On quantum statistics transmutation via flux attachment

Nicolas Rougerie
UMPA
CNRS & ENS Lyon

DPP-fermions-2022, Lyon, June 2022

Joint work with Gaultier Lambert and Douglas Lundholm

Quantum mechanics/quantum statistics

- Many-body non-relativistic quantum mechanics in $\mathbb{R}^d$
- Action of many-body Schrödinger Hamiltionian on $L^2(\mathbb{R}^{dN})$

$$H_N = \sum_{j=1}^{N} -\Delta_{\mathbf{x}_j} + V(\mathbf{x}_j) + \sum_{1\leq i<j\leq N} w(\mathbf{x}_i - \mathbf{x}_j)$$

- Add symmetry : indistinguishable particles $\rightarrow$ quantum statistics
- Two possible classes for fundamental particles

$$\Psi_N(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = \Psi_N(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N) \text{ **Bosons**}$$
$$\Psi_N(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = -\Psi_N(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N) \text{ **Fermions**}$$

- Bosons, work on $L^2_{\mathrm{sym}}(\mathbb{R}^{dN}) = \bigotimes_{\mathrm{sym}} L^2(\mathbb{R}^d)$
- Fermions, work on $L^2_{\mathrm{asym}}(\mathbb{R}^{dN}) = \bigotimes_{\mathrm{asym}} L^2(\mathbb{R}^d)$

This talk: **how to change quantum statistics by playing with magnetic fields in 2D.**

Magnetic fields, change of gauge (in 2D)

- External magnetic field $B(\mathbf{x}) \in \mathbb{R}$, vector potential $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^2$

$$\mathrm{curl}\,\mathbf{A} = \partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$$

- Change canonical momentum $\mathbf{p} = -\mathrm{i}\nabla_{\mathbf{x}}$ to $\mathbf{p}_A = -\mathrm{i}\nabla_{\mathbf{x}} + \mathbf{A}$
- Schrödinger operator becomes

$$H_N = \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_{\mathbf{x}_j} + \mathbf{A}(\mathbf{x}_j)\right)^2 + V(\mathbf{x}_j) + \sum_{1 \leq i < j \leq N} w(\mathbf{x}_i - \mathbf{x}_j)$$

- Gauge invariance: joint change of wave-function/vector potential

$$\Psi_N \to \Psi_N \prod_{j=1}^{N} e^{\mathrm{i}\varphi(\mathbf{x}_j)} \text{ and } \mathbf{A} \to \mathbf{A} - \nabla\varphi$$

- Indeed, in density/phase reprensation $\psi = \sqrt{\rho} e^{\mathrm{i}\phi}$

$$\langle \psi, (-\mathrm{i}\nabla + \mathbf{A})^2 \psi \rangle = \int |\nabla \sqrt{\rho}|^2 + \int \rho \, |\nabla\phi + \mathbf{A}|^2$$

$\Rightarrow$ **in 2D, one can switch between bosonic and fermionic statistics.**

Switching between bosonic and fermionic statistics

- Many-body vector potential and magnetic field (Aharonov-Bohm flux)

$$\mathbf{A}(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_k|^2} \qquad B(\mathbf{x}_j) = 2\pi \sum_{k \neq j} \delta_{\mathbf{x}_j = \mathbf{x}_k}$$

- Many-body phase factor (arg = angle of a vector)

$$\varphi_1(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{1 \leq j < k \leq N} \arg(x_j - x_k)$$

- If Ψ_N is bosonic, $\Phi_N = e^{\mathrm{i}\varphi_1} \Psi_N$ is fermionic and

$$\left\langle \Psi_N, \sum_{j=1}^{N} -\Delta_{\mathbf{x}_j} \Psi_N \right\rangle = \left\langle \Phi_N, \sum_{j=1}^{N} \left(-\mathrm{i}\nabla_{\mathbf{x}_j} + \mathbf{A}(\mathbf{x}_j) \right)^2 \Phi_N \right\rangle$$

2D bosons $\leftrightarrow$ 2D fermions with attached (integer) magnetic flux

(and vice-versa)

Emergent fractional statistics/anyons

- What if the attached flux is fractional ? Leinaas-Myrheim, Wilczek, Goldin-Menikoff-Sharp 77-82 invent ANYONS
- Pick any $\alpha \in \mathbb{R}$ and set

$$\mathbf{A}_\alpha(\mathbf{x}_j) := \alpha \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \qquad B(\mathbf{x}_j) = 2\pi\alpha \sum_{k \neq j} \delta_{\mathbf{x}_j = \mathbf{x}_k}$$

- Many-body phase factor (arg = angle of a vector)

$$\varphi_\alpha(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \alpha \sum_{1 \leq j < k \leq N} \arg(x_j - x_k)$$

- If Ψ_N is bosonic/fermionic, set

$$\Phi_N = e^{\mathrm{i}\varphi_\alpha} \Psi_N$$

$$\left\langle \Phi_N, \sum_{j=1}^N -\Delta_{\mathbf{x}_j} \Phi_N \right\rangle = \left\langle \Psi_N, \sum_{j=1}^N \left(-\mathrm{i}\nabla_{\mathbf{x}_j} - \mathbf{A}_\alpha(\mathbf{x}_j)\right)^2 \Psi_N \right\rangle$$

- FORMALLY, Φ_N has exotic exchange phase

$$\Phi_N(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = e^{\mathrm{i}\pi\alpha} \Phi_N(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N)$$

attaching non-integer magnetic flux ↔ realization of non-standard quantum statistics

Attaching magnetic flux: quantum Hall physics

Main ingredients (for the original effect in Gallium-Arsenide heterostructures)

- A gas of electrons trapped in a 2D plane
- A strong perpendicular magnetic field $\mathfrak{b}$
- The Pauli principle and/or interparticle repulsion

Usual approximation

- All particles forced to ground eigenspace of one-body kinetic energy

$$\left(-\mathrm{i}\nabla - \mathfrak{b}x^{\perp}\right)^2$$

- Lowest Landau Level in symmetric gauge

$$\mathrm{LLL} := \left\{\psi \in L^2(\mathbb{R}^2), \psi(x) = f(z)e^{-\frac{\mathfrak{b}}{2}|z|^2}, f \text{ analytic}\right\}$$

- Replace $L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$ by $\bigotimes^N_{\mathrm{asym}} \mathrm{LLL}$
- Family of Laughlin states (1983): with m odd integer

$$\Psi^{(m)}(z_1, \ldots, z_N) = c_m \prod_{1\leq i<j\leq N} (z_i - z_j)^m e^{-\frac{\mathfrak{b}}{2}|z_j|^2}$$

- $m = 1$: free fermions, locally filled Landau level
- $m > 1$: account for interactions in a partially filled Landau level

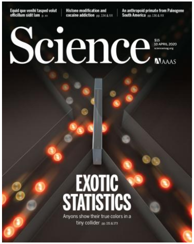
Facts in quantum Hall physics

- Excitations/impurities in Laughlin-like states

$$\Psi_{\mathrm{qh}}^{(m)}(z_1, \ldots, z_N) = \Psi^{(m)}(z_1, \ldots, z_N) \prod_{j=1}^{N} \prod_{k=1}^{K} (z_j - a_k)$$

  $a_1, \ldots, a_K$ location of **QUASI-HOLES**

- Are the effective charge carriers (Laughlin 83)
- Carry charge $1/m$ (Laughlin 83, Saminadayar-Glattli-Jin-Etienne 97)
- Have anyon statistics $1/m$ (Halperin, Arovas-Schrieffer-Wilcek 84, Bartolomei-et al-Fève, Nakamura-et al-Manfra 20)



Probes of such behavior ? Mathematical results in this direction ?

Thought experiment in bath-tracers system

- N fermions in the LLL (bath), n quantum impurities of charge q (tracers)
- Joint Hilbert space $\mathfrak{H}^{n\oplus N} = L^2(\mathbb{R}^{2n}) \otimes \mathrm{LLL}^{\otimes_{\mathrm{asym}} N}$
- Joint Hamiltonian with strong repulsive interactions, large $g > 0$

$$H_{n\oplus N} := g \sum_{k=1}^{N} \sum_{j=1}^{n} w(x_k - y_j) + \sum_{j=1}^{n} \left(-\mathrm{i}\nabla_{y_j} - q\mathfrak{b} y_j^{\perp}\right)^2 + \dots$$

- Natural trial state "cancels" interactions ($\mathbf{w}, \mathbf{z} = y, x$ in complex notation)

$$\Psi_\Phi(\mathbf{w};\mathbf{z}) := \Phi(\mathbf{y}) c(\mathbf{w}) \left(\prod_{j=1}^{n} \prod_{k=1}^{N} (w_j - z_k)^p \right) \left(\prod_{1\leq k<\ell\leq N} (z_k - z_\ell)^m \prod_{k=1}^{N} e^{-\mathfrak{b}|z_k|^2/2} \right)$$

Statement (Lundholm-NR 16)

Large $\mathfrak{b} = N$ limit. Let $\mathbf{A}^{\mathrm{tot}}(y_j)$ the total vector potential

$$\mathbf{A}^{\mathrm{tot}}(y_j) := -\left(q - \frac{p}{m}\right) \mathfrak{b} y_j^{\perp} - \frac{p^2}{m} \sum_{\ell\neq j} \frac{(y_j - y_\ell)^{\perp}}{|y_j - y_\ell|^2}.$$

$$\int_{\mathbb{R}^{2(n+N)}} \left| \left(-\mathrm{i}\nabla_{y_j} - q\mathfrak{b} y_j^{\perp}\right) \Psi_\Phi \right|^2 = 2\mathfrak{b}\frac{p}{m} + \int_{\mathbb{R}^{2n}} \left| \left(-\mathrm{i}\nabla_{y_j} + \mathbf{A}^{\mathrm{tot}}(y_j)\right) \Phi \right|^2 + \mathrm{Errors}$$

Mathematical challenges

1. Prove energy estimate rigorously (appropriate smallness of Errors)
2. Consider contact/delta interactions $w = \delta_0$ (makes sense on $\mathfrak{H}^{n \oplus N}$). Is the trial state a quasi-mode ?
3. Optimality of the trial state: depends on the FQH-spectral gap conjecture (work by Nachtergaele-Warzel-Young)

This talk: deal with 1 in the special case $m = p = 1$.

$$\Psi_\Phi(\mathbf{w};\mathbf{z}) := \Phi(\mathbf{y})c(\mathbf{w}) \left(\prod_{j=1}^{n} \prod_{k=1}^{N} (w_j - z_k) \right) \left(\prod_{1 \leq k < \ell \leq N} (z_k - z_\ell) \prod_{k=1}^{N} e^{-\mathfrak{b}|z_k|^2/2} \right)$$

- Bath $=$ free fermions in LLL. Scale $\mathfrak{b} = N$ (no loss)

$$\left| \Psi^{(1)}(z_1, \ldots, z_N) \right|^2 = c_1^2 \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 e^{-N|z_j|^2}$$

- Law for eigenvalues of random matrices from Ginibre's ensemble.
- Moments of associated characteristic polynomial

$$\int_{\mathbb{R}^{2N}} \left| \prod_{j=1}^{n} \prod_{k=1}^{N} (w_j - z_k) \right|^2 \left| \Psi^{(1)}(z_1, \ldots, z_N) \right|^2$$

Statistics transmutation

- Turn bosons into fermions (and vice-versa)

$$\Phi(y_1,\dots,y_n) = \left(\prod_{1\le i<j\le n} e^{\mathrm{i}\arg(y_i-y_j)}\right)\widetilde{\Phi}(y_1,\dots,y_n)$$

- Set $\mathfrak{b} = N$, fix κ large enough. Droplet: extension of the fermionic bath

$$\mathscr{D}_n := \left\{(y_1,\dots,y_n)\in\mathbb{R}^{2n} : |y_j| \le 1-\kappa\sqrt{\frac{\log N}{N}}\right\}$$

- No merging set

$$\mathscr{D}_n^{\varnothing} := \left\{(y_1,\dots,y_n)\in\mathscr{D}_n : |y_i-y_j| \ge 2\kappa\sqrt{\frac{\log N}{N}}\right\}$$

Theorem (Lambert-Lundholm-NR, 22)

Let $\widetilde{\Phi}$ have support in $\mathscr{D}_n$. Assume $\left|\widetilde{\Phi}(\mathbf{y})\right| \le C|y_i-y_j|$.

$$\int_{\mathbb{R}^{2(n+N)}} \left|\left(-\mathrm{i}\nabla_{y_j} - q\mathfrak{b}y_j^{\perp}\right)\Psi_{\Phi}\right|^2 = 2\mathfrak{b} + \int_{\mathbb{R}^{2n}} \left|\left(-\mathrm{i}\nabla_{y_j} - (q-1)\mathfrak{b}y_j^{\perp}\right)\widetilde{\Phi}\right|^2$$

$$+O\left(\kappa^4\frac{(\log N)^3}{N}\right) + O\left(\kappa^2\frac{(\log N)^{3/2}}{N^{1/2}}\left(\int_{\mathscr{D}_n\setminus\mathscr{D}_n^{\varnothing}}\left|\left(-\mathrm{i}\nabla_{y_j}-(q-1)\mathfrak{b}y_j^{\perp}\right)\widetilde{\Phi}\right|^2\right)^{1/2}\right)$$

Illustration: equal charges, bosonic tracers

- Joint Hamiltonian, smooth potential W

$$H_{n\oplus N} := g \sum_{k=1}^{N} \sum_{j=1}^{n} \delta_0(x_k - y_j) + \sum_{j=1}^{n} \left(-\mathrm{i}\nabla_{y_j} - q\mathfrak{b} y_j^{\perp}\right)^2 + W(y_1, \ldots, y_n)$$

on $\mathfrak{H}^{n+N} = L^2_{\mathrm{sym}}(\mathbb{R}^{2n}) \otimes \mathrm{LLL}^{\otimes_{\mathrm{asym}} N}$

- Effective Hamiltonian for tracers

$$H_n^{\mathrm{eff}} = \sum_{j=1}^{n} -\Delta_{y_j} + W(y_1, \ldots, y_n)$$

- **Fermionic** ground state energy ($\mathcal{D}_\alpha$ disk of radius α)

$$E^{\mathrm{eff}}(n) := \inf \left\{ \langle U_n | H_n^{\mathrm{eff}}\, U_n \rangle, U_n \in H_0^1(\mathcal{D}_\alpha^n), \int_{\mathcal{D}_\alpha^n} |U_n|^2 = 1, U_n \text{ anti-symmetric} \right\}$$

Corollary (Lambert-Lundholm-NR 22)

Set $q = 1$ and let $E(n \oplus N)$ be the lowest eigenvalue of $H_{n\oplus N}$. Fix $\alpha < 1$. We have

$$E(n \oplus N) \leq 2n\mathfrak{b} + E^{\mathrm{eff}}(n) + C_n \frac{(\log N)^3}{N}$$

in the limit $\mathfrak{b} = N \to \infty$.

Emerging potentials and the plasma analogy

- Direct/lengthy calculation (probably folklore in adiabatic theory)

$$\int_{\mathbb{R}^{2(n+N)}} \left| \left(-\mathrm{i}\nabla_{y_j} - \mathfrak{b} y_j^{\perp} \right) \Psi_{\Phi} \right|^2 = \int_{\mathbb{R}^{2n}} \left| \left(-\mathrm{i}\nabla_{y_j} - \mathfrak{b} y_j^{\perp} + \mathcal{A}_j \right) \Phi \right|^2 + \int_{\mathbb{R}^{2n}} |\Phi|^2 \mathcal{V}_j$$

- Using LLL properties (analyticity)

$$\boxed{\mathcal{A}_j(\mathbf{w}) = \frac{1}{2} \nabla_{w_j}^{\perp} \log c^{-2}(\mathbf{w}) \quad \mathcal{V}_j(\mathbf{w}) = \frac{1}{2} \Delta_{w_j} \log c^{-2}(\mathbf{w})}$$

- $c(\mathbf{w}) = L^2$ normalization constant

$$c(\mathbf{w})^{-2} := \int_{\mathbb{C}^N} \left| \left(\prod_{j=1}^{n} \prod_{k=1}^{N} (w_j - z_k) \right) \left(\prod_{1 \le k < \ell \le N} (z_k - z_\ell) \prod_{k=1}^{N} e^{-\frac{\mathfrak{b}}{2} |z_k|^2 / 2} \right) \right|^2 d\mathbf{z}$$

- Partition function of Gibbs ensemble for fictitious 2D Coulomb system

$$\mu(\mathbf{z}) = \frac{1}{\mathcal{Z}(\mathbf{w})} \exp\left(-\mathfrak{b} H(\mathbf{w}; \mathbf{z}) \right) \text{ probability measure for all } \mathbf{w}$$

$$H(\mathbf{w}; \mathbf{z}) = \sum_{j=1}^{N} |z_j|^2 + 2\mathfrak{b}^{-1} \sum_{1 \le i < j \le N} \log \frac{1}{|z_i - z_j|} + 2\mathfrak{b}^{-1} \sum_{j=1}^{N} \sum_{k=1}^{n} \log \frac{1}{|z_j - w_k|}.$$

Screening in the 2D one-component plasma

- In the regime of interest, there should be screening in the plasma
- Local charge neutrality

$$\varrho^{\mathrm{tot}} = \sum_{j=1}^{N} \delta_{z_j} + \sum_{j=1}^{n} \delta_{w_j} - \frac{1}{\pi} \mathbb{1}_{D(0,R)} \simeq 0$$

- Free energy satisfies (completing a square + neglecting entropy)

$$F(\mathbf{w}) = -\mathfrak{b}^{-1} \log \mathcal{Z}(\mathbf{w}) = 2\mathfrak{b}^{-1} \log c(\mathbf{w})$$
$$\simeq -\sum_{j=1}^{n} |w_j|^2 - 2\mathfrak{b}^{-1} \sum_{1 \le i < j \le N} \log \frac{1}{|w_i - w_j|} + C.$$

where C does not depend on $\mathbf{w}$.

- Differentiating yields the desired expressions

Issue: Validity and precision of screening heuristics far from obvious in general plasmas (Bauerschmidt-Bourgade-Nikula-Yau, Leblé-Serfaty, Armstrong-Serfaty, Leblé 2015-2021+...)

Using the determinantal structure

- Ginibre correlation kernel/LLL projector

$$K_N(z,x) := \sum_{0 \le j < N} \frac{\mathfrak{b}^{j+1}}{\pi j!} z^j \overline{x}^j e^{-\mathfrak{b}(|z|^2+|x|^2)/2}$$

- Interest in the characteristic polynomial lead to

Theorem (G. Lambert + other authors)

For any $\mathbf{w} \in \mathbb{C}^n$ *with* $w_1 \neq \cdots \neq w_n$*, we have*

$$c(\mathbf{w})^{-2} = \mathfrak{b}^{-\frac{(n+N)(n+N-1)}{2}} \pi^{n+N} \frac{(n+N-1)!}{n!} \frac{\prod_{j=1}^n e^{\mathfrak{b}|w_j|^2}}{|\triangle(\mathbf{w})|^2} \det_{n\times n}\left[K_{n+N}(w_i,w_j)\right]$$

with the Vandermonde determinant.

$$\triangle(\mathbf{w}) := \prod_{1 \le i < j \le n} (w_j - w_i)$$

- Proof: reduced density matrices of free fermions, Wick's theorem.
- Use: reduce to control derivatives of

$$\log\left(\det_{n\times n}\left[K_{n+N}(w_i,w_j)\right]\right)$$

Correlation kernel

$$\mathbf{M}(\mathbf{w}) := \left[\pi K_{n+N}(w_i, w_j)\right]_{m\times m} \qquad \Upsilon(\mathbf{w}) := \pi^m \mathfrak{b}^{-m} \det \mathbf{M}(\mathbf{w})$$

Goal: $\Upsilon \simeq 1$ **for large** $N = \mathfrak{b}$

- Jacobi's formula for the derivatives. Control of

$$\partial_{w_j} \log \Upsilon(\mathbf{w}) = \mathrm{Tr}\left(\mathbf{M}(\mathbf{w})^{-1}\partial_{w_j}\mathbf{M}(\mathbf{w})\right)$$

$$4\Delta_{w_j} \log \Upsilon(\mathbf{w}) = 4\mathrm{Tr}\left(\mathbf{M}(\mathbf{w})^{-1}\Delta_{w_j}\mathbf{M}(\mathbf{w})\right) - \mathrm{Tr}\left(\frac{1}{\mathbf{M}(\mathbf{w})}\partial_{\overline{w_j}}\mathbf{M}(\mathbf{w})\frac{1}{\mathbf{M}(\mathbf{w})}\partial_{w_j}\mathbf{M}(\mathbf{w})\right)$$

- Correlation kernel for large N

$$K_N(z,w) \simeq K_\infty^{(\mathfrak{b})}(z,w) = \frac{\mathfrak{b}}{\pi} e^{-\frac{\mathfrak{b}}{2}\left(|z|^2+|w|^2-2z\overline{w}\right)}$$

$$|K_\infty^{(\mathfrak{b})}(z,w)| = \frac{\mathfrak{b}}{\pi} e^{-\mathfrak{b}|z-w|^2/2}$$

- Hence, for separated w_i, w_j

$$\mathbf{M}(\mathbf{w}) \simeq N^{-1} \times (\text{ Identity matrix })$$

Dealing with tracer encounters

- Single quasi-holes merging: $w_1 \simeq w_2$ but all other points "far away"
- With $\mathbf{M}_{3-n} \simeq$ Identity

$$\mathbf{M} = \begin{pmatrix} M_{12} & V \\ V^* & \mathbf{M}_{3-n} \end{pmatrix}$$

- Schur's complement formula

$$\partial_{w_1} \log \Upsilon(\mathbf{w}) = \mathrm{Tr}\left(\frac{1}{\mathbf{M}_{12} - V\mathbf{M}_{3-n}^{-1}V^*} \partial_{w_1} \left(\mathbf{M}_{12} - V\mathbf{M}_{3-n}^{-1}V^* \right) \right)$$

- Explicit calculations to handle contributions from $\mathbf{M}_{12}$
- Multiple tracer encounters/merging: global rough bounds from integral formulae for emerging potentials e.g.

$$\mathcal{A}_j(\mathbf{w}) = N\Im\left(\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \left(\int_{\mathbb{C}} \frac{\Upsilon(\mathbf{w}, z)}{\Upsilon(\mathbf{w})} \frac{dz}{\pi(w_j - z)} - \overline{w_j} \right) \right)$$

- Schur again for $\Upsilon(\mathbf{w}, z)$.

Conclusions

- Unique to 2D quantum mechanics: one can represent bosons as fermions using gauge freedom, and vice-versa
- Does not seem a very good idea for calculations, but suggests funny mechanisms using magnetic flux attachment
- Impurities in quantum-Hall systems couple peculiarly to magnetic fields
- Can lead to modified effective charge and statistics transmutation
- We rigorously proved that natural trial state exhibits such an effect
- Much remains to be done to investigate emergence of trial state
- Main prospect: the same type of mechanism leads to exotic quantum statistics (anyons)
- In that case, even rigorously computing with trial state seems challenging (non-integrable classical 2D Coulomb system, β-ensemble)

Thank you for your attention !