Distribution of zeros of the spectrogram of noisy signals

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Université de Poitiers

Mathématiques

Université de Poitiers

Outline

1. Zeros of the spectrogram of complex white noise
2. Intensity of zeros of the spectrogram of a noisy signal
3. Behaviour of zeros in the presence of specific signals
   - Hermite functions
   - Linear chirps
   - Pairs of parallel linear chirps
4. Some open problems and conclusion

Definition

The short time Fourier transform (STFT) of a signal $x : \mathbb{R} \to \mathbb{C}$ *with Gaussian window*

$$g(t) = \frac{1}{2^{1/4}} e^{-\pi t^2}$$

is defined by

$$STFT(x)(\tau, \omega) := \int_{\mathbb{R}} x(t) e^{-\pi(t-\tau)^2} e^{-2i\pi\omega t} \mathrm{d}t.$$

Definition

The spectrogram of a signal $x : \mathbb{R} \to \mathbb{C}$ *is defined by*

$$\forall z = \tau + i\omega \in \mathbb{C}, \quad Spec(x)(z) := |STFT(x)(\tau, \omega)|^2.$$

Proposition

The STFT can be rewritten as

$$STFT(x)(\tau,\omega) = e^{-\pi\frac{\tau^2+\omega^2}{2}} B(x)(\tau - i\omega) \Rightarrow Spec(x)(z) = e^{-\pi|z|^2} \left|B(x)(\bar{z})\right|^2$$

where $B(x)$ is the Bargmann transform defined by

$$B(x)(z) := 2^{1/4} \int_{\mathbb{R}} x(t) e^{2\pi tz - \pi t^2 - \pi z^2/2} \mathrm{d}t.$$

Definition

Let $\xi : \mathbb{R} \to \mathbb{C}$ be the standard complex white noise defined by

$$\xi(t) = \sum_{k \geqslant 0} a_k h_k(t)$$

where $(a_k)_{k \in \mathbb{N}}$ is a sequence of i.i.d. $\mathcal{N}_{\mathbb{C}}(0, Id)$ random variables and h_k is the k-th Hermite function.

Since $(h_k)_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$ then:

- $\forall t \in \mathbb{R}$, $\xi(t) \sim \mathcal{N}_{\mathbb{C}}(0, Id)$.
- $\forall t, t' \in \mathbb{R}$ with $t \neq t'$, $\xi(t) \perp\!\!\!\perp \xi(t')$.

Proposition

Let h_k be the k-th Hermite function. Then,

$$B(h_k)(z) = \frac{\pi^{k/2} z^k}{\sqrt{k!}}$$

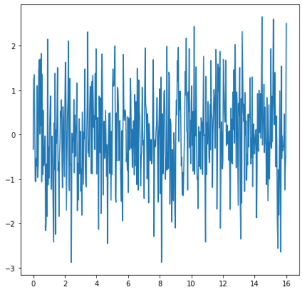
The spectrogram of ξ writes

$$Spec(\xi)(z) = e^{-\pi|z|^2} \left|F(\bar{z})\right|^2, \quad \text{where} \quad F(z) := \sum_{k \geqslant 0} a_k \frac{\pi^{k/2} z^k}{\sqrt{k!}}, \; z \in \mathbb{C}.$$

F is the planar Gaussian analytic function with covariance kernel

$$K(z,w) = \mathbb{E}[F(z)\overline{F(w)}] = \sum_{j,k \geqslant 0} \mathbb{E}[a_j a_k] \frac{\pi^{(z+k)/2} z^j \bar{w}^k}{\sqrt{j!k!}} = \sum_{j,k \geqslant 0} \frac{(z\bar{w})^k}{k!} = e^{\pi z \bar{w}}$$

Spectrogram of complex white noise



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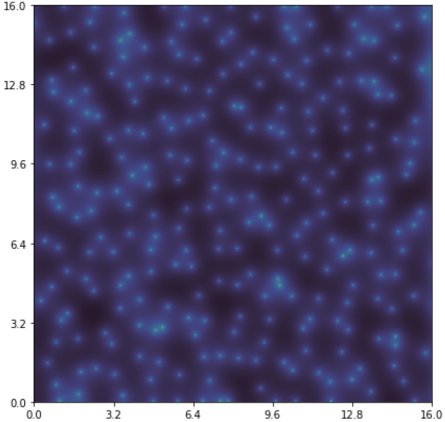

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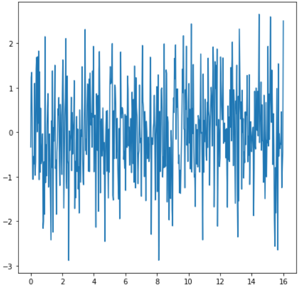

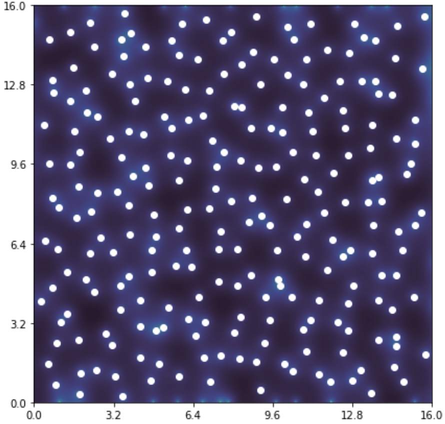

Definition

Let X be a point process. We define its intensity function $\rho : \mathbb{C} \to \mathbb{R}_+$ by

$$\rho(z)\mathrm{d}z = \mathbb{P}(X \textit{ has a point in a ball centered in } z \textit{ with volume } \mathrm{d}z)$$

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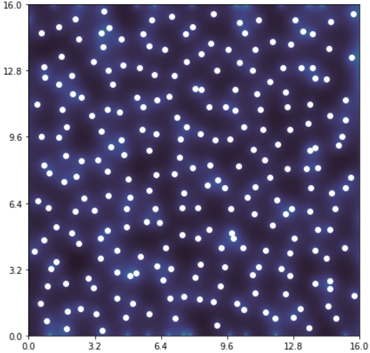
If X is the point process of zeros of the spectrogram of white noise then

- X is stationary;
- X is isotropic;
- $\forall z \in \mathbb{C}, \quad \rho(z) = 1$;
- X is NOT a DPP.



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If X is the point process of zeros of the spectrogram of a noisy signal then

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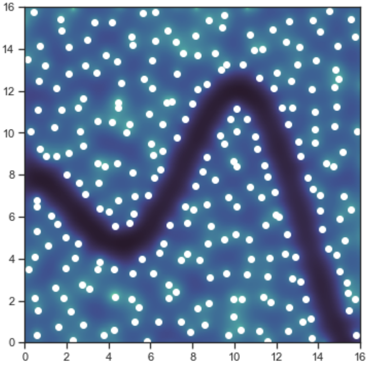

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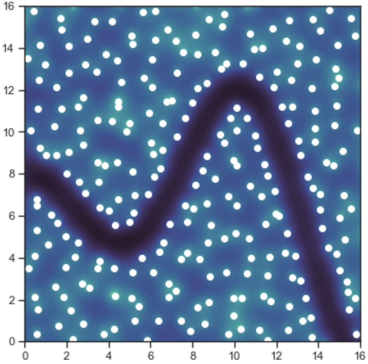
Main problem: What are the properties of the zeros of the spectrogram of a noisy signal?



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Theorem ([1])

The intensity $\rho(z)$ of the point process of zeros of $Spec(x+\xi)(z)$ satisfies

$$\rho(z) = \left(1 + Spec(x)(z) + \frac{\Delta Spec(x)(z)}{4\pi}\right) \exp\left(-Spec(x)(z)\right)$$

[1] Formula appears in a different form in: *Luis Alberto Escudero, Naomi Feldheim, Günther Koliander, José-Luis Romero.* Efficient computation of the zeros of the Bargmann transform under additive white noise. (2021)

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Attempt of interpretation:

- $(1 + Spec(x)(z))\exp(-Spec(x)(z))$ decreasing function of $Spec(x)(z) \Longrightarrow$ zeros avoid locations where $Spec(x)(z)$ has a high value.

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- $\frac{\Delta Spec(x)(z)}{4\pi} \exp\left(-Spec(x)(z)\right) \implies$ zeros prefers locations where $Spec(x)(z)$ has large variations.

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Proof

The spectrogram of a noisy signal $x + \xi$ writes

$$Spec(x+\xi)(z) = e^{-\pi|z|^2} \left|B(x+\xi)(\bar{z})\right|^2 = e^{-\pi|z|^2} \left|F(\bar{z}) + B(x)(\bar{z})\right|^2 .$$

Let $\rho(z)$ be the intensity function of zeros of the Gaussian field $F(\bar{z}) + B(x)(\bar{z})$.

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The spectrogram of a noisy signal $x + \xi$ writes

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<u>Kac-Rice formula:</u>

$$\rho(\bar{z}) = \mathbb{E}\left[\det\begin{pmatrix} \partial_z(F(z)+B(x)(z)) & \partial_z\overline{(F(z)+B(x)(z))} \\ \partial_{\bar{z}}(F(z)+B(x)(z)) & \partial_{\bar{z}}\overline{(F(z)+B(x)(z))} \end{pmatrix} \middle| F(z)+B(x)(z)=0\right] \times p_{F(z)+B(x)(z)}(0)$$

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$$\Rightarrow \rho(\bar{z}) = \mathbb{E}\left[\left|\partial_z(F(z)+B(x)(z))\right|^2 \middle| F(z)+B(x)(z) = 0\right] p_{F(z)}(-B(x)(z))$$

$$F(z) \sim \mathcal{N}_{\mathbb{C}}(0, e^{\pi|z|^2}) \;\Rightarrow\; p_{F(z)}(-B(x)(z)) = \frac{\exp\left(-e^{-\pi|z|^2}|B(x)(z)|^2\right)}{\pi e^{\pi|z|^2}},$$

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hence

$$\rho(\bar{z}) = \mathbb{E}\left[|\nabla'_z(F(z) + B(x)(z))|^2 \middle| F(z) + B(x)(z) = 0\right] \times \frac{\exp\left(-e^{-\pi|z|^2}|B(x)(z)|^2\right)}{\pi e^{\pi|z|^2}}$$

where $\nabla'_z = \partial_z - \pi\bar{z}$.

Note that

$$\mathbb{E}\left[\nabla'_z F(z)\overline{F(w)}\right] = \nabla'_z K(z,w) = \nabla'_z(e^{\pi z \bar{w}}) = \pi(\bar{\omega} - \bar{z})e^{\pi z \bar{w}},$$

hence

$$\mathbb{E}\left[\nabla'_z F(z)\overline{F(z)}\right] = 0 \;\Rightarrow\; F(z) \text{ and } \nabla'_z F(z) \text{ are independent.}$$

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Thus,

$$\rho(\bar{z}) = \left(\mathbb{E}[|\nabla'_z F(z)|^2] + |\nabla'_z B(x)(z)|^2\right) \frac{\exp\left(-e^{-\pi|z|^2}|B(x)(z)|^2\right)}{\pi e^{\pi|z|^2}}$$

Finally,

$$\mathbb{E}[\nabla'_z F(z)\overline{\nabla'_\omega F(w)}] = \nabla'_z \nabla'_{\bar{\omega}}(e^{\pi z \bar{w}}) = \pi(1 + \pi|z - w|^2)e^{\pi z \bar{\omega}}$$

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$$\Longrightarrow \boxed{\rho(z) = \left(1 + \frac{|\nabla'_{\bar{z}} B(x)(\bar{z})|^2}{\pi e^{\pi|z|^2}}\right)\exp\left(-e^{-\pi|z|^2}|B(x)(\bar{z})|^2\right)}$$

Expressing $\rho(z)$ as a function of $Spec(x)(z)$

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- $\Delta Spec(x)(z) = 4 \frac{|\nabla'_{\bar{z}}(B(x)(\bar{z}))|^2}{e^{\pi|z|^2}} - 4\pi Spec(x)(z)$

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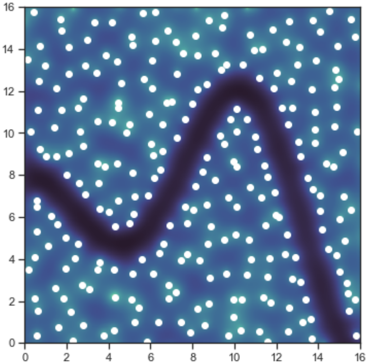
Summary

Theorem

$$\rho(z) = \left(1 + Spec(x)(z) + \frac{\Delta Spec(x)(z)}{4\pi}\right) \exp\left(-Spec(x)(z)\right)$$

$Spec(x)(z)$	$\Delta Spec(x)(z)$	$\rho(z)$



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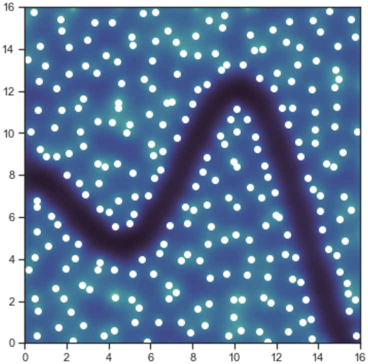
$Spec(x)(z)$	$\Delta Spec(x)(z)$	$\rho(z)$
≈ 0	≈ 0	1



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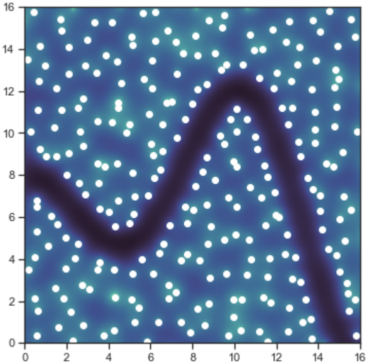
$\gg 0$	Any	≈ 0



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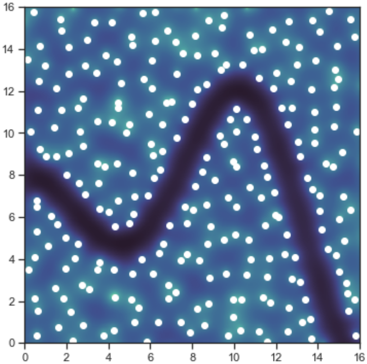
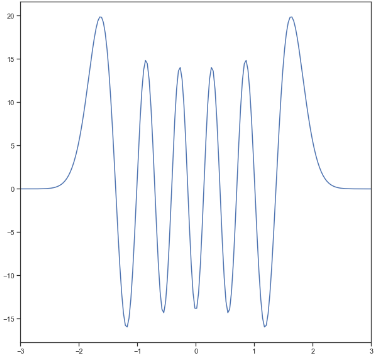


Outline

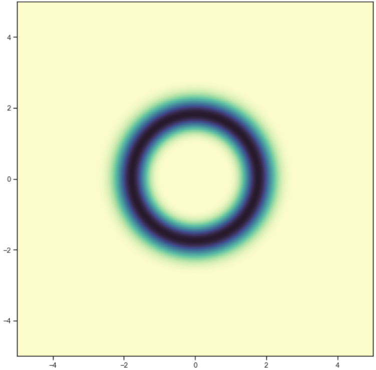
Proposition

Let h_k be the k-th Hermite function ($k \in \mathbb{N}\backslash\{0\}$). The spectrogram of h_k satisfies

$$\mathrm{Spec}(h_k)(z) = \frac{\pi^k |z|^{2k}}{k!} e^{-\pi |z|^2}.$$



(a) Plot of h_{10}

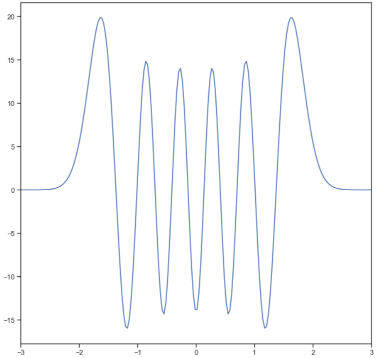


(b) Spectrogram of h_{10}

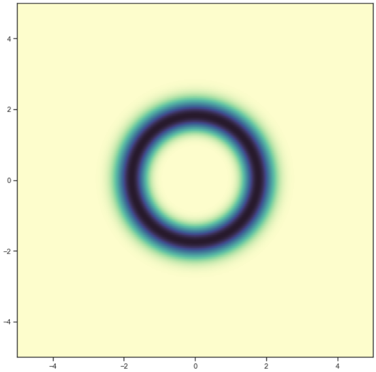
Proposition

Let h_k be the k-th Hermite function ($k \in \mathbb{N} \setminus \{0\}$). The spectrogram of h_k only depends on $r = \pi|z|^2$ and satisfies

$$\mathrm{Spec}(h_k)(r) = \frac{r^k}{k!} e^{-r}.$$



(c) Plot of h_{10}

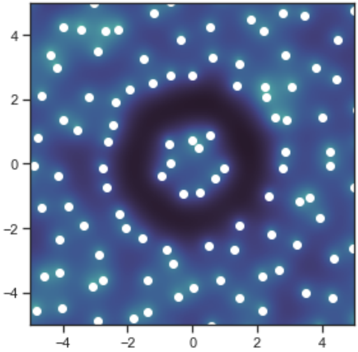


(d) Spectrogram of h_{10}

Proposition

Let $\gamma \in \mathbb{R}_+$. The intensity of zeros $\rho(z)$ of $\mathrm{Spec}(\xi + \sqrt{\gamma} h_k)$ *only depends on $r = \pi |z|^2$ and is equal to*

$$\rho(r) = \left(1 + \gamma \frac{r^{k-1}}{k!} (k-r)^2 e^{-r}\right) \exp\left(-\gamma \frac{r^k}{k!} e^{-r}\right)$$



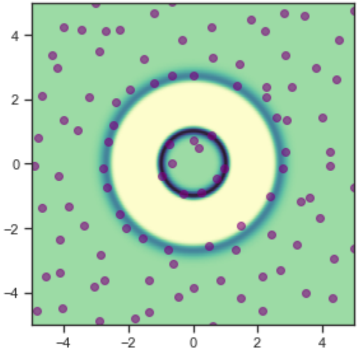


Figure: **Left**: Spectrogram of a noisy hermite function for $\gamma = 1000$ and $k = 10$. Zeros are shown in white. **Right**: Heatmap of $\rho(z)$. The zeros of the figure on the left are shown in purple.

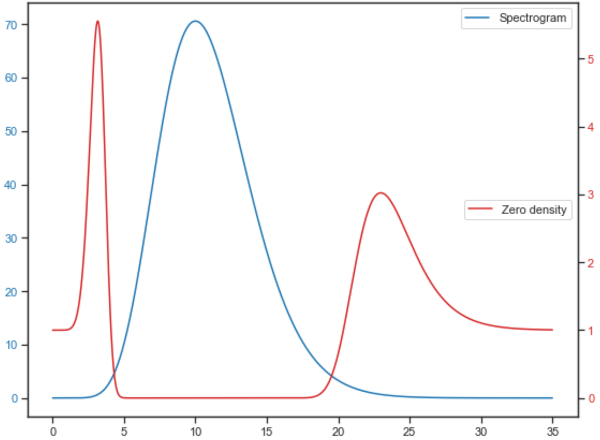


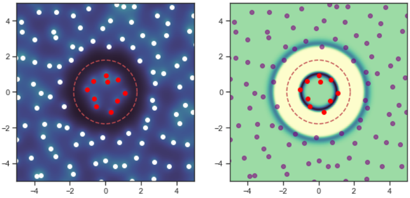
Figure: Blue: $Spec(\sqrt{1000}h_{10})(z)$ with respect to $r = \pi|z|^2$. Red: Density of zeros of $Spec(\xi + \sqrt{1000}h_{10})(z)$ with respect to $r = \pi|z|^2$.

Proposition

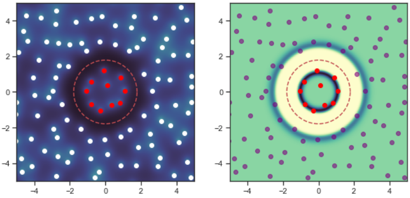
Let $k \in \mathbb{N}\backslash\{0\}$. Let $N(B(0,R))$ be the number of zeros of $Spec(\xi + \sqrt{\gamma}h_k)$ in a centered ball of radius R. Then,

$$\mathbb{E}[N(B(0,R))] = k - (k - \pi R^2)\exp\left(-\gamma \frac{\pi^k R^{2k}}{k!} e^{-\pi R^2}\right).$$

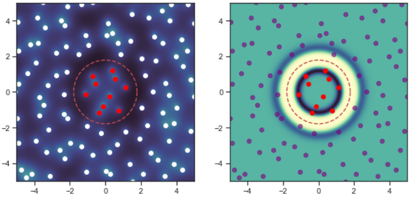
An interesting consequence is that the average number of zeros in $B\left(0, \sqrt{\frac{k}{\pi}}\right)$ is equal to k and does not depends on γ.



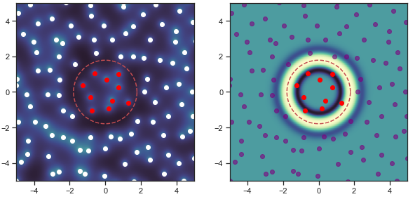
(a) $\gamma = 1000$



(b) $\gamma = 500$



(c) $\gamma = 100$



(d) $\gamma = 50$

Figure: $Spec(\xi + \sqrt{\gamma} h_{10})$ and $\rho(z)$ for various values of γ. The red line show the ball with radius $\sqrt{10/\pi}$. Zeros inside the ball are shown in red.

Theorem (Rouché's theorem)

Let C be a closed, simple curve of $\mathbb{C}$. Let f and g be holomorphic functions on the interior of C. If

$$\forall z \in C, \quad |f(z) - g(z)| < |f(z)|,$$

then f and g have the same number of zeros in the interior of C, where each zero is counted as many times as its multiplicity.

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Applying this result to $f(z) = B(x)(z)$ and $g(z) = F(z) + B(x)(z)$ yields

Corollary (The trapping of zeros)

Let C be a closed, simple curve of $\mathbb{C}$ and let $x : \mathbb{R} \to \mathbb{C}$. If

$$\forall z \in C, \quad \mathrm{Spec}(\xi)(z) < \mathrm{Spec}(x)(z),$$

then $\mathrm{Spec}(\xi + x)(z)$ and $B(x)(\bar{z})$ have the same number of zeros in the interior of C, where each zero is counted as many times as its multiplicity.

Recall that $B(h_k)(z) = \frac{\pi^{k/2} z^k}{k!} \Rightarrow 0$ is a zero with multiplicity k.

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Proposition

Let $k \in \mathbb{N}$ and $\varepsilon > 0$. Let $N(B(0,R))$ be the number of zeros of $Spec(\xi + \sqrt{\gamma} h_k)$ in a centered ball of radius R.
If $\gamma > e^{-k}$ and

$$\gamma > \frac{k!}{k^k}\left(-\log(\varepsilon) + \frac{k}{2} + \log\left(2\sqrt{k+k^2} + \frac{1}{\sqrt{\pi}}\right)\right)$$

then

$$\mathbb{P}\left(N\left(B\left(0, \sqrt{k/\pi}\right)\right) = k\right) \geqslant 1 - \varepsilon.$$

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then

$$\mathbb{P}\left(N\Big(B\Big(0, \sqrt{k/\pi}\Big)\Big) = k \right) \geqslant 1 - \varepsilon.$$

In other words,

$$\mathbb{P}\left(N\Big(B\Big(0, \sqrt{k/\pi}\Big)\Big) = k \right) = 1 - O(e^{-\gamma}).$$

Outline

We consider a linear chirp $x(t) := e^{2i\pi t(a+bt)}$.

Proposition

For all $z = \tau + i\omega \in \mathbb{C}$,

$$Spec(x)(z) = \sigma_b e^{-\pi\sigma_b^2(\omega-(a+b\tau))^2}, \ \textit{where } \sigma_b := \sqrt{\frac{2}{1+4b^2}}.$$

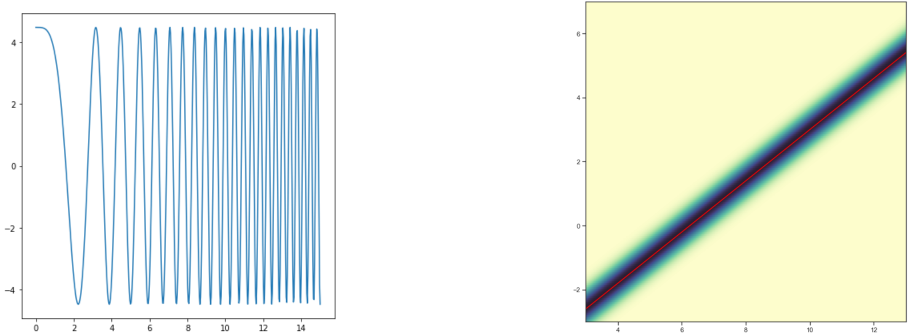


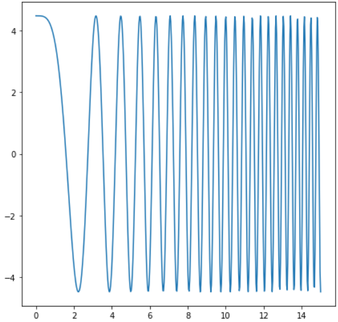
Figure: **Left**: Plot of a linear chirp. **Right**: Spectrogram of $x(t) = e^{2i\pi t(-5+0.4t)}$. The line $y = -5 + 0.8x$ is shown in red.

We consider a linear chirp $x(t) := e^{2i\pi t(a+bt)}$.

Proposition

For all $z = \tau + i\omega \in \mathbb{C}$, Spec$(x)(r)$ only depends on $r := \frac{\sigma_b}{\sqrt{2}}(\omega - (a + 2b\tau))$, the distance between z and the line $y = a + 2bx$, and

$$Spec(x)(r) = \sigma_b e^{-\pi\sigma_b^2 r^2}, \text{ where } \sigma_b := \sqrt{\frac{2}{1+4b^2}}.$$



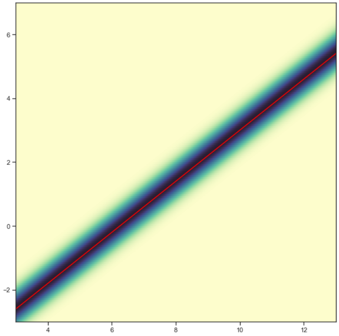


Figure: **Left**: Plot of a linear chirp. **Right**: Spectrogram of $x(t) = e^{2i\pi t(-5+0.4t)}$. The line $y = -5 + 0.8x$ is shown in red.

Proposition

The density $\rho(z)$ *of zeros of* $Spec(\xi + \sqrt{\gamma}x)(z)$ *only depends on* $r := \frac{\sigma_b}{\sqrt{2}}(\omega - (a + 2b\tau))$ *and is equal to*

$$\rho(r) = \left(1 + 4\pi\gamma\sigma_b r^2 e^{-2\pi r^2}\right) \exp\left(-\gamma\sigma_b e^{-2\pi r^2}\right), \text{ where } \sigma_b := \sqrt{\frac{2}{1+4b^2}}.$$

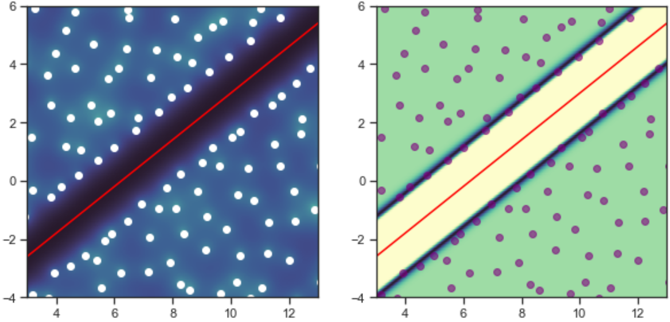


Figure: **Left**: Spectrogram of a noisy chirp for $\gamma = 1000$, $a = -5$ and $b = 0.4$. Zeros are shown in white. **Right**: Heatmap of $\rho(z)$. The zeros of the figure on the left are shown in purple.

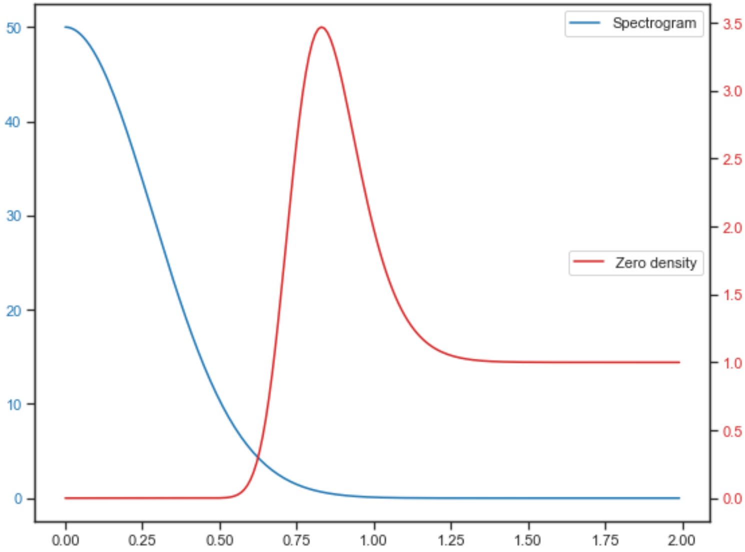


Figure: Blue: Spectrogram of the chirp with respect to the distance with the central axis. Red: Density of zeros of the noisy chirp with respect to the distance with the central axis.

Let $R \in \mathbb{R}_+$. Consider the rectangle

$$B = \left\{ z = \tau + i\omega \in \mathbb{C} : r \in [0, R], \frac{\sigma_b}{\sqrt{2}}(\tau + 2b\omega) \in [0, 1] \right\}.$$

Proposition

Let $N(B)$ be the number of zeros of $\mathrm{Spec}(\xi + \sqrt{\gamma}x)$ in B. Then,

$$\mathbb{E}[N(B)] = R \exp\left(-\gamma\sigma_b e^{-2\pi R^2}\right)$$

Outline

In this section, we consider two parallel linear chirps:

$$x(t) = \sqrt{\gamma_1} e^{2i\pi t(a_1+bt)} + \sqrt{\gamma_2} e^{2i\pi t(a_2+bt)}.$$

Proposition

Let $z = \tau + i\omega \in \mathbb{C}$ *and define*

$$\begin{cases} r := \frac{\sigma_b}{\sqrt{2}}(\omega - 2b\tau - a_1); \\ a := \frac{\sigma_b}{\sqrt{2}}(a_2 - a_1); \\ s := \frac{\sigma_b}{\sqrt{2}}(\tau + 2b\omega - (a_1 + a_2)b); \end{cases}$$

Then, $Spec(x)(z)$*, written as a function of* r *and* s *is equal to*

$$Spec(x)(r,s) = \sigma_b \left(\gamma_1 e^{-2\pi r^2} + \gamma_2 e^{-2\pi (r-a)^2} \right. \\ \left. + 2\sqrt{\gamma_1 \gamma_2} e^{-\pi r^2 - \pi (r-a)^2} \cos(2\pi a s) \right).$$

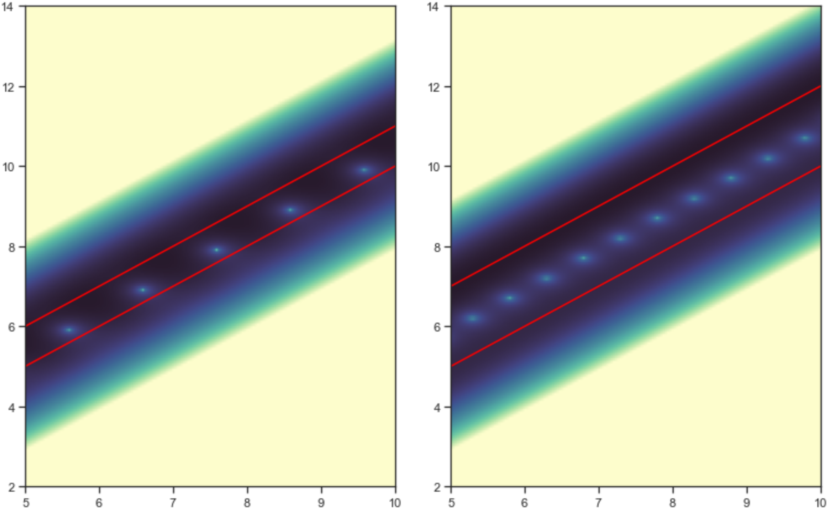


Figure: Spectrogram of two chirps with $\gamma_1 = 10$, $\gamma_2 = 30$, $b = 1/2$, $a_1 = 0$ and a_2 being either 1 (left figure) or 2 (right figure).

The spectrogram of the superposition of chirps vanish at some points.

Proposition

Spec$(x)(r,s) = 0$ *if and only if*

$$r = \frac{a}{2} - \frac{\log(\gamma_2) - \log(\gamma_1)}{4a\pi} \text{ and } \exists k \in \mathbb{Z} \text{ s.t. } s = \frac{1+2k}{2a}$$

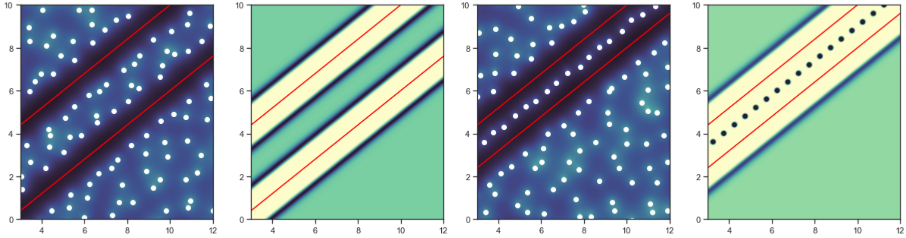
Recall that we consider two parallel linear chirps:

$$x(t) = \sqrt{\gamma_1} e^{2i\pi t(a_1+bt)} + \sqrt{\gamma_2} e^{2i\pi t(a_2+bt)}.$$

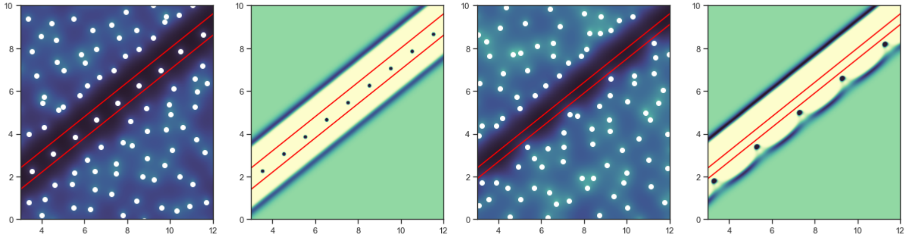
Proposition

The density $\rho(z)$ of the zeros of $\mathrm{Spec}(\xi + x)$ *can be expressed as a function of r and s by*

$$\rho(r,s) = \Big(1 + 4\pi\sigma_b \Big(\gamma_1 r^2 e^{-2\pi r^2} + \gamma_2 (r-a)^2 e^{-2\pi(r-a)^2} + 2\sqrt{\gamma_1\gamma_2}\, r(r-a) e^{-\pi r^2 - \pi(r-a)^2} \cos(2\pi a s)\Big)\Big) \exp\Big(-\sigma_b \Big(\gamma_1 e^{-2\pi r^2} + \gamma_2 e^{-2\pi(r-a)^2} + 2\sqrt{\gamma_1\gamma_2}\, e^{-\pi r^2 - \pi(r-a)^2} \cos(2\pi a s)\Big)\Big).$$



(a) $\gamma_1 = \gamma_2 = 100,\ a = 4,\ b = 0.4$ (b) $\gamma_1 = \gamma_2 = 100,\ a = 2,\ b = 0.4$



(c) $\gamma_1 = 100,\ \gamma_2 = 50,\ a = 1,\ b = 0.4$ (d) $\gamma_1 = 500,\ \gamma_2 = 50,\ a = 0.5,\ b = 0.4$

Figure: For each figure; **Left**: Spectrogram of two noisy parallel chirps. Zeros are shown in white. **Right**: Heatmap of $\rho(z)$. The zeros of the figure on the left are shown in purple.

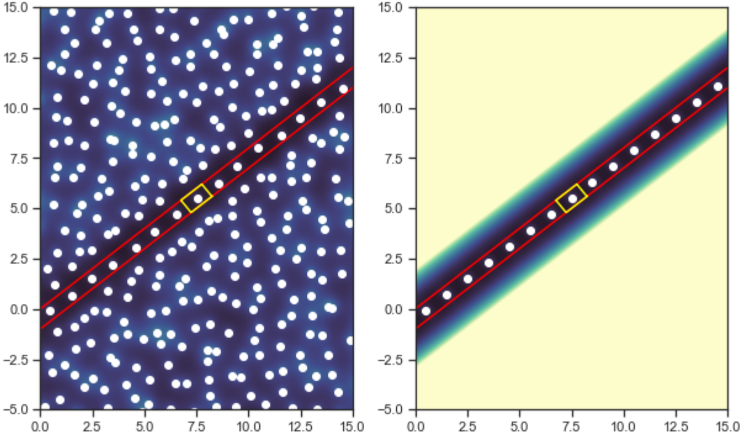


Figure: Spectrogram of two chirps with noise (on the left) and without noise (on the right). In each case, the zeros are shown as white circles and one of the curve C_N considered in the following Proposition is shown in yellow.

Proposition

Let $\varepsilon > 0$ and $N \in \mathbb{N}$. We consider the parallelogram C_N whose extremities are the points such that $(r, s) \in \{(0, N/a), (0, (N+1)/a), (a, N/a), (a, (N+1)/a)\}$ where N is any integer. We then consider the following assumptions:

1. $a \leqslant \sqrt{\frac{2}{\pi}}$ or $\frac{1}{2}|\log(\gamma_1) - \log(\gamma_2)| \geqslant -2\text{arccosh}\left(\sqrt{\frac{\pi}{2}}a\right) + a\sqrt{\pi^2 a^2 - 2\pi}$;
2. $|\log(\gamma_1) - \log(\gamma_2)| < 2\pi a^2$
3. $4\frac{e^{-u^2}}{\sqrt{2\pi}}\left(a + \frac{1}{a} + \frac{2\sqrt{2}}{u}\right) \leqslant \varepsilon$; *where* $u = \sqrt{\sigma}_b \min\left(|\sqrt{\gamma}_1 - \sqrt{\gamma}_2 e^{-\pi a^2}|, |\sqrt{\gamma}_2 - \sqrt{\gamma}_1 e^{-\pi a^2}|\right)$.

Then, under these three assumptions, $Spec(\xi + x)$ has a unique zero inside C_N with probability $\geqslant 1 - \varepsilon$.

If $\gamma_1 = \gamma_2 = \gamma$ and $a \leqslant \sqrt{\frac{2}{\pi}}$,

$$\mathbb{P}\left(\text{unique zero inside } C_N\right) = 1 - O\left(e^{-\gamma\sigma_b\left(1 - e^{-\pi a^2}\right)^2}\right).$$

Outline

Open problems

- Higher order correlation between zeros of the spectrogram of a noisy signal.
- Density of local maxima of the spectrogram of a noisy signal.
- Correlation of zeros and local maxima of the spectrogram of white noise.

Problems I've tried to tackle but gave up because it looked too complicated for me.

- Higher order correlation between zeros of the spectrogram of a noisy signal.
- Density of local maxima of the spectrogram of a noisy signal.
- Correlation of zeros and local maxima of the spectrogram of white noise.

Proposition

Let $\rho_{cross}(z, w)$ be the cross correlation between zeros and local maxima of the spectrogram of complex Gaussian white noise at points $z, w \in \mathbb{C}$. Then, $\rho_{cross}(z, w)$ only depends on $r = |z - w|\sqrt{\pi}$ and is equal to

$$\int_{\mathbb{C}^3} \frac{|u|^2(|w|^2 - |v|^2)\mathbb{1}_{|w|>|v|}}{\det(\Lambda(r))\pi^3} \exp\left(- \left\langle \begin{pmatrix} 0 \\ u \\ 0 \\ v \\ w \end{pmatrix}, \Lambda(r)^{-1} \begin{pmatrix} 0 \\ u^* \\ 0 \\ v^* \\ w^* \end{pmatrix} \right\rangle \right) \mathrm{d}u\mathrm{d}v\mathrm{d}w,$$

where

$$\Lambda(r) := \begin{pmatrix} 1 & 0 & -r & r^2 & -1 \\ 0 & 1 & 1 - r^2 & -r(2 - r^2) & -r \\ -r & 1 - r^2 & e^{r^2} & 0 & 0 \\ r^2 & -r(2 - r^2) & 0 & 2e^{r^2} & 0 \\ -1 & -r & 0 & 0 & e^{r^2} \end{pmatrix}.$$

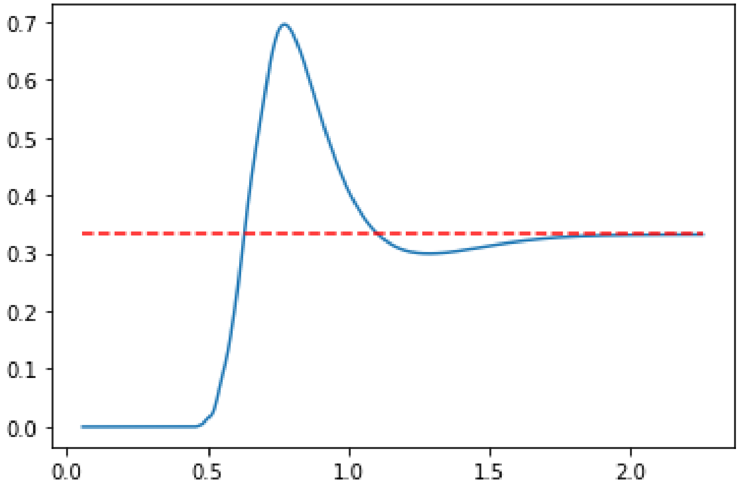


Figure: Monte-Carlo approximation of the cross-correlation between zeros and local maxima with respecting to their relative distance.

Conclusion

- Expression of the density of zeros of Spec($x + \xi$) as a function of Spec(x).
- Rouché's theorem explains the "trapping" of zeros.
- We can mathematically describe the intensity of zeros when the signal is Hermite, a linear chirp or two parallel linear chirps.

Conclusion

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Thank you for your attention!