Signal analysis via the stochastic geometry of spectrogram level sets

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Outline

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- The spectrogram of a signal associating with each point in the time-frequency plane is a localised measure of the energy of the signal at that time and that frequency.
- A particularly significant domain of application is in the field of acoustics.
- For instance, AM-FM-type signals with a small number of components admit spectrogram representations that are sparse.

Spectrogram analysis

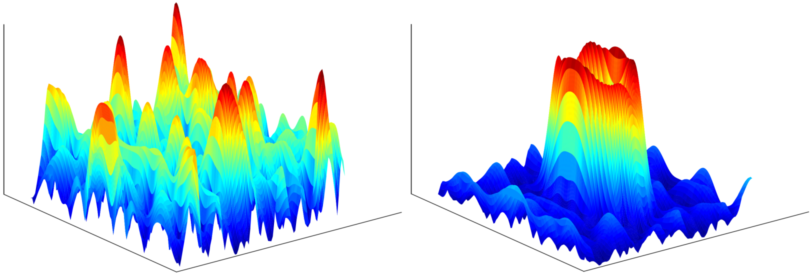

Fig. Left : spectrogram of white noise. Right : spectrogram of one fundamental mode corrupted by white noise.

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- Techniques such as synchrosqueezing, reassignment and ridge extraction have gained prominence in the context of identifying and processing the maxima of the spectrogram.

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- The zeros of the Gabor spectrogram of white noise exhibit a spatial distribution that is highly uniform on the time-frequency plane.
- The presence of a nonzero signal creates distortions in the highly uniform spatial distribution of points.

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- The STFT of white noise is connected to the Gaussian Analytic Functions (abbrv., GAFs).
- The hyperuniformity of the GAF zero sets provides a cogent explanation for the empirical observation that these zeros have a highly homogeneous spatial distribution.

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- We propose to investigate signals via the (upper) level sets $\Lambda(\theta)$ of the spectrogram X, rather than its zeros or critical points.

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- Level sets are more robust to noise and numerical perturbations, and accord a richer mathematical theory.

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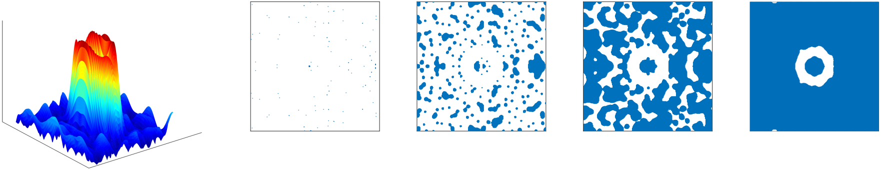

Lower level sets $\Lambda(\theta)^{\complement} = \{(u,v) \in \mathbb{R}^2 : |X(u,v)| \leq \theta\}$ with increasing level θ.

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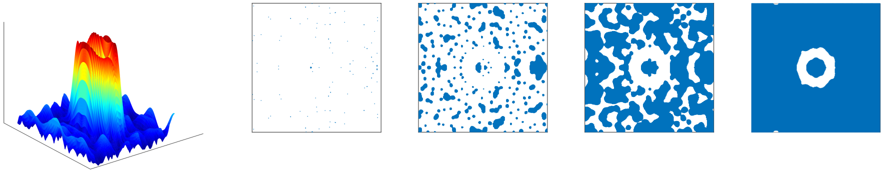

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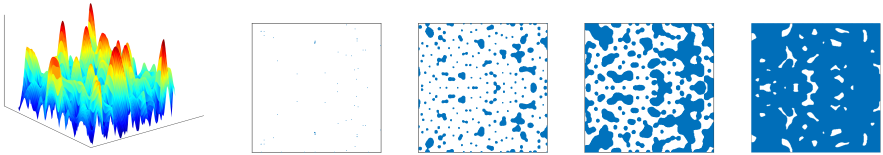

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- a natural hypothesis testing problem to decide between pure white noise and the presence of a fundamental mode.
- an efficient test of hypothesis for this problem, and provide theoretical guarantees for its effectiveness.
- an estimation procedure for a fundamental mode if it is present, and provide error bounds for the accuracy of such estimation.

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- This procedure is able to effectively perform detection and estimation for linear combinations of fundamental modes.
- The signal estimation turns out to be highly accurate as long as the fundamental modes being combined are reasonably well separated.

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Definition

Fix a window function $\phi \in L^2(\mathbb{R})$. The STFT of $f \in L^2(\mathbb{R})$ w.r.t. ϕ is

$$V_\phi f(u,v) := \langle f, M_v T_u \phi \rangle = \int_R f(t)\overline{\phi(t-u)}e^{-2i\pi tv}\mathrm{d}t,$$

where $M_v f = e^{2i\pi v\cdot} f(\cdot)$ and $T_u f = f(\cdot - u)$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R})$ w.r.t. the Lebesgue measure.

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- The Bargmann transform of $f \in L^2(\mathbb{R})$ is

$$Bf(z) = 2^{1/4} \int_{\mathbb{R}} f(t) \exp(2\pi t z - \pi t^2 - \frac{\pi}{2} z^2) \mathrm{d}t, \quad z \in \mathbb{C}.$$

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- For $f \in L^2(\mathbb{R})$, the Gabor transform of f can be computed as

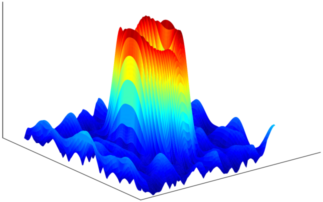
$$V_g f(u, v) = \exp(-\pi i u v - \frac{\pi}{2} |z|^2) Bf(\bar{z}), \quad u, v \in \mathbb{R},$$

where $z = u + iv$.

Gabor transform of a noisy fundamental mode

Suppose we consider the generative model where

$$\begin{array}{rl} \text{Observation} & = \text{Signal strength} \cdot \underbrace{\text{Fundamental mode}}_{\text{Peaked structure}} + \underbrace{\text{White noise}}_{\text{Approximately controlled}} \\ \text{Gabor spectrogram:} & \end{array}$$



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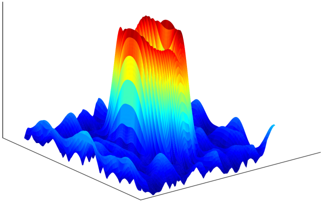

- We focus our attention on the most fundamental setting for a signal, namely Hermite functions, which form an orthonormal basis for $L^2(\mathbb{R})$ and is central to Gabor analysis.

Gabor transform of Hermite functions

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- For any $u, v \in \mathbb{R}$ such that

$$\frac{\sqrt{u^2+v^2} - \sqrt{k/\pi}}{\sqrt{k/\pi}} = r,$$

we have

$$\frac{|V_g h_k(u,v)|}{\max_{u,v\in\mathbb{R}} |V_g h_k(u,v)|} = \frac{(1+r)^k}{e^{k(r+r^2/2)}}.$$

Gaussian white noise

Definition

(1) Schwartz space $\mathcal{S}(\mathbb{R})$ is the function space consisting of rapidly decreasing smooth functions from $\mathbb{R}$ to $\mathbb{C}$.

(2) The space of tempered distributions on $\mathbb{R}$, denoted as $\mathcal{S}'(\mathbb{R})$, is the continuous dual of $\mathcal{S}(\mathbb{R})$.

- Define the action $\langle \psi, \phi \rangle := \psi(\phi)$ for any $\psi \in S'(\mathbb{R})$, $\phi \in S(\mathbb{R})$.
- The STFT of $f \in \mathcal{S}'(\mathbb{R})$ w.r.t. a window function $\phi \in \mathcal{S}(\mathbb{R})$ is

$$V_\phi f(u,v) := \langle f, M_v T_u \phi \rangle, \quad u, v \in \mathbb{R}.$$

- White noise measure μ_1 : unique probability on $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ satisfying

$$\mathbb{E}_{\mu_1}\left[e^{i\langle \cdot, \phi \rangle}\right] := \int_{S'(\mathbb{R})} e^{i\langle \xi, \phi \rangle} \mathrm{d}\mu_1(\xi) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}, \quad \phi \in \mathcal{S}(\mathbb{R}),$$

Gabor transform of Gaussian white noise

- Let ξ be a random variable with distribution μ_1, i.e.,

$$\xi = \sum_{k=0}^{\infty} \langle \xi, h_k \rangle h_k,$$

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- Let $u, v \in \mathbb{R}$ and write $z = u + iv \in \mathbb{C}$. Then the Gabor transform of ξ is

$$V_g \xi(u,v) = \sqrt{\pi} \exp(i\pi uv - \frac{\pi}{2}|z|^2) \sum_{k=0}^{\infty} \langle \xi, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}},$$

where convergence is in $L^2(\mu_1)$.

- The series on R.H.S. is the standard planar Gaussian Analytic Function.

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- The metric is given by

$$d^2((u_1, v_1), (u_2, v_2)) := \left\{\mathbb{E}\left[|V_g\xi(u_1, v_1) - V_g\xi(u_2, v_2)|^2\right]\right\}^{1/2}$$
$$= 2\pi\left[1 - \cos\left(\pi(u_1 + u_2)(v_1 - v_2)\right)\exp(-\frac{\pi}{2}\|(u_1, v_1) - (u_2, v_2)\|^2)\right].$$

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Gabor spectrogram of white noise

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Theorem

For $L \geq \pi$, we have that for any $\tau > 0$,

$$\mathbb{P}\left[\sup_{(u,v)\in\mathbb{B}_L} |V_g\xi(u,v)| \leq \sqrt{2}(14K+\tau)\sqrt{\log L}\right] \geq 1 - 4\exp\left(-\frac{\tau^2}{2\pi}\cdot\log L\right),$$

where $K > 0$ is a constant and the parameter set

$$\mathbb{B}_L := \{(u,v) \in \mathbb{R}^2 : \max\{|u|,|v|\} \leq L\}.$$

Sketch of the proof

With probability $\geq 1 - \exp(-\frac{\rho^2}{2\pi})$, we have

$$\sup_{(u,v)\in\mathbb{B}_L} |V_g\xi(u,v)| \leq \mathbb{E}\left[\sup_{(u,v)\in\mathbb{B}_L} |V_g\xi(u,v)|\right] + \rho \quad \text{(By Borell-TIS inequality)}$$

$$\leq K \int_0^\infty \sqrt{\log\left(N(\mathbb{B}_L, d, \varepsilon)\right)} \mathrm{d}\varepsilon + \rho, \quad \text{(By Dudley's entropy integral)}$$

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- the first equality holds since $\max\{|u_1 - u_2|, |v_1 - v_2|\} \leq \frac{\varepsilon}{3\sqrt{2\pi^3}L}$ implies $d((u_1, v_1), (u_2, v_2)) \leq \varepsilon$ when $0 \leq \varepsilon \leq L^{-2}$;
- $2\sqrt{\pi}$ comes from the fact that $d^2(\cdot, \cdot) \leq 4\pi$.

Spectrogram level sets of a noisy fundamental mode

The spectrogram level set of a signal y, restricted to $\mathbb{B}_L$, with threshold γ is

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Assume the signal y is generated as $y = \lambda h_k + \xi \in \mathcal{S}(\mathbb{R})$. Let

$$|\lambda| \geq \frac{5\sqrt{2}(14K+\tau)\sqrt{\log L}}{\prod_{t=1}^{k} \sqrt{k/(et)}},$$

where $\tau > 0$ is a parameter. Then, for $L \geq \max\{\sqrt{k/\pi}, \pi\}$, we have

$$\emptyset \neq \Lambda\left(3\sqrt{2}(14K+\tau)\sqrt{\log L}\right) \subseteq \left\{(u,v) \in \mathbb{R}^2 : \frac{|V_g h_k(u,v)|}{\prod_{t=1}^{k} \sqrt{k/(et)}} > \alpha\right\},$$

with prob. $\geq 1 - 4\exp\left(-\frac{\tau^2}{2\pi} \cdot \log L\right)$, where $\alpha := \frac{\sqrt{2}(14K+\tau)\sqrt{\log L}}{|\lambda| \prod_{t=1}^{k} \sqrt{k/(et)}} \in (0, \frac{1}{5}]$.

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- This makes conducting signal detection via spectrogram level sets possible !

Outline

Motivations and the big picture

The spectrogram of white noise

Signal analysis via spectrogram level sets

Empirical investigations

Generative and observational models

Generative model : The observation y is generated as

$$y = \lambda h_k + \xi,$$

where

- the signal h_k is an Hermite function, $1 \leq k \leq k_0$, where $k_0 \in \mathbb{N}$ is given.
- white noise ξ is a random variable with distribution μ_1,
- λ is the signal strength.

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H_1 : The observation $y = \lambda h_k + \xi$ with $\lambda \neq 0$ and some integer $k \in [0, k_0]$.

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A *test* ψ_θ is a measurable function of the observed level set $\Lambda(\theta)$ that maps $\Lambda(\theta)$ to the set $\{0, 1\}$, with the understanding :

- the value 0 corresponds to acceptance of H_0,
- the value 1 pertains to rejecting the null and accepting the alternative H_1.

Signal detection – A hypothesis testing perspective

Denote

- $\mathbb{P}^0$ as the distribution of $\Lambda(\theta)$ under H_0
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We say that we detect the signal at strength λ if, for any $\delta > 0$, there exists a test $\psi_{\theta(\delta)}$ such that

$$\mathbb{P}^0[\psi_{\theta(\delta)} = 1] \vee \max_{1 \leq k \leq k_0} \mathbb{P}_k[\psi_{\theta(\delta)} = 0] \leq \delta.$$

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- We want an algorithm for signal detection when the probabilities of there two types of errors are less than δ.

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Here,

$$\mathfrak{M}(k_0) := \min_{1 \leq k \leq k_0} \prod_{t=1}^{k} \sqrt{k/(et)}.$$

Signal estimation

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Definition

Define the following statistics :

- $\hat{\theta} := \max\{\theta : \Lambda(\theta) \neq \emptyset\}$;
- $\hat{k} := [[\pi \cdot (\min\{|z|^2 : z \in \Lambda(\hat{\theta})\})]]$, where $[[x]]$ denotes the nearest integer to $x \in \mathbb{R}$;
- $\hat{\lambda} := \hat{\theta} / \prod_{t=1}^{\hat{k}} \sqrt{\hat{k}/(et)}$.

Signal estimation – Mode

Theorem

Given parameter $\tau > 0$, for $L \geq \max\{\sqrt{k_0/\pi}, \pi\}$, and signal strength

$$|\lambda| \geq t_{\mathrm{mode}}(\tau) := 5C(k_0)\sqrt{2}(14K + \tau)\mathfrak{M}(k_0)^{-1}\sqrt{\log L},$$

we have

$$\inf_{1 \leq k \leq k_0} \mathbb{P}_k[\hat{k} = k] \geq 1 - 4\exp\left(-\frac{\tau^2}{2\pi} \cdot \log L\right).$$

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- This theorem provides an error bound for the accuracy of the mode obtained by our estimation in terms of the quantity of data available.

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Given parameter $1 \geq \delta > 0$, for $L > \max\{\sqrt{k_0/\pi}, \pi, \exp(14K/\delta)^2\}$, and signal strength

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4.2 Moreover, suppose the average of the radii of the large circle and the smaller one is η, the unknown index k could be estimated as $\lfloor \pi\eta^2 \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x ;

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- Note that we only care about the level set itself, not the values of the spectrograms in this area.
- As a side note, the procedure in the algorithm could be extended to learn linear combinations of fundamental modes as long as the signals being combined are reasonably well separated.

Outline

Modified accuracy

- In order to measure the performance of our algorithm for learning linear combinations of fundamental modes $\sum_{i=1}^{m} \lambda_i h_{k_i}$ corrupted with noise, where $1 \leq k_i \leq k_0$, we need to provide a reasonable metric.

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- Define the modified accuracy (mACC) of the estimation $\{\hat{k_j}\}_{j=1}^{\hat{m}}$ as

$$\text{mACC} = \begin{cases} 0, & \text{if } \hat{m} \neq m \\ \max\left\{0, 1 - \sum_{i=1}^{m} \left| \frac{\hat{k_i}}{k_i} - 1 \right| \right\} & \text{if } \hat{m} = m,\ \max_i |\hat{k_i} - k_i| \leq 1, \\ 0, & \text{if } \hat{m} = m,\ \max_i |\hat{k_i} - k_i| > 1. \end{cases}$$

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- By definition, we can see that mACC $=1$ means the perfect estimation, and mACC $=0$ represents the worst.

One fundamental mode

Consider the generative model

$$y = \lambda h_k + \sigma \xi,$$

where

- the integer k is uniformly generated from $\{1, 2, \cdots, 112\}$;
- the signal strength λ is uniformly generated in $[1, 2]$;
- the noise strength $\sigma = \frac{1}{10\sqrt{\log L}}$.

Set the observation radius $L = 8$.

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Define the spectrogram level set of the data y with threshold $0.2m_L$ as

$$\Lambda(0.2m_L) = \{(u, v) \in \mathbb{B}_L \mid \ |V_g y(u, v)| \geq \gamma m_L\},$$

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Empirical result – One fundamental mode

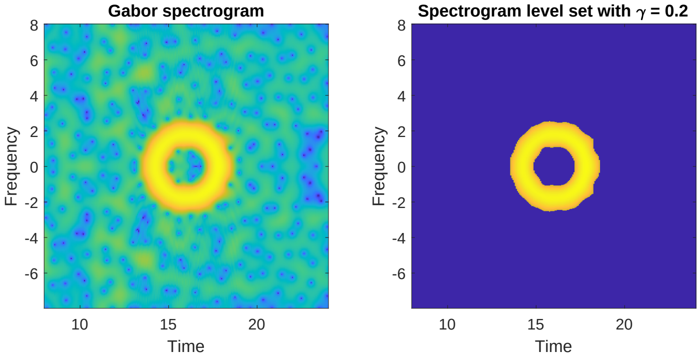

Fig. Gabor spectrogram and spectrogram level set of one fundamental mode corrupted by noise with $k = 10$.

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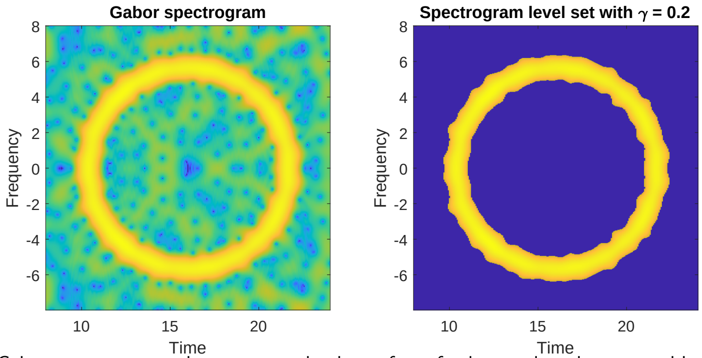

Fig. Gabor spectrogram and spectrogram level set of one fundamental mode corrupted by noise with $k = 100$.

- It is promising to detect the fundamental mode through the ring in the spectrogram level set.

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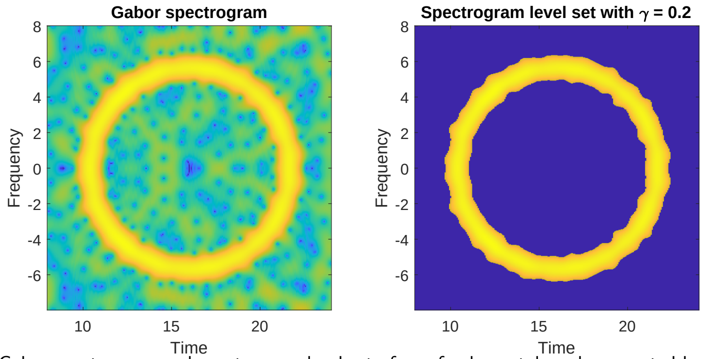

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- It is promising to detect the fundamental mode through the ring in the spectrogram level set.
- Level sets at 20% of maximum lead to signal estimation of one fundamental mode with accuracy $>$ 99%.

Two or more fundamental modes

We test the generative model

$$y = \sum_{i=1}^{m} \lambda_i h_{k_i} + \sigma\xi,$$

where the integers k_i's are uniformly generated from $\{1, 2, \cdots, 112\}$ such that each k_i are well separated as :

$$\sqrt{\frac{k_i}{\pi}} \notin [\sqrt{\frac{k_j}{\pi}} - w, \sqrt{\frac{k_j}{\pi}} + w], \quad 1 \leq i \neq j \leq m.$$

Here $w > 0$ is a preset parameter.

Empirical result – Two or more fundamental modes

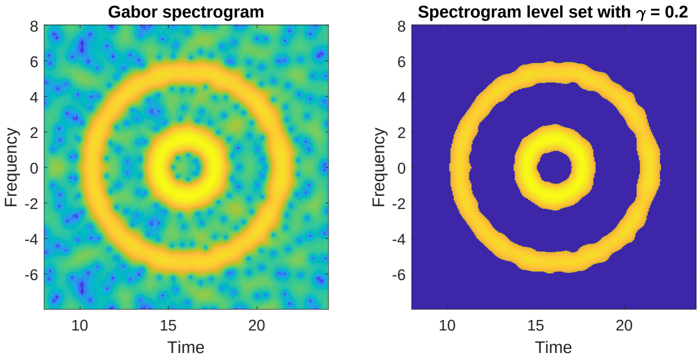

Fig. Gabor spectrogram and spectrogram level set of linear combination of fundamental modes corrupted by noise with $k_1 = 8$, $k_2 = 90$.

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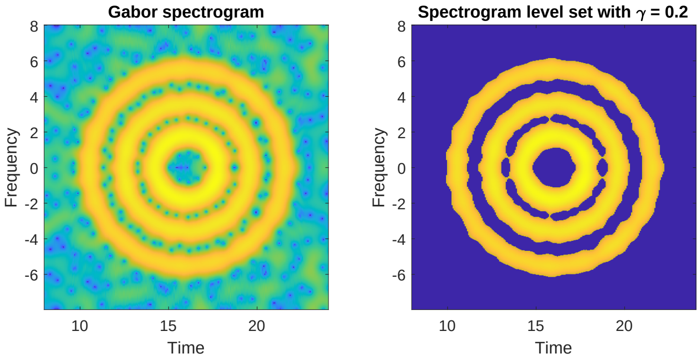

Fig. Gabor spectrogram and spectrogram level set of linear combination of fundamental modes corrupted by noise with $k_1 = 10$, $k_2 = 40$, $k_3 = 95$.

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TABLE – Empirical performance of our algorithm for learning the linear combination of fundamental modes over 10000 trials with different parameters.

Number of modes (m)	The parameter w	Average modified accuracy
2	1.5	94.96%
2	2	99.85%
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- Overall, our algorithm is quite effective for learning linear combinations of fundamental modes as long as these modes are reasonably well separated.

Conclusion

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