–Chow (1981), Myrvold (1992): one can compute $|\mathcal{F}_k(\mathrm{G})|$ in polynomial time.

Question: can one compute $|\mathcal{C}_\ell(\mathrm{G})|$ in polynomial time when G is non planar?

The circular matroid and associates

Let $(k, \ell) \in \{0, \ldots, |\mathrm{V}| - 1\} \times \{0, \ldots, b_1(\mathrm{G})\}$.
Define a matroid $\mathcal{M}_{k,\ell}$ with set of bases $\mathcal{B}_{k,\ell}(\mathrm{G})$ given by the collection of subgraphs X of G satisfying the conditions

- $\chi(X) := b_0(X) - b_1(X) = k - \ell + 1$
- $\max(0, \ell - k) \leq b_1(X) \leq \ell$

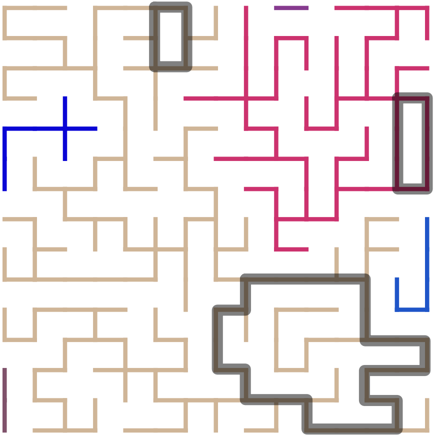


$b_0(X)$: number of connected components of X

$b_1(X) = |\mathrm{E}(X)| - |\mathrm{V}(X)| + b_0(X)$: number of independent cycles (first Betti number of X)

The circular matroid and associates

An element of $\mathcal{B}_{5,3}$:



Here $b_0(X) = 6$ and $b_1(X) = 3$, and $\chi(X) = b_0(X) - b_1(X) = 3$.

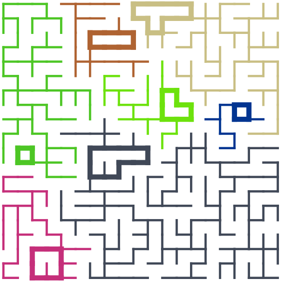
The bicircular matroid

There are other important matroids on graphs.

S set of edges of a finite connected graph, $\mathcal{I}$ the collection of subgraphs each of whose connected components has at most one cycle.

Bases: forests of unicycles

Circuits: minimal connected subgraphs with $b_1 = 2$ ($\ominus$, ∞,$\circ\!\!-\!\!\circ$)



Counting bases: it is $\#P$-hard (*Giménez–Noy, 2006*), however there is an approximate counting method (*Guo–Jerrum, 2019*)

Measured combinatorial geometries

Vector spaces of forms on a graph

A 0-form is a function on vertices.
A 1-form is a function on edges, antisymmetric.
Let Ω^0 and Ω^1 be the corresponding vector spaces.

Let $d : \Omega^0 \to \Omega^1$ be the discrete derivative
and $d^* : \Omega^1 \to \Omega^0$ the discrete divergence.

$$df(e) = f(\overline{e}) - f(\underline{e}) \quad \text{and} \quad d^*\omega(v) = \sum_{e:\overline{e}=v} \omega(e)$$

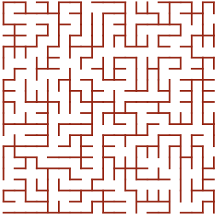
- Exact forms: $\operatorname{im} d \subset \Omega^1$
- Co-closed forms: $\ker d^* \subset \Omega^1$

$$\Omega^1 = \operatorname{im} d \oplus \ker d^*$$

Uniform spanning tree

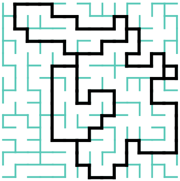
The circular matroid is linearly representable: with $H = \operatorname{im} d$, a subset $J \subset S$ is independent iff $\{\Pi^H(e_i), i \in J\} \subset \Omega^1$ is linearly independent.

Burton–Pemantle (1993): The uniform spanning tree is determinantal on $E = \Omega^1$ associated with the subspace $H = \operatorname{im} d$.



Random connected subgraphs

An element of $\mathcal{C}_4(\mathsf{G})$:



Let $\theta_1, \ldots, \theta_k$ be linearly independent 1-forms in $\ker d^*$. For each $X \in \mathcal{C}_k(\mathsf{G})$, choose $(\gamma_1, \ldots, \gamma_k)$ a basis of cycles of X, and define

$$\mathbb{P}(X) \propto \left| \det \left(\theta_i(\gamma_j)\right)_{1 \leq i,j \leq k} \right|^2$$

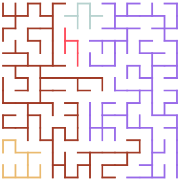
where $\theta(\gamma) = \sum_{e \in \gamma} \theta_e$.

Theorem 1 ([K.–Lévy, 2022])

This probability measure is determinantal associated with the linear subspace $H = \operatorname{im} d \oplus \mathrm{Vect}(\theta_1, \ldots, \theta_k)$.

Random spanning forests

An element of $\mathcal{F}_5(\mathrm{G})$:



Let $\varphi_1, \ldots, \varphi_k$ be linearly independent 1-forms in $\operatorname{im} d$. For each $X \in \mathcal{F}_{k+1}(\mathrm{G})$, choose a basis $(\kappa_1, \ldots, \kappa_k)$ of cuts determined by X, and define

$$\mathbb{P}(X) \propto \left| \det\left(\varphi_i(\kappa_j)\right)_{1\leq i,j\leq k} \right|^2$$

where $\varphi(\kappa) = \sum_{e\in\kappa} \varphi_e$.

Theorem 2 ([K.–Lévy, 2022])

This probability measure is determinantal associated with the linear subspace $H = \operatorname{im} d \cap \mathrm{Vect}(\varphi_1, \ldots, \varphi_k)^\perp$.

The partition function $\sum_{X\in\mathcal{B}} w(X) \prod_{e\in X} x_e$ generalizes the second Symanzik polynomial ($k=1$).

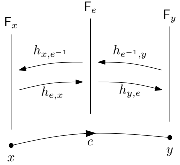
All these real polynomials are stable (*Borcea–Brändén–Liggett, 2009*).

Combinatorial geometries in higher rank

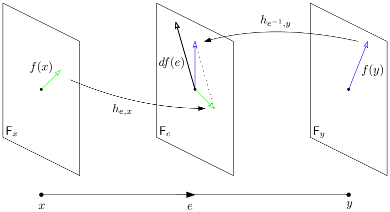
Vector-valued 1-forms

We consider vector-valued 1-forms in $\mathbb{C}^N$, and a unitary connection h.

Motivation: graph embedded in a manifold, seen as the base of a vector bundle of rank N; h is the parallel transport along an edge.



There is a discrete covariant derivative d_h and its adjoint d_h^*.



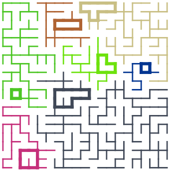
$N = 1$

For each forest of unicycles F, define

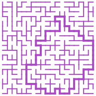
$$\mathbb{P}(F) \propto \prod_{\text{cycle } c} |1 - \mathrm{hol}_h(c)|^2$$

where $\mathrm{hol}_h(c)$ is the holonomy.

Kenyon (2009): This probability measure is determinantal associated with the linear subspace $H = \mathrm{im} d_h$.

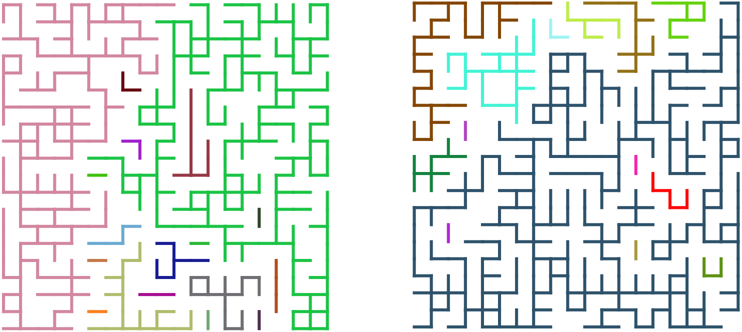


When h tends to 1 along θ (i.e. $h_t = e^{it\theta}$, with $t \to 0$), we recover Theorem 1 about $\mathcal{C}_1(\mathrm{G})$.



$N = 2$

One can define a determinantal process associated with $H = \mathrm{im}\, d_h$, by fixing an orthonormal basis of each fiber.



We represent the sample as two subgraphs (X_1, X_2)

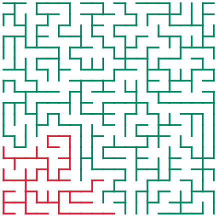
$N = 2$: the case $h = \mathrm{id}$

Choose a vector $u_e^1 \in \mathbb{C}^2$ for each edge e. Define the kernel

$$K_{e,e'} = \langle u_e^1, u_{e'}^1 \rangle \, T_{e,e'}$$

where T is the kernel for the uniform spanning tree.

This kernel satisfies $0 \leq K \leq 1$ and the associated determinantal measure is the law of the subgraph X_1.



$N = 2$: limit $h \to \mathrm{id}$

- Consider the connection $h = \exp(tA)$ in the limit $t \to 0$, where A is a matrix-valued 1-form in $\mathfrak{u}(2)$.

- For each couple of connected spanning subgraphs (X_1, X_2) such that $b_1(X_1 \sqcup X_2) = 2$, choose a basis (γ_1, γ_2) of cycles of $X_1 \sqcup X_2$, and define

$$\mathbb{P}\left((X_1, X_2)\right) \propto \left| \det A\left(\gamma_1\right)_{i_1} A\left(\gamma_2\right)_{i_2} \right|^2$$

where $A(\gamma)_i$ is the i-th column of $A(\gamma) = \sum_{e \in \gamma} A_e$.

Theorem 3 ([K.–Lévy, 2022])

This probability measure is determinantal associated with the linear subspace $H = \mathrm{im} d_{\mathrm{id}} \oplus \mathrm{Vect}(A_1, A_2)$.

$N = 2$: general h

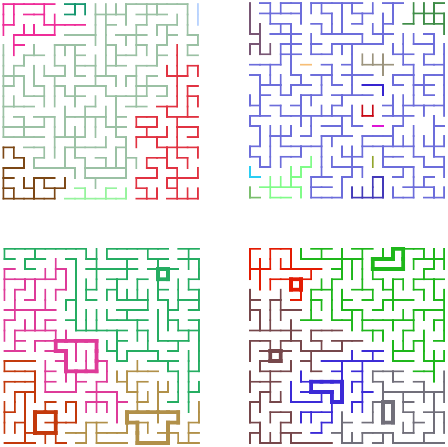
Trace : $\mathrm{E(G)} \to \{0,1,2\}$, total occupation number

Theorem 4 ([K.–Lévy, 2022])

The trace of the determinantal measure associated with $H = \mathrm{imd}_h$ is the sum of two coupled random forests of unicycles.

- Recall: In rank $N = 1$, it is simply a forest of unicycles with distribution proportional to $\prod_{c:\text{cycle}} |1 - \mathrm{hol}_h(c)|^2$
- In rank $N \geq 2$, we obtained more complicated combinatorial expressions involving traces of holonomies

$N = 2$: the case of holonomies in SU(2)



Distribution of the trace:
sum of two i.i.d. copies of random forests of unicycles, the law of which is the quaternion-determinantal measure associated with $\operatorname{im} d_h$

Thank you for your attention

References on this work

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Combinatorial geometries and determinantal measures on graphs
in preparation

A. K. and Thierry Lévy
Quantum spanning forests
in preparation