

The Kravchuk transform:

A novel covariant representation for discrete signals amenable to zero-based detection tests

June 8th 2022

Barbara Pascal

Determinantal and permanental point processes, quantum physics, and signal processing

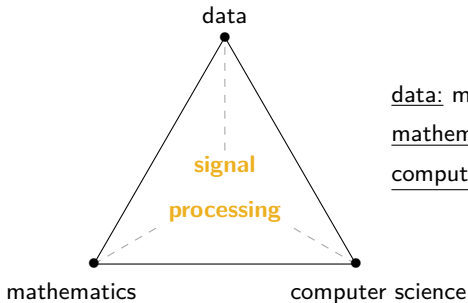
DPP fermions 2022

Signal processing aims to extract **information** from **real** data.

Data of very diverse types:

- measurements of a physical quantity,
- biological or epidemiological indicators,
- data produced by human activities.

The Golden triangle of signal processing



data: modeling of phenomena

mathematics: formalization & evaluation

computer science: efficient implementation

inspired from P. Flandrin

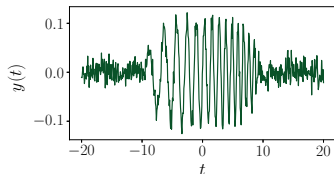
- Signal detection: the role of representations
-

- Time-frequency analysis: the Short-Time Fourier Transform
 - Signal detection based on the spectrogram zeros - I
 - Covariance principle and stationary point processes
-

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

Time and frequency: two dual descriptions of temporal signals

A continuous finite energy **signal** is a function of time $y(t)$ with $y \in L^2_{\mathbb{C}}(\mathbb{R})$.



- electrical cardiac activity,
- audio recording,
- seismic activity,
- light intensity on a photosensor
- ...

Information of interest:

- time events, e.g., an earthquake and its replica
- frequency content, e.g., monitoring of the heart beating rate

time

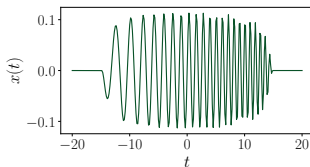
ever-changing world
marker of events and evolutions

frequency

waves, oscillations, rhythms
intrinsic mechanisms

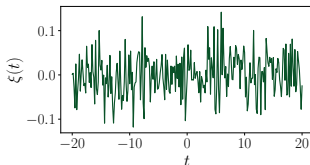
A **chirp** is a transient waveform modulated in amplitude and frequency:

$$x(t) = A_{\nu}(t) \sin \left(2\pi \left(f_1 + (f_2 - f_1) \frac{t + \nu}{2\nu} \right) t \right)$$



White noise is a random variable $\xi(t)$ such that

$$\mathbb{E}[\xi(t)] = 0 \quad \text{and} \quad \mathbb{E}[\overline{\xi(t)\xi(t')}] = \delta(t - t')$$

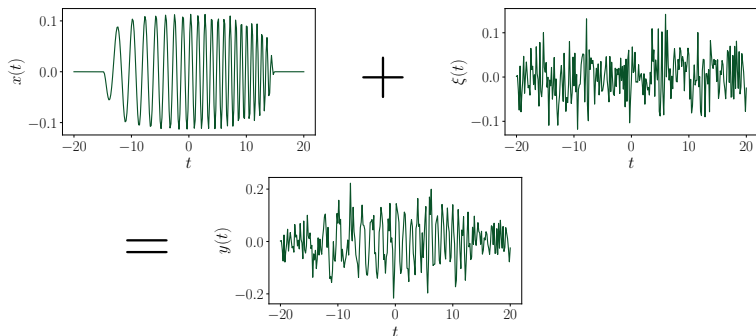


P. Flandrin: 'A signal is characterized by a structured organization.'

Signal-plus-noise observation model

Noisy observations

$$y(t) = \text{snr} \times x(t) + \xi(t)$$

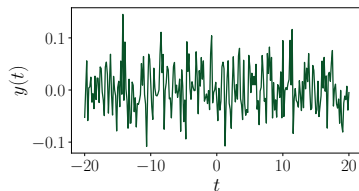
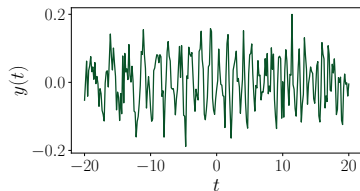


Signal processing task:

Given an observation $y(t)$

detection decides whether there is an underlying signal or only noise.

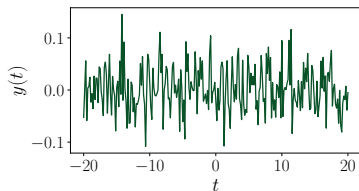
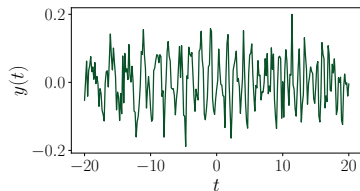
Direct observation



X hard to distinguish between oscillations and fluctuations

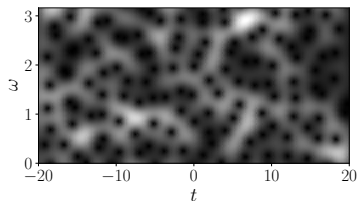
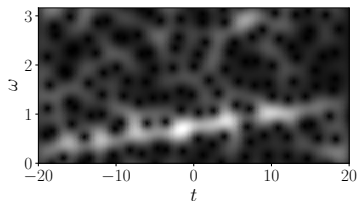
The role of representations in signal processing

Direct observation



✗ hard to distinguish between oscillations and fluctuations

Time-frequency representation



✓ lines of local maxima: structures are simpler to capture

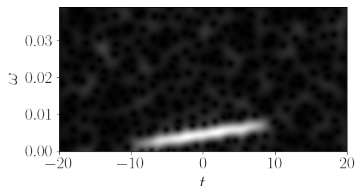
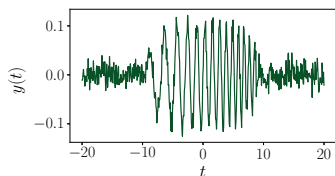
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Time and frequency Short-Time Fourier Transform with window h :

$$V_h y(t, \omega) \triangleq \int_{-\infty}^{\infty} \overline{y(u)} h(u - t) \exp(-i\omega u) du$$



Energy density interpretation $S_h y(t, \omega) = |V_h y(t, \omega)|^2$ the *spectrogram*

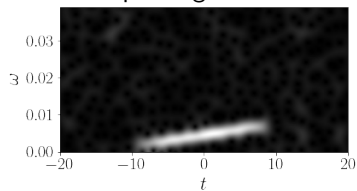
$$\int \int_{-\infty}^{+\infty} S_h y(t, \omega) dt \frac{d\omega}{2\pi} = \int_{-\infty}^{+\infty} |x(t)|^2 dt \quad \text{if } \|h\|_2^2 = 1$$

Signal, i.e., information of interest: regions of **maximal** energy.

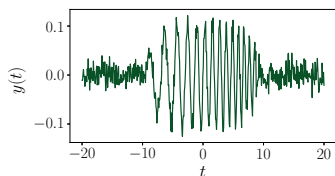
Denoising in the time-frequency plane: $y = \text{snr} \times x + \xi$,

$\text{snr} = 2$

spectrogram

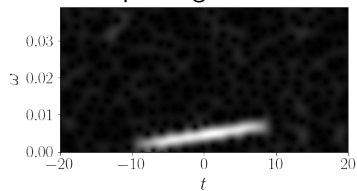


noisy observation

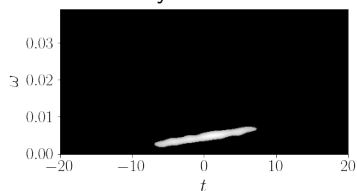


(Meignen et al., 2017)

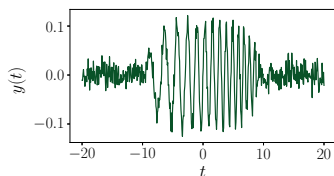
spectrogram



only maxima

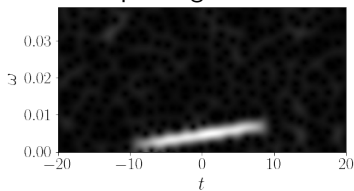


noisy observation

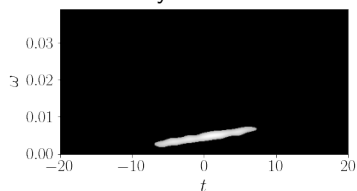


Inversion formula
$$y(t) = \int \int_{-\infty}^{+\infty} \overline{V_h y(u, \omega)} h(t - u) \exp(i\omega u) du \frac{d\omega}{2\pi}$$

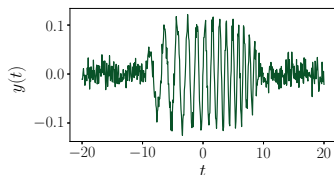
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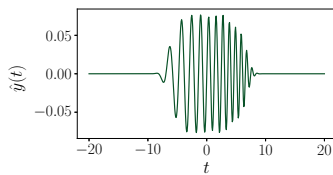
only maxima



noisy observation

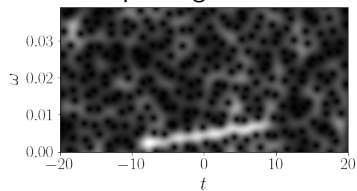


estimate

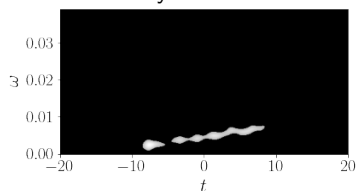


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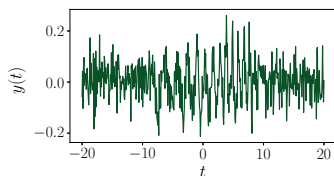
spectrogram



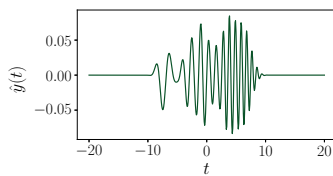
only maxima



noisy observation



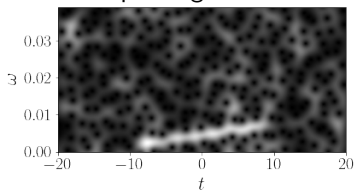
estimate



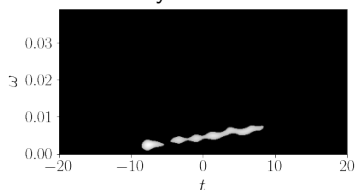
Denosing in the time-frequency plane: $y = \text{snr} \times x + \xi$, $\text{snr} = 0.5$

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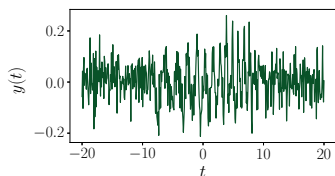
spectrogram



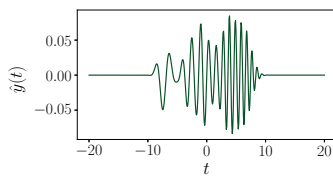
only maxima



noisy observation



estimate



Maxima extraction: *reassignment, synchrosqueezing, ridge extraction*

(Meignen et al., 2017)

Outline of the presentation

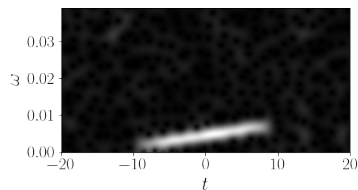
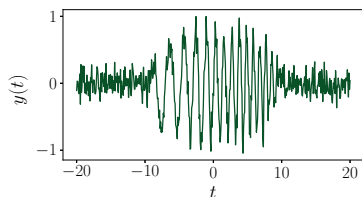
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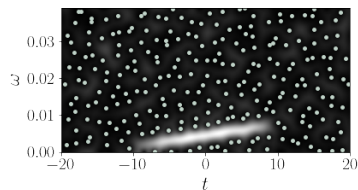
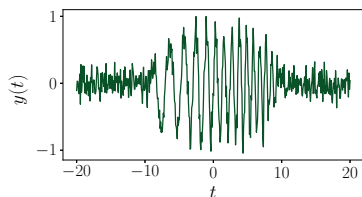
Restriction to the **circular Gaussian window**: $g(t) = \pi^{-1/4} e^{-t^2/2}$

Look for the zeros, i.e., the points (t_i, ω_i) such that $|V_g y(t_i, \omega_i)|^2 = 0$.



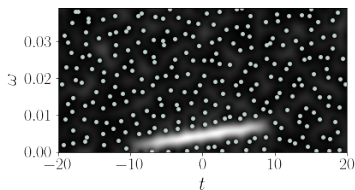
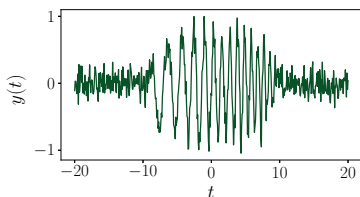
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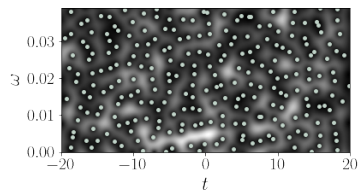
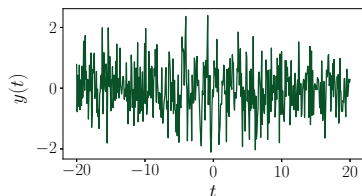


Observations: (Gardner & Magnasco, 2006), (Flandrin, 2015)

- Zeros are repelled by the signal.
- In the noise region zeros are evenly spread.
- There exists a short-range repulsion between zeros.

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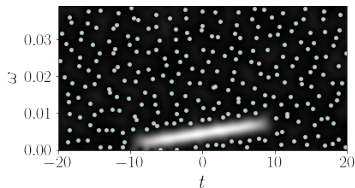
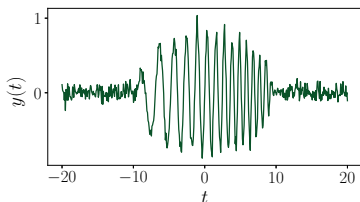


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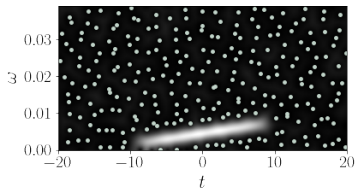
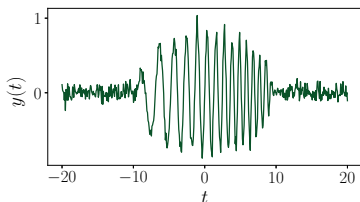


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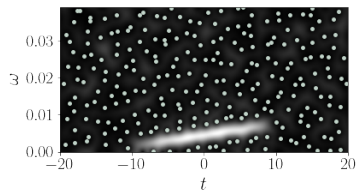
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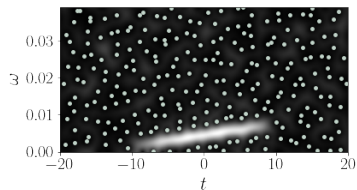
What can be said theoretically about the zeros of the spectrogram?

Unorthodox time-frequency analysis: spectrogram zeros

Idea assimilate the time-frequency plane with \mathbb{C} through $z = (\omega + it)/\sqrt{2}$



Idea assimilate the time-frequency plane with \mathbb{C} through $z = (\omega + it)/\sqrt{2}$



Bargmann factorization

$$V_g y(t, \omega) = e^{-|z|^2/2} e^{-i\omega t/2} B_y(z)$$

g the circular Gaussian window

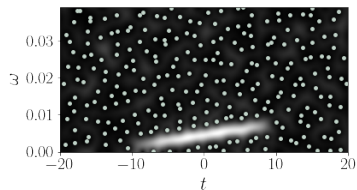
Bargmann transform of the signal y

$$B_y(z) \triangleq \pi^{-1/4} e^{-z^2/2} \int_{\mathbb{R}} \overline{y(u)} \exp\left(\sqrt{2}uz - u^2/2\right) du,$$

B_y is an **entire** function, almost characterized by its infinitely many zeros:

$$B_y(z) = z^m e^{C_0 + C_1 z + C_2 z^2} \prod_{n \in \mathbb{N}} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \frac{1}{2} \left(\frac{z}{z_n}\right)^2\right).$$

Idea assimilate the time-frequency plane with \mathbb{C} through $z = (\omega + it)/\sqrt{2}$



Bargmann factorization

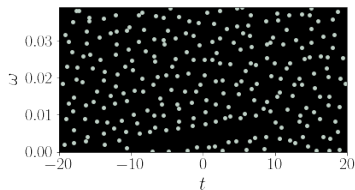
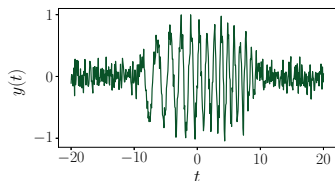
$$V_g y(t, \omega) = e^{-|z|^2/2} e^{-i\omega t/2} B_y(z)$$

g the circular Gaussian window

Theorem The zeros of the Gaussian spectrogram $V_g y(t, \omega)$

- coincide with the zeros of the **entire** function B_y ,
- hence are **isolated** and constitute a **Point Process**,
- which almost completely **characterizes** the spectrogram.

(Flandrin, 2015)



Advantages of working with the zeros

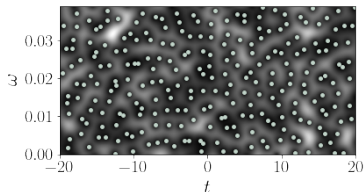
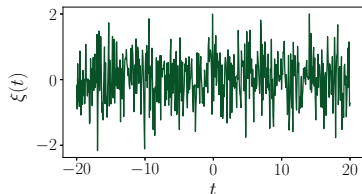
- Easy to find compared to relative maxima.
- Form a robust pattern in the time-frequency plane.
- Require little memory space for storage.
- Efficient tools were recently developed in **stochastic geometry**.

Signal detection based on the spectrogram zeros

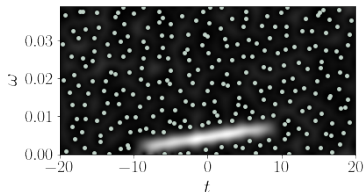
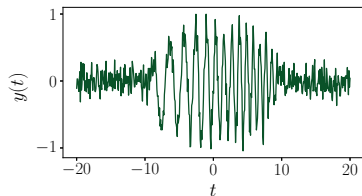
(Bardenet, Flamant & Chainais, 2020)

- H_0 white noisy only, i.e., $y(t) = \xi(t)$
- H_1 presence of a signal, i.e., $y(t) = \text{snr} \times x(t) + \xi(t)$, $\text{snr} > 0$

null hypothesis



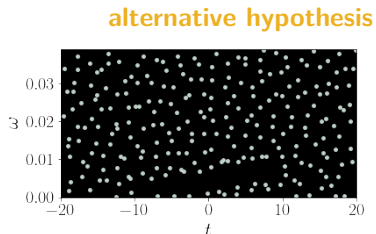
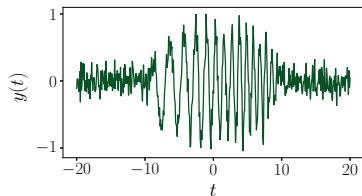
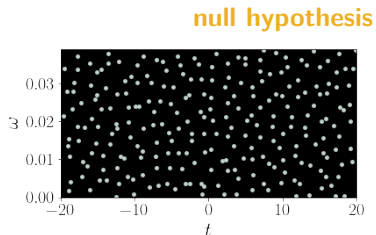
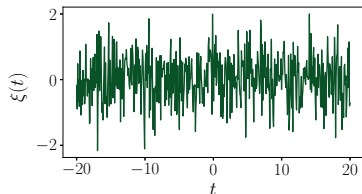
alternative hypothesis



Signal detection based on the spectrogram zeros

(Bardenet, Flamant & Chainais, 2020)

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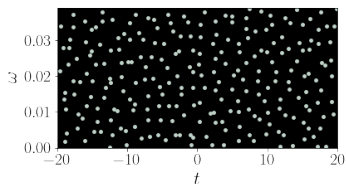
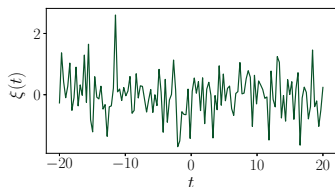


The zeros of the spectrogram of white noise

Continuous complex white Gaussian noise

(Bardenet et al., 2020), (Bardenet & Hardy, 2020)

$$\xi(t) = \sum_{n=0}^{\infty} \xi[n] h_n(t), \quad \xi[n] \sim \mathcal{N}_{\mathbb{C}}(0, 1), \quad \{h_n, k = 0, 1, \dots\} \text{ Hermite functions}$$

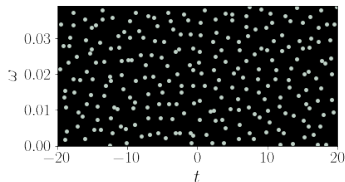
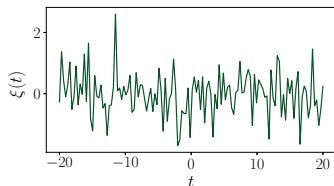


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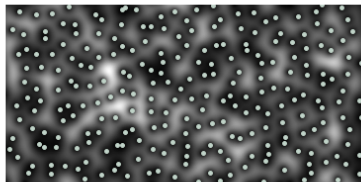


Theorem

$$V_g \xi(t, \omega) = e^{-|z|^2/4} e^{-i\omega t/2} \text{GAF}_{\mathbb{C}}(z) \quad (\text{Bardenet \& Hardy, 2021})$$

$$\text{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}} \text{ the planar Gaussian Analytic Function and } z = \frac{\omega + it}{\sqrt{2}}.$$

The zeros of the planar Gaussian Analytic Function



$$V_g \xi(t, \omega) \stackrel{\text{non-vanishing}}{\propto} \text{GAF}_{\mathbb{C}}(z)$$

$$z = (\omega + it)/\sqrt{2}$$

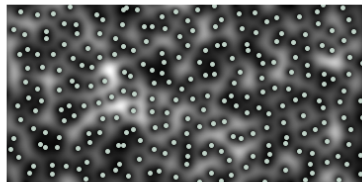
Zeros of $\text{GAF}_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations

Properties of the Point Process of the zeros of $\text{GAF}_{\mathbb{C}}$:

- invariant under the isometries of \mathbb{C} , i.e., **stationary**,
- has a uniform density $\rho^{(1)}(z) = \rho^{(1)} = 1/\pi$,
- explicit two-point correlation function $\rho^{(2)}(z, z') = \rho^{(2)}(|z - z'|)$,
- scaling of the *hole probability*: $r^{-4} \log p_r \rightarrow -3e^2/4$, as $r \rightarrow \infty$

$$p_r = \mathbb{P}(\text{no point in the disk of center } 0 \text{ and radius } r)$$

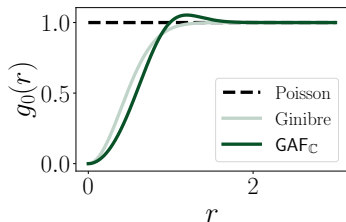
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Zeros of $\text{GAF}_{\mathbb{C}}$: random set of points forming a **Point Process** characterized by a probability distribution on point configurations



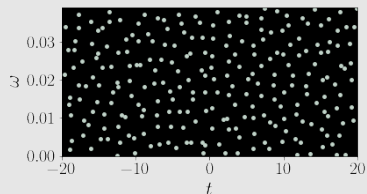
Pair correlation $\rho^{(2)}(z, z') dz dz' =$

$$\mathbb{P}(\text{1 point in } B(z, dz) \text{ and 1 in } B(z', dz'))$$

The point process of the zeros of the spectrogram is not **determinantal**.

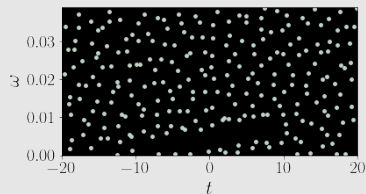
Purpose: summary statistic s , such that for $y = \text{snr} \times x + \xi$

$\mathbb{E}[s(y)|\mathbf{H}_0]$ small



$\text{snr} = 0$

$\mathbb{E}[s(y)|\mathbf{H}_1]$ large

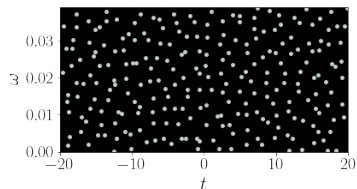


$\text{snr} > 0$

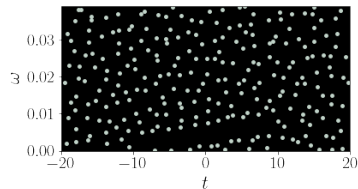
'Large value of $s(y)$ is a strong indication that there is a signal.'

Tools from **stochastic geometry** to capture spatial statistics of the zeros.

Unorthodox path: zeros of Gaussian Analytic Functions



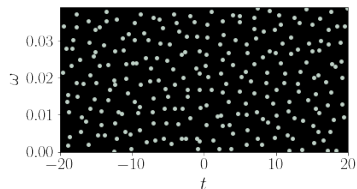
$\text{snr} = 0$



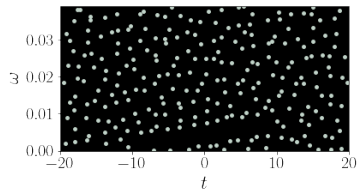
$\text{snr} > 0$

The signal creates **holes** in the zeros pattern: **second order** statistics.

Unorthodox path: zeros of Gaussian Analytic Functions



$\text{snr} = 0$



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The signal creates **holes** in the zeros pattern: **second order** statistics.

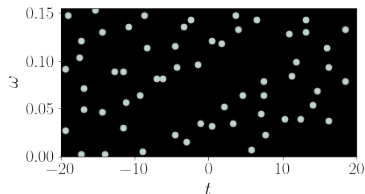
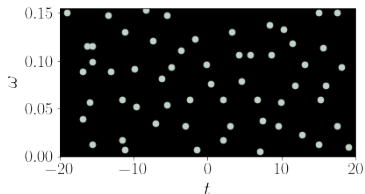
A functional statistic: the empty space function

Z a stationary point process, z_0 any reference point

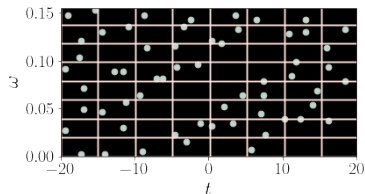
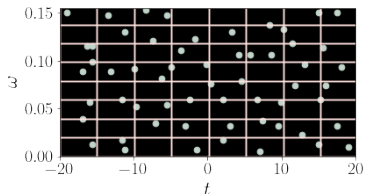
$$F(r) = \mathbb{P} \left(\inf_{z_i \in Z} d(z_0, z_i) < r \right)$$

→ probability to find a zero at distance less than r from z_0

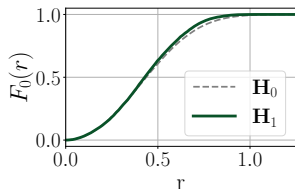
$$F(r) = \mathbb{P} \left(\inf_{z_i \in Z} d(z_0, z_i) < r \right) : \text{empty space function}$$



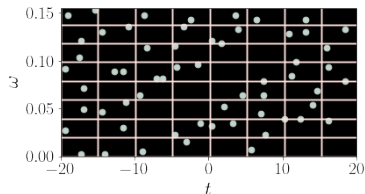
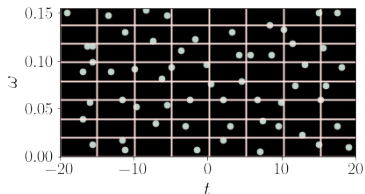
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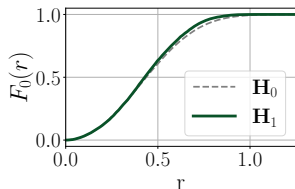
$$\hat{F}(r) = \frac{1}{N_{\#}} \sum_{j=1}^{N_{\#}} \mathbf{1} \left(\inf_{z \in \mathbf{Zeros}} d(z_j, z) < r \right)$$



$$F(r) = \mathbb{P} \left(\inf_{z_i \in Z} d(z_0, z_i) < r \right) : \text{empty space function}$$

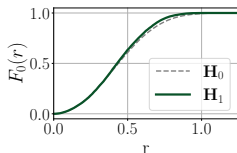
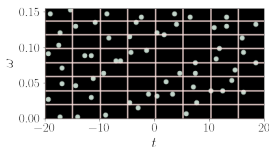
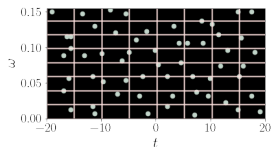


$$\hat{F}(r) = \frac{1}{N_{\#}} \sum_{j=1}^{N_{\#}} \mathbf{1} \left(\inf_{z \in \text{Zeros}} d(z_j, z) < r \right)$$



- Monte Carlo envelope test based on the discrepancy between \hat{F} and F_0

Monte Carlo envelope test

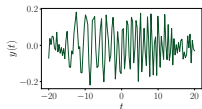
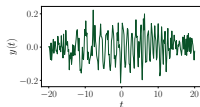
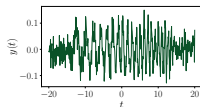
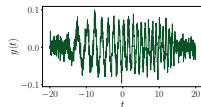


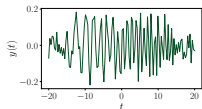
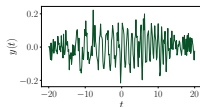
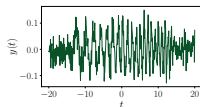
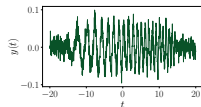
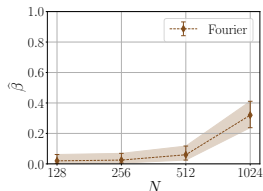
$$s(\mathbf{y}) = \sqrt{\int_0^{r_{\max}} |\hat{F}_{\mathbf{y}}(r) - F_0(r)|^2 dr},$$

Test settings: α level of significance, m number of samples under \mathbf{H}_0

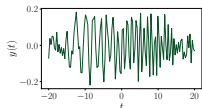
Index k , chosen so that $\alpha = k/(m + 1)$

- (i) generate m independent samples of complex white Gaussian noise;
- (ii) compute their summary statistics $s_1 \geq s_2 \geq \dots \geq s_m$;
- (iii) compute the summary statistic of the observation \mathbf{y} under concern;
- (iv) if $s(\mathbf{y}) \geq s_k$, then reject the null hypothesis with confidence $1 - \alpha$.

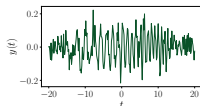
Detection of a noisy chirp of duration $2\nu = 30$ s $N = 128$  $N = 256$  $N = 512$  $N = 1024$

Detection of a noisy chirp of duration $2\nu = 30$ s $N = 128$  $N = 256$  $N = 512$  $N = 1024$ **Performance:** power of the test computed over 200 samples

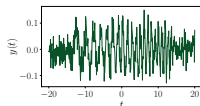
Detection of a noisy chirp of duration $2\nu = 30$ s



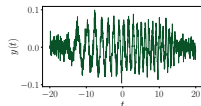
$N = 128$



$N = 256$

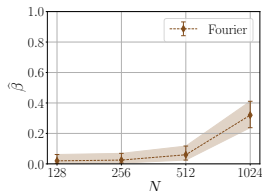


$N = 512$

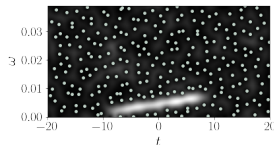


$N = 1024$

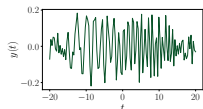
Performance: power of the test computed over 200 samples



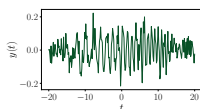
- ✓ Fast Fourier Transform
- ✗ low detection power
- ✗ requires large number of samples



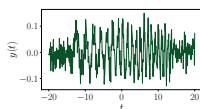
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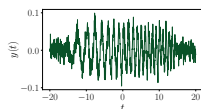
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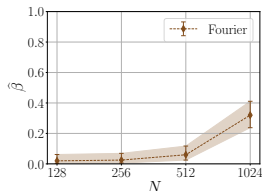


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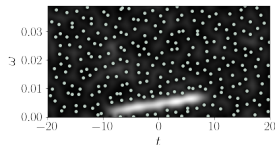
Performance: power of the test computed over 200 samples



- ✓ Fast Fourier Transform
- ✗ low detection power
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Limitations:

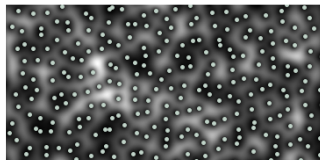
- necessary discretization of the STFT: arbitrary resolution
- observe only a bounded window: edge corrections to compute $\hat{F}(r)$



Short-Time Fourier Transform

$$V_g \xi(t, \omega) \propto \text{GAF}_{\mathbb{C}}(z) = \sum_{n=0}^{\infty} \xi[n] \frac{z^n}{\sqrt{n!}}$$

Unbounded phase space \mathbb{C}

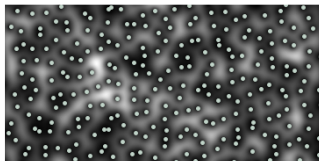


→ edge corrections

Short-Time Fourier Transform

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→ edge corrections

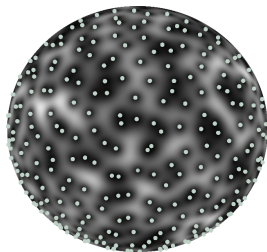
New transform?

$$? \propto \text{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^N \xi[n] \sqrt{\binom{N}{n}} z^n$$

stereographic projection $z = \cot(\vartheta/2)e^{i\varphi}$

→ spherical coordinates $(\vartheta, \varphi) \in S^2$

Compact phase space S^2 ?



→ no border!

Outline of the presentation

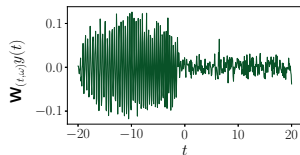
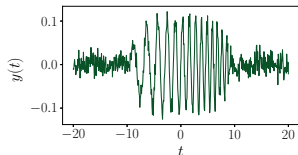
- Signal detection: the role of representations
-

- Time-frequency analysis: the Short-Time Fourier Transform
 - Signal detection based on the spectrogram zeros - I
 - Covariance principle and stationary point processes
-

- The Kravchuk transform and its zeros
- Numerical implementation of the Kravchuk transform
- Signal detection based on the spectrogram zeros - II

Time and frequency shifts

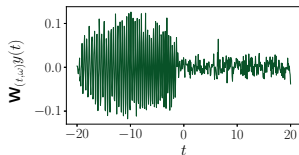
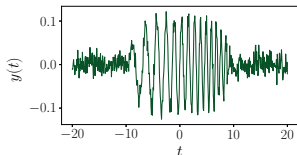
$$W_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$



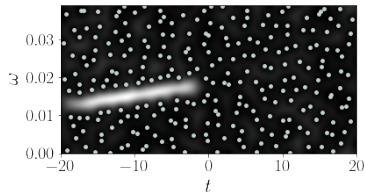
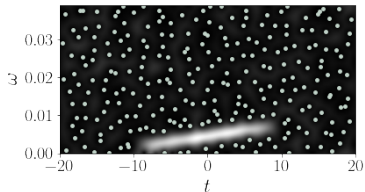
Algebraic interpretation: covariance under a symmetry group

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$



$$V_h[\mathbf{W}_{(t,\omega)}y](t', \omega') \stackrel{\text{(covariance)}}{=} e^{-i(\omega' - \omega)t} V_h y(t' - t, \omega' - \omega),$$



Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$

$$|V_h[\mathbf{W}_{(t,\omega)}y](t', \omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t' - t, \omega' - \omega)|^2,$$

Time and frequency shifts

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Complex white Gaussian noise

$$\tilde{\xi} = \mathbf{W}_{(t,\omega)}\xi$$

- $\mathbb{E}[\tilde{\xi}(u)] = e^{-i\omega u}\mathbb{E}[\xi(u - t)] = 0$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u-t)$$

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- $\mathbb{E}[\overline{\tilde{\xi}(u)}\tilde{\xi}(u')] = e^{i\omega(u-u')}\mathbb{E}[\overline{\xi(u)}\xi(u')] = \delta(u-u')$

Time and frequency shifts

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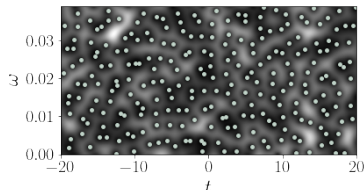
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Invariance under time-frequency shifts:

$$\tilde{\xi} = \mathbf{W}_{(t,\omega)}\xi \stackrel{\text{(law)}}{=} \xi$$



Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u-t)$$

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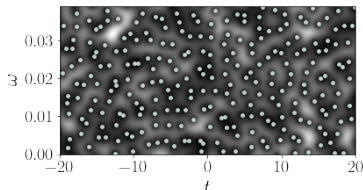
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Invariance under time-frequency shifts:

$$\tilde{\xi} = \mathbf{W}_{(t,\omega)}\xi \stackrel{\text{(law)}}{=} \xi$$



Covariance is the key to get **stationarity**: how to get covariant transforms?

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u - t)$$

$$|V_h[\mathbf{W}_{(t,\omega)}y](t', \omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t' - t, \omega' - \omega)|^2,$$

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u-t)$$

$$|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t'-t,\omega'-\omega)|^2,$$

Weyl-Heisenberg group $\{e^{i\gamma} \mathbf{W}_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

$$\mathbf{W}_{(t',\omega')} \mathbf{W}_{(t,\omega)} = e^{i\omega t'} \mathbf{W}_{(t+t',\omega+\omega')}.$$

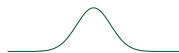
Time and frequency shifts

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Weyl-Heisenberg group $\{e^{i\gamma} \mathbf{W}_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

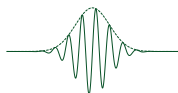
$$\mathbf{W}_{(t',\omega')} \mathbf{W}_{(t,\omega)} = e^{i\omega t'} \mathbf{W}_{(t+t', \omega+\omega')}.$$



$$g(t) = \pi^{-1/4} \exp(-t^2/2)$$



$$\mathbf{T}_u g(t) = g(t - u)$$



$$\mathbf{M}_\omega g(t) = g(t) \exp(-i\omega t)$$

Algebraic interpretation: covariance under a symmetry group

Time and frequency shifts

$$\mathbf{W}_{(t,\omega)}y(u) = e^{-i\omega u}y(u-t)$$

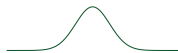
$$|V_h[\mathbf{W}_{(t,\omega)}y](t',\omega')|^2 \stackrel{\text{(covariance)}}{=} |V_hy(t'-t,\omega'-\omega)|^2,$$

Weyl-Heisenberg group $\{e^{i\gamma} \mathbf{W}_{(t,\omega)}, (\gamma, t, \omega) \in [0, 2\pi] \times \mathbb{R}^2\}$

$$\mathbf{W}_{(t',\omega')} \mathbf{W}_{(t,\omega)} = e^{i\omega t'} \mathbf{W}_{(t+t',\omega+\omega')}.$$

Coherent state interpretation $\{\mathbf{W}_{(t,\omega)}h, t, \omega \in \mathbb{R}\}$ covariant family

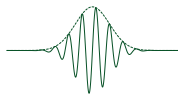
$$V_hy(t, \omega) = \int_{-\infty}^{\infty} \overline{y(u)} h(u-t) \exp(-i\omega u) du = \langle y, \mathbf{W}_{(t,\omega)}h \rangle$$



$$g(t) = \pi^{-1/4} \exp(-t^2/2)$$



$$\mathbf{T}_u g(t) = g(t-u)$$



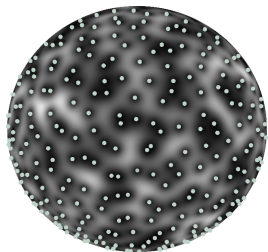
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Outline of the presentation

- Signal detection: the role of representations
-

- Time-frequency analysis: the Short-Time Fourier Transform
 - Signal detection based on the spectrogram zeros - I
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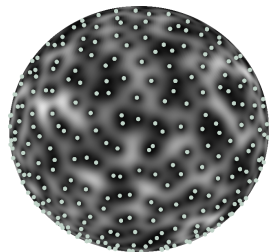


The Kravchuk transform: covariance under $SO(3)$

Coherent state interpretation $\mathbf{y} \in \mathbb{C}^{N+1}$

$$T\mathbf{y}(\vartheta, \varphi) = \langle \mathbf{y}, \Psi_{(\vartheta, \varphi)} \rangle$$

$$\vartheta \in [0, \pi], \varphi \in [0, 2\pi]$$

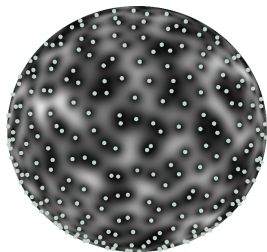


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SO(3) **coherent states** (Gazeau, 2009)

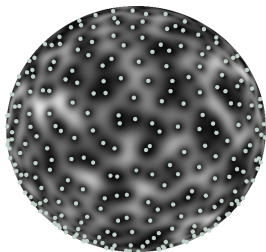
$$\Psi_{\vartheta, \varphi} = \sum_{n=0}^N \sqrt{\binom{N}{n}} \left(\cos \frac{\vartheta}{2} \right)^n \left(\sin \frac{\vartheta}{2} \right)^{N-n} e^{in\varphi} \mathbf{q}_n = \mathbf{R}_{\mathbf{u}(\vartheta, \varphi)} \Psi_{(0,0)},$$

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Kravchuk transform

$\{\mathbf{q}_n, n = 0, 1, \dots, N\}$ the *Kravchuk functions*

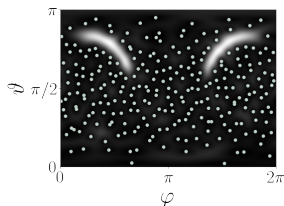
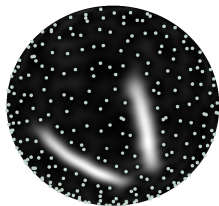
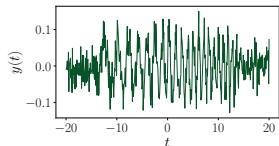
$$T\mathbf{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \langle \mathbf{y}, \mathbf{q}_n \rangle \sqrt{\binom{N}{n}} z^n, \quad z = \cot(\vartheta/2)e^{i\varphi}$$

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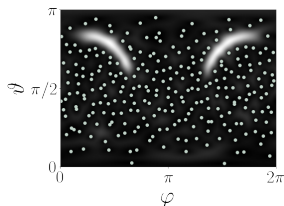
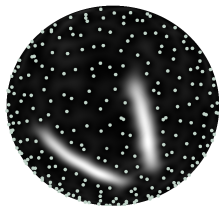
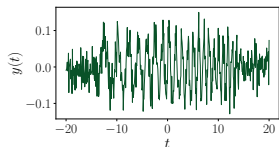


The Kravchuk transform: covariance under SO(3)

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Theorem

$$T\xi(\vartheta, \varphi) = \sqrt{(1+|z|^2)^{-N}} \text{GAF}_{\mathbb{S}}(z), \quad z = \cot(\vartheta/2) e^{i\varphi}$$

$$\text{GAF}_{\mathbb{S}}(z) = \sum_{n=0}^N \xi[n] \sqrt{\binom{N}{n}} z^n \text{ the spherical Gaussian Analytic Function}$$

(Pascal & Bardenet, 2022)

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Kravchuk transform

$\{\mathbf{q}_n, n = 0, 1, \dots, N\}$ the Kravchuk basis

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→ first: basis change, i.e., computation of $\langle \mathbf{y}, \mathbf{q}_n \rangle = \sum_{\ell=0}^N \overline{\mathbf{y}[\ell]} q_n(\ell; N)$

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Evaluation of Kravchuk functions $q_n(\ell; N) = \frac{1}{\sqrt{2^N}} \sqrt{\binom{N}{n}} Q_n(\ell; N) \sqrt{\binom{N}{\ell}}$

$$(N-n)Q_{n+1}(t; N) = (N-2t)Q_n(t; N) - nQ_{n-1}(t; N),$$

$\{Q_n(t; N), n = 0, 1, \dots, N\}$ orthogonal family of Kravchuk polynomials

$$\sum_{\ell=0}^N \binom{N}{\ell} Q_n(\ell; N) Q_{n'}(\ell; N) = 2^N \binom{N}{n}^{-1} \delta_{n,n'}$$

Evaluation of Kravchuk functions

(i) recursion to compute the Kravchuk polynomials

$$(N - n)Q_{n+1}(t; N) = (N - 2t)Q_n(t; N) - nQ_{n-1}(t; N),$$

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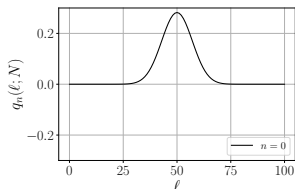
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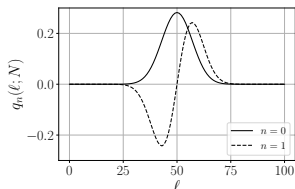
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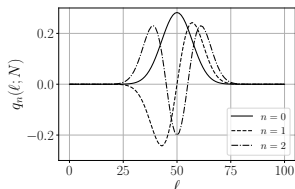
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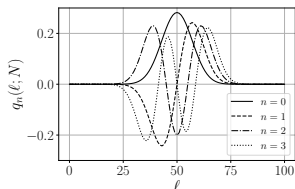
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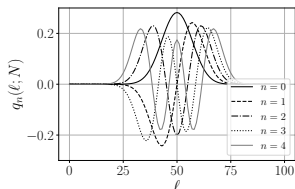
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Instability of the computation of Kravchuk polynomials

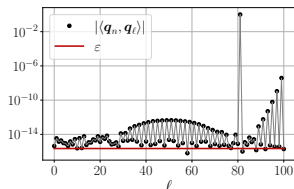
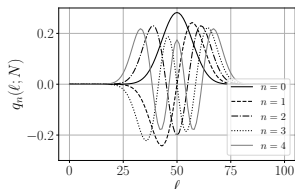
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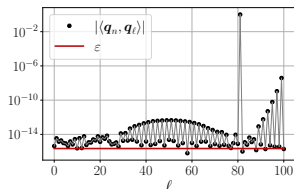
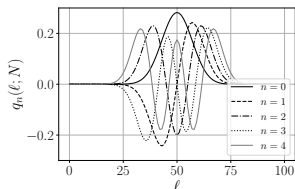
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→ estimated basis is **not orthogonal!** Not possible to compute $\langle \mathbf{y}, \mathbf{q}_n \rangle$.

Kravchuk transform

$\{q_n, n = 0, 1, \dots, N\}$ the Kravchuk basis

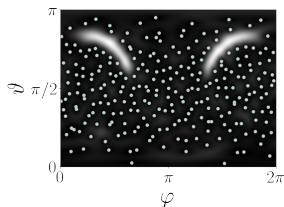
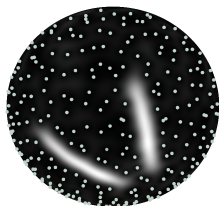
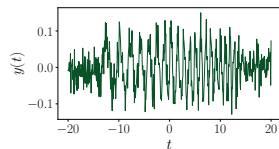
$$T\mathbf{y}(z) = \frac{1}{\sqrt{(1+|z|^2)^N}} \sum_{n=0}^N \left(\sum_{\ell=0}^N \overline{\mathbf{y}[\ell]} q_n(\ell; N) \right) \sqrt{\binom{N}{n}} z^n \rightarrow \text{intractable}$$

A generative function for Kravchuk polynomials

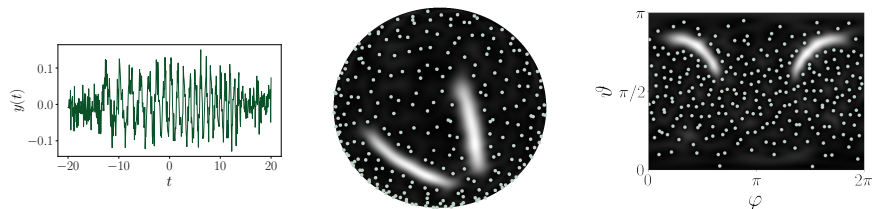
$$\begin{aligned} \sum_{n=0}^N \binom{N}{n} Q_n(\ell; N) z^n &= (1-z)^\ell (1+z)^{N-\ell} \\ \Rightarrow \sum_{n=0}^N \sqrt{\binom{N}{n}} q_n(\ell; N) z^n &= \sqrt{\binom{N}{\ell}} \frac{(1-z)^\ell (1+z)^{N-\ell}}{\sqrt{2^N}} \end{aligned}$$

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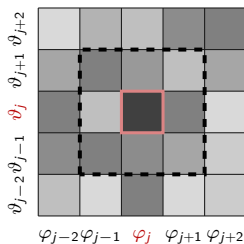
✗ no more Fast Fourier Transform algorithm using $z^n = \cot(\vartheta/2)^n e^{in\varphi}$



Advantage compared to Fourier: can tune the resolution of phase space.

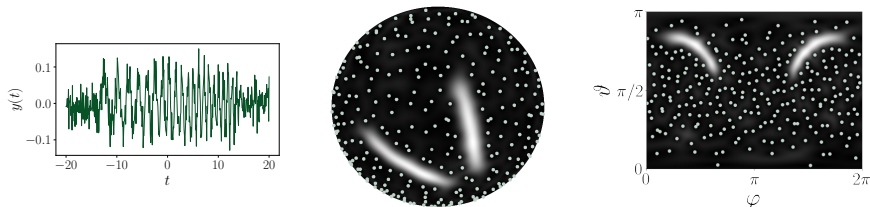


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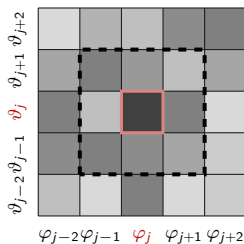


Minimal Grid Neighbors

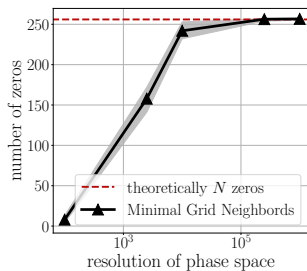
Detection of the zeros of the Kravchuk spectrogram $|Ty(z_i)|^2 = 0$



Advantage compared to Fourier: can tune the resolution of phase space.



Minimal Grid Neighbors



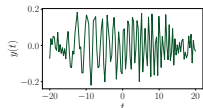
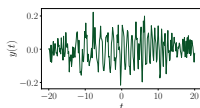
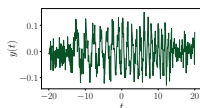
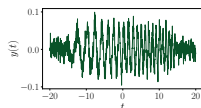
Proposition: all local minima of $|Ty(z)|^2$ are zeros.

Outline of the presentation

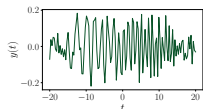
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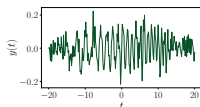
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Detection of a noisy chirp of duration $2\nu = 30$ s $N = 128$  $N = 256$  $N = 512$  $N = 1024$

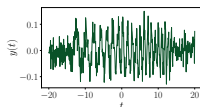
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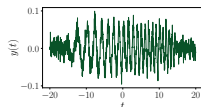
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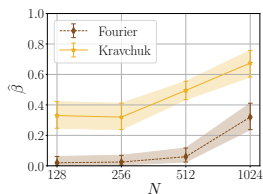


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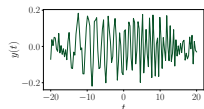


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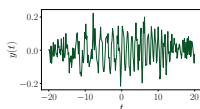
Performance: power of the test computed over 200 samples



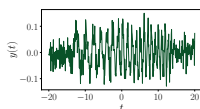
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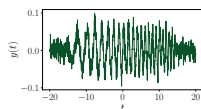
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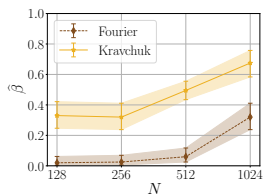


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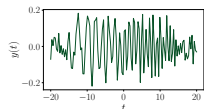
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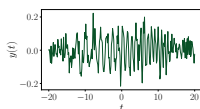


- ✓ higher detection power
- ✓ more robust to small N
- ✗ no fast algorithm yet

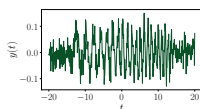
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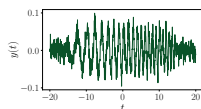
$N = 128$



$N = 256$

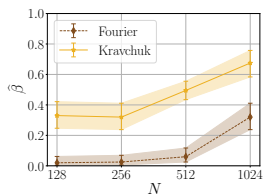


$N = 512$



$N = 1024$

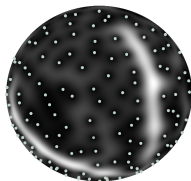
Performance: power of the test computed over 200 samples



- ✓ higher detection power
- ✓ more robust to small N
- ✗ no fast algorithm yet

Advantages of using Kravchuk vs. Fourier spectrogram

- intrinsically encoded resolution: no need for prior knowledge
- compact phase space: no edge correction



Take home messages

- Novel covariant discrete Kravchuk transform $T\mathbf{y}(\vartheta, \varphi)$
 - * Interpreted as a coherent state decomposition
 - * Representation on a compact phase space
 - * Zeros of the Kravchuk spectrogram of white noise fully characterized
- Signal detection based on spectrogram zeros
 - * Preliminary work using the zeros of the Fourier spectrogram
 - * Significant improvement using the Kravchuk spectrogram

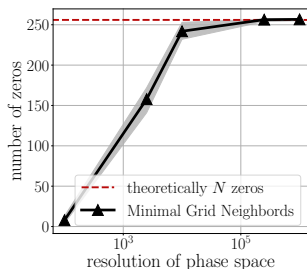
Pascal & Bardenet, 2022: [arxiv:2202.03835](https://arxiv.org/abs/2202.03835)

GitHub: [bpascal-fr/kravchuk-transform-and-its-zeros](https://github.com/bpascal-fr/kravchuk-transform-and-its-zeros)

Work in progress and perspectives

- Interpretation of the action of $SO(3)$ on \mathbb{C}^{N+1}
- Implementation of the inversion formula: denoising based on zeros
- Design of a Kravchuk FFT counterpart
- Convergence of Kravchuk toward the Fourier spectrogram as $N \rightarrow \infty$

Opening: can the Kravchuk spectrogram have multiple zeros?



Spherical Gaussian Analytic Function

$$\text{GAF}_S(z) = \sum_{n=0}^N \xi[n] \sqrt{\binom{N}{n}} z^n$$

with $\xi[n] \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ i.i.d.

→ only **simple** zeros

.....

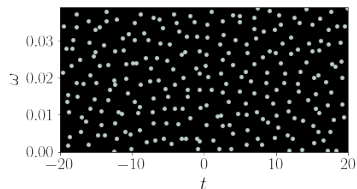
General case $T\mathbf{y}(z) = \sqrt{(1 + |z|^2)}^{-N} \sum_{n=0}^N \sqrt{\binom{N}{n}} (\mathbf{Q}\mathbf{y})[n] z^n$

If \mathbf{y} deterministic, such that $(\mathbf{Q}\mathbf{y})[n] = \sqrt{\binom{N}{n}} a^{N-n} b^n, a \in \mathbb{C}, b \in \mathbb{C}^*$,

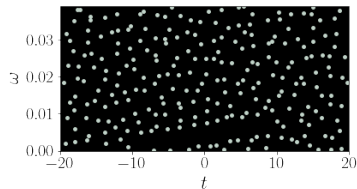
$$\sqrt{(1 + |z|^2)}^{-N} \sum_{n=0}^N \sqrt{\binom{N}{n}} (\mathbf{Q}\mathbf{y})[n] z^n = (a + bz)^N$$

→ $-a/b$ multiple root of order of degeneracy N

Unorthodox path: zeros of Gaussian Analytic Functions



$\text{snr} = 0$



$\text{snr} > 0$

The signal creates **holes** in the zeros pattern: **second order** statistics.

Functional statistics:

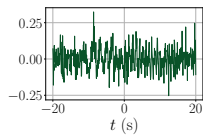
- the empty space function

$$F(r) = \mathbb{P} \left(\inf_{z_i \in Z} d(z_0, z_i) < r \right) : \text{probability to find a zero at less than } r$$

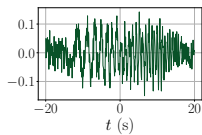
- Ripley's K -function

$$K(r) = 2\pi \int_0^r s g_0(s) ds : \text{expected \# of pairs at distance less than } r$$

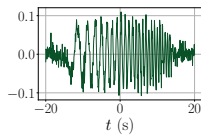
Detection test: choice of the functional statistic



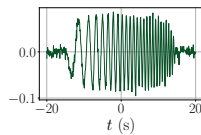
snr = 0.5



snr = 1

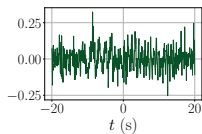


snr = 2

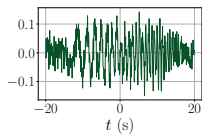


snr = 5

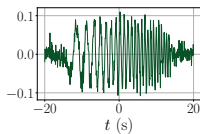
Detection test: choice of the functional statistic



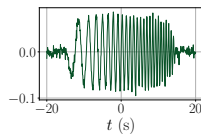
$\text{snr} = 0.5$



$\text{snr} = 1$

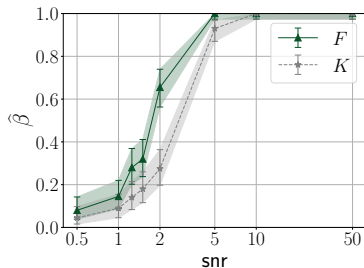


$\text{snr} = 2$

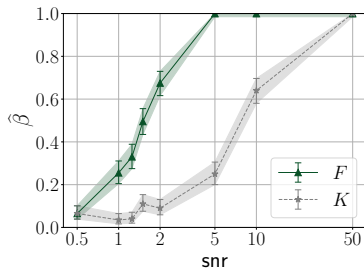


$\text{snr} = 5$

Ripley's K functional vs. empty space functional F



$N + 1 = 257$ points

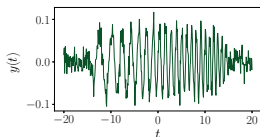


$N + 1 = 513$ points

Detection test: snr and relative duration of the signal

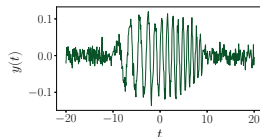
Fixed observation window of 40 s

long time event



duration $2\nu = 30$ s

short time event

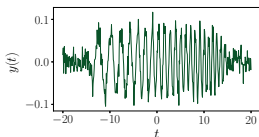


duration $2\nu = 20$ s

Detection test: snr and relative duration of the signal

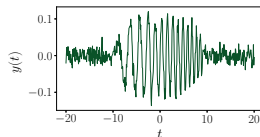
Fixed observation window of 40 s

long time event



duration $2\nu = 30$ s

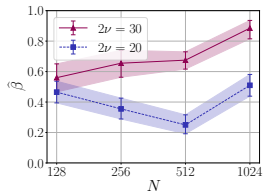
short time event



duration $2\nu = 20$ s

Robustness to small number of samples and short duration.

medium noise level

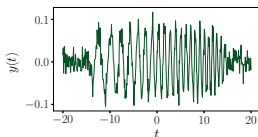


snr = 2

Detection test: snr and relative duration of the signal

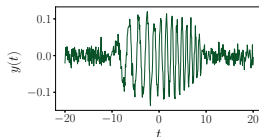
Fixed observation window of 40 s

long time event



duration $2\nu = 30$ s

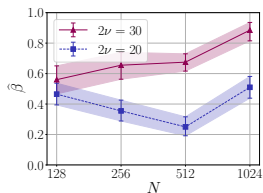
short time event



duration $2\nu = 20$ s

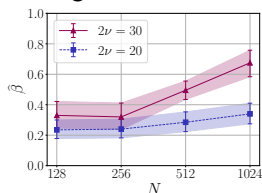
Robustness to small number of samples and short duration.

medium noise level



snr = 2

high noise level



snr = 1.5