# Sampling, interpolation, and repulsive point processes

José Luis Romero

University of Vienna
and Austrian Academy of Sciences

Der Wissenschaftsfonds.

# Bandlimited functions

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$$\mathrm{PW}(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [-1/2, 1/2] \right\}, \qquad \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$$

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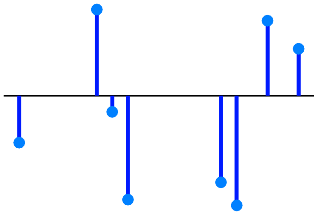

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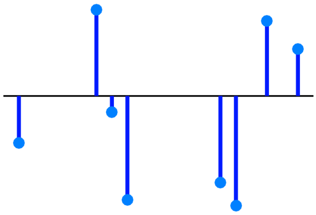

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$$|f(x + iy)| \lesssim e^{\pi |y|}$$

## The density theorems for Bargmann-Fock spaces

$$\mathcal{F}(\mathbb{C}) := \left\{ f : \mathbb{C} \to \mathbb{C} \text{ analytic } : \|f\|^2 = \int_{\mathbb{C}} |f(x+iy)|^2 e^{-\pi(x^2+y^2)}\, dxdy < \infty \right\}$$

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**Theorem (Seip, Wallstén, 1992)**

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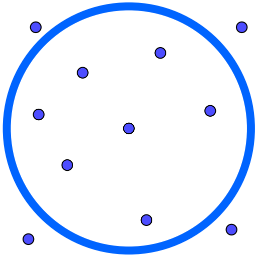

* K Seip, R Wallstén, J. für die Reine und Angew. Math., 1992.

# The density theorems for Bergman spaces

$$\mathcal{A}_\alpha(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic } : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \left(1 - |z|^2\right)^\alpha \frac{dxdy}{(1-|z|^2)^2} < \infty \right\}$$

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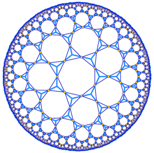

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* K Seip, Beurling type density theorems in the unit disk, Invent. Math., 1993.

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**Theorem (Gröchenig, R-, Stöckler, 2018)**

*Let* $\hat{g}(\xi) = e^{-a\xi^2}/p(\xi)$, $p \in \mathbb{C}[\xi]$ *with only imaginary zeros and* $\Lambda \subseteq \mathbb{R}$ *separated. Then*

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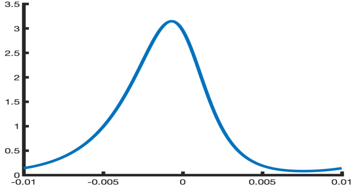
Totally positive functions of Gaussian-type (Schönberg, 1950's)

Unlike $\frac{\sin(\pi x)}{\pi x}$, $g$ decays exponentially

$\hat{g}(\xi) = e^{-\xi^2}(1+0.5\pi i\xi)^{-1}(1+1.1\pi i\xi)^{-1}(1+1.25\pi i\xi)^{-1}(1+2.5\pi i\xi)^{-1}$

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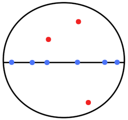


$$\#\{z \in B_r(0) : f(z) = 0\} \sim 2r$$

* Gröchenig, K., Romero, J. L., Stöckler, J. Invent. Math., 2018

# The density theorems for shift-invariant spaces

$$\mathrm{V}_g(\mathbb{R}) := \left\{ f(x) = \sum_{k\in\mathbb{Z}} c_k g(x-k) : c \in \ell^2(\mathbb{Z}) \right\}, \qquad g : \mathbb{R} \to \mathbb{C} \text{ smooth and decaying}$$

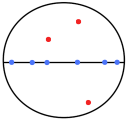
**Theorem (Gröchenig, R-, Stöckler, 2018)**

*Let $\hat{g}(\xi) = e^{-a\xi^2}/p(\xi)$, $p \in \mathbb{C}[\xi]$ with only imaginary zeros and $\Lambda \subseteq \mathbb{R}$ separated. Then*

$$D^-(\Lambda) > 1 \quad \Rightarrow \quad \|f\|_2^2 \asymp \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \quad (f \in \mathrm{V}_g(\mathbb{R})) \quad \Rightarrow \quad D^-(\Lambda) \geq 1$$

$$D^+(\Lambda) < 1 \quad \underset{\text{(for } p \equiv 1)}{\Rightarrow} \quad \forall a \in \ell^2(\Lambda), \exists f \in \mathrm{V}_g(\mathbb{R}) \, : \, f(\lambda) = a_\lambda \quad \Rightarrow \quad D^+(\Lambda) \leq 1$$

$$f \in \mathrm{PW}(\mathbb{R}) : |f(x+iy)| \leq Ce^{\pi|y|}$$



$$\#\{z \in B_r(0) : f(z) = 0\} \sim 2r$$

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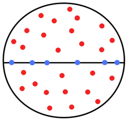

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# Disconnected spectra

$\mathrm{PW}_\Omega := \{ f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subseteq \Omega \}$, with $\Omega \subseteq \mathbb{R}$ be compact and $|\Omega| = 1$

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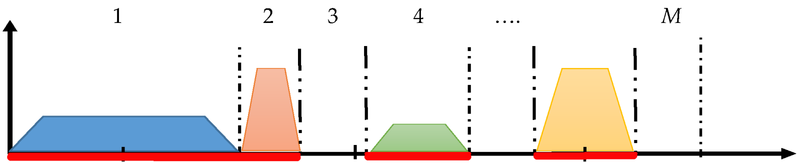
Relevance in telecommunications

Multi-band spectra: union of intervals



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Soft theorems:

$$D^-(\Lambda) > C \Rightarrow \text{sampling on } PW_\Omega \Rightarrow D^-(\Lambda) \geq c$$

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**Beurling's problem**: Sampling $\Rightarrow D^-(\Lambda) \geq 1$, interpolation $\Rightarrow D^+(\Lambda) \leq 1$

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**Theorem (Landau, 1967)**

*Let* $\Omega \subseteq \mathbb{R}$ *be compact with* $|\Omega| = 1$, *and* $\Lambda \subseteq \mathbb{R}$ *arbitrary. Then*

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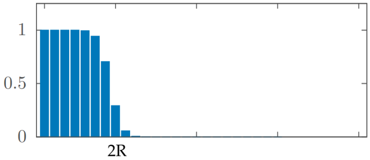
Concentration operator

$$T_{[-R,R]}\, f = P_{PW_\Omega}\big(f \cdot 1_{[-R,R]}\big)$$

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Ex. $\Lambda = \{k + \{\alpha k\} : k \in \mathbb{Z}\}$, $\alpha$ irrational, $D^-(\Lambda) = D^+(\Lambda) = 1$


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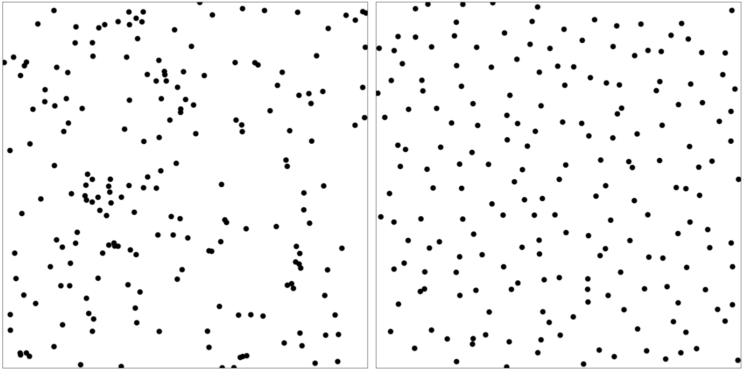
# Repulsive point processes



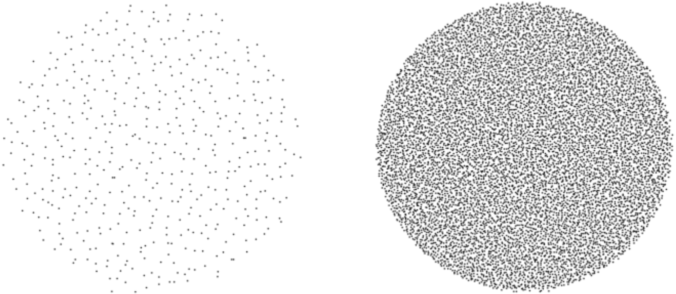
# The planar Coulomb gas

Random configuration of points $\{\zeta_1, \dots, \zeta_n\} \subseteq \mathbb{C}$ from the Boltzmann-Gibbs distribution:

$$P_n^{\beta}(\zeta_1, \dots, \zeta_n) \propto \exp\left(-\beta \left[\sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j)\right]\right)$$

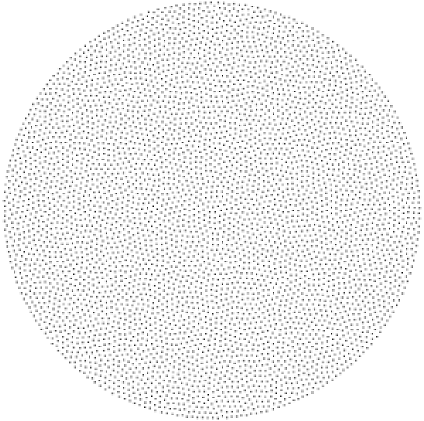
- $Q : \mathbb{C} \to \mathbb{R}$ is the external potential (large at $\infty$)
  Confines the configuration to a compact region
  Model case: $Q(z) = |z|^2$
- $\beta > 0$ is the inverse temperature
  High temperature (small $\beta$) $\Rightarrow$ more disorder
  Low temperature ($\beta = \beta(n) \longrightarrow \infty$) $\Rightarrow$ more order
- Frozen gas: $\beta = \infty \Rightarrow$ Fekete sets

# Warm temperature: $\beta = 1$



$Q(z) = |z|^2 \qquad \beta = 1 \qquad n \longrightarrow \infty$

# Very low temperature: $\beta \gg 1$



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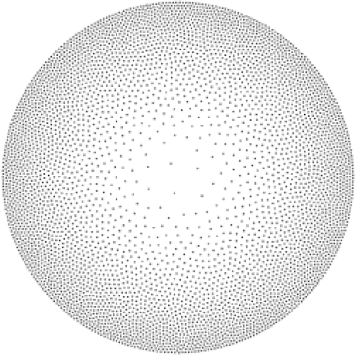

$Q(z) = |z|^4$

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There exists a compact set $S$, called droplet, such that

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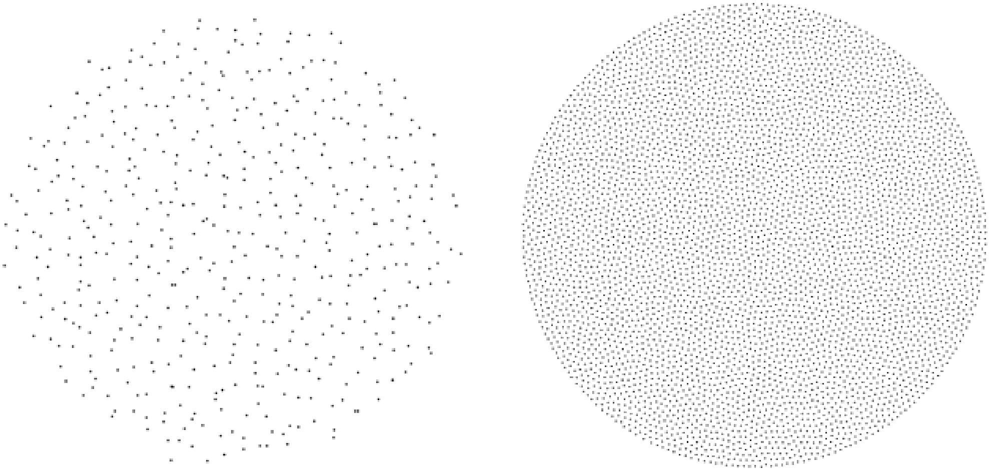
Freezing regime: $\beta = \beta_n \to \infty$ (at which rate?)

# High versus low temperature, $Q(z) = |z|^2$



$\beta = 1$ $\qquad$ $\beta \gg 1$

# Separation of Coulomb gases at low temperature

Draw $\{\zeta_{n,1}, \ldots, \zeta_{n,n}\}$ from Boltzmann-Gibbs

**Theorem (Ameur, R-, 2022)**

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For $\beta = \infty$:
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- Lieb,E.H.,Rougerie,N.,Yngvason,J, Comm. Math. Phys., 2019;

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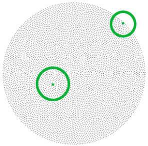

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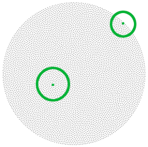

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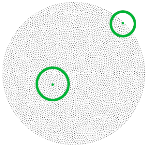

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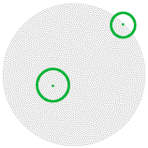

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- Relies on: Hedenmalm, H., Wennman, A., Acta Math., 2021
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# Random sampling and interpolation of weighted polynomials

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*($C = C(Q)$ deterministic, valid for $n \geq n_0$ with random $n_0$)*

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## Theorem (To compare)

*For fixed $\beta_n = \beta$ and each $\varepsilon > 0$*

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* P. Bourgade, L. Erdös, Duke Math. J., 2014.

* P. Wong. Comm. Math. Phys., 2012.

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**Theorem (Ameur, Marceca, R-)**

*Under $\beta_n \gtrsim \log n$, almost surely:*

- *(Separation) With $\sigma_n(x) = \max\{\sigma(x), n^{-\frac{1}{3}}\}$,*

$$\liminf_{n\to\infty} \min_{j\neq k} \{n\sigma_n(x_j)\cdot|x_j - x_k|\} \geq s_0 > 0$$

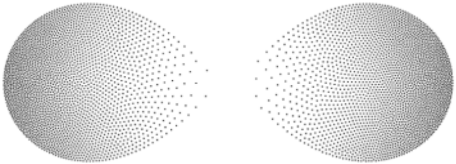
- *(Bulk) For every $y_n \to x \in (-2,2)$,*

$$\limsup_{n\to\infty} \left|\#\left[B(y_n, L/n)\right] - 2\sigma(x)L\right| \leq D\log^2 L$$

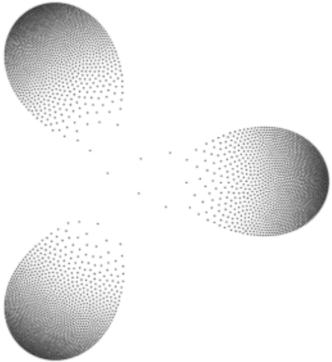
- *(Boundary) For $x = \pm 2$,*

$$\limsup_{n\to\infty} \left|\#\left[B(x,(L/n)^{\frac{2}{3}})\right] - \frac{2}{3\pi}L\right| \leq DL^{\frac{4}{5}}\log^{\frac{1}{5}} L$$

## Very low temperature: $\beta \gg 1$



$$Q(z) = |z|^4 - \frac{2}{\sqrt{2}}\mathrm{Re}(z^2)$$



$$Q(z) = |z|^6 - \frac{2}{\sqrt{5}}\mathrm{Re}(z^3)$$