Point Processes in Time-Frequency Analysis

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Overview

References

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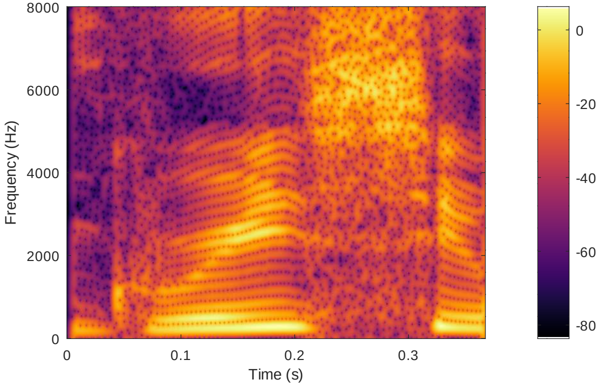
Overview

Motivation

- Many scientific fields use time-frequency representations
- Usually they look at spectrograms that can easily be visualized and offer an intuitive interpretation of the time-frequency content of a signal



Motivation

- Anyone interested in point processes will immediately wonder about the many dark dots
- Only recently researchers began to analyze the mathematical properties of this point process
- To introduce randomness and indeed obtain a point process and not just a point pattern, the first obvious choice is to consider white noise as signal
- Even this “simple” case is not yet fully understood

Motivation

- There are three classes of point patterns that directly emerge from the study of white noise
- The zeros of the short-time Fourier transform (STFT) of white noise
- The local extrema of the STFT of white noise
- The intersection of the STFT of white noise with another complex-valued function
- All these cases can be reduced to finding zeros of different random STFTs

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Short-Time Fourier Transform

- **Window function** $h \in L^2(\mathbb{R})$
- Function (or signal) $x \in L^2(\mathbb{R})$
- The **short-time Fourier transform** (STFT) of x with respect to window h is

$$F_x^h(u,v) = \int_{\mathbb{R}} x(s)\overline{h(s-u)}e^{-2\pi ivs}ds,$$

  for all $(u,v) \in \mathbb{R}^2$
- We can reinterpret the pair (u,v) as a complex number $w = u + iv$

Overview

White Noise

- Heuristically, complex white noise is
  - A random signal $\mathcal{N}$ on $\mathbb{R}$
  - For each time t the random variable $\mathcal{N}(t)$ is a complex Gaussian random variable
  - For different times t_1 and t_2 the random variables $\mathcal{N}(t_1)$ and $\mathcal{N}(t_2)$ are independent
- Mathematically, these properties have some issues...
- We can define it in a weak sense, i.e., make sense of integrals

$$\langle \mathcal{N}, x \rangle = \int_{\mathbb{R}} \mathcal{N}(s) \overline{x(s)} ds$$

White Noise

- Specifically, $\langle \mathcal{N}, x \rangle$ is for each signal $x \in L^2(\mathbb{R})$ a complex Gaussian random variable with zero mean and variance $\|x\|^2$
- Furthermore, for orthogonal x_1 and x_2, we have independence of $\langle \mathcal{N}, x_1 \rangle$ and $\langle \mathcal{N}, x_2 \rangle$, i.e.,

$$\mathbb{E}\left[\langle \mathcal{N}, x_1 \rangle \overline{\langle \mathcal{N}, x_2 \rangle}\right] = 0$$

- Covariance structure

$$\mathbb{E}\left[\langle \mathcal{N}, x_1 \rangle \overline{\langle \mathcal{N}, x_2 \rangle}\right] = \langle x_2, x_1 \rangle$$

STFT of White Noise

- For fixed u and v the STFT corresponds to the linear functional

$$F_x^h(u,v) = \langle x, \mathbf{\Phi}(u,v)h \rangle,$$

where $\mathbf{\Phi}(u,v)h$ denotes the *time-frequency shift*

$$\mathbf{\Phi}(u,v)h(t) = e^{2\pi i v t} h(t-u), \qquad t \in \mathbb{R}.$$

- Define the STFT of white noise as $F_{\mathcal{N}}^h(u,v) = \langle \mathcal{N}, \mathbf{\Phi}(u,v)h \rangle$
- For $w_1, w_2 \in \mathbb{C}$, the covariance is

$$\mathbb{E}\big[F_{\mathcal{N}}^h(u_1,v_1)\overline{F_{\mathcal{N}}^h(u_2,v_2)}\big] = \langle \mathbf{\Phi}(u_2,v_2)h, \mathbf{\Phi}(u_1,v_1)h \rangle$$

Covariance Structure

Defining $F(w) := e^{-\pi i u v} F^{h}_{\mathcal{N}}(u, -v)$ the covariance structure simplifies to

$$\mathbb{E}\left[F(w_1)\overline{F(w_2)}\right] = e^{-\pi i u_1 v_1} e^{\pi i u_2 v_2} \langle \mathbf{\Phi}(u_2, -v_2)h, \mathbf{\Phi}(u_1, -v_1)h \rangle$$
$$= e^{-\pi i u_1 v_1} e^{\pi i u_2 v_2} \int e^{-2\pi i v_2 s} h(s - u_2) \overline{e^{-2\pi i v_1 s} h(s - u_1)} ds$$
$$= e^{-\pi i u_1 v_1} e^{\pi i u_2 v_2} \int h(s) \overline{h(s - (u_1 - u_2))} e^{-2\pi i v_2 (s + u_2) + 2\pi i v_1 (s + u_2)} ds$$
$$= e^{\pi i(-u_1 v_1 - u_2 v_2 + 2 u_2 v_1)} \int h(s) \overline{h(s - (u_1 - u_2))} e^{-2\pi i (v_2 - v_1) s} ds$$
$$= e^{\pi i((u_1 - u_2)(v_2 - v_1) + u_2 v_1 - u_1 v_2)} F^{h}_{h}\big(u_1 - u_2, -(v_1 - v_2)\big)$$
$$= e^{\pi i \Im(w_1 \overline{w_2})} A_h\big(u_1 - u_2, -(v_1 - v_2)\big)$$

Gaussian Weyl-Heisenberg Function

- We say $F\colon \mathbb{C} \to \mathbb{C}$ is a Gaussian Weyl-Heisenberg function (GWHF) if:
- For any $w_1, \dots, w_n \in \mathbb{C}$, $(F(w_1), \dots, F(w_n))$ is a normally distributed complex random vector.
- $F(w)$ is circularly symmetric, i.e., $F \sim e^{i\theta} F$, for all $\theta \in \mathbb{R}$
- The stochastics of F are invariant under *twisted shifts*

$$F(w) \mapsto e^{\pi i \Im(w\bar{\zeta})} \cdot F(w-\zeta), \qquad \zeta \in \mathbb{C}.$$

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General Assumptions

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- For convenience, we will abbreviate $A_h(u,-v) = H(w)$
- Covariance structure is given by

$$\mathbb{E}\left[F(w_1) \cdot \overline{F(w_2)}\right] = e^{\pi i \Im(w_1 \bar{w}_2)} H(w_1 - w_2)$$

- Positive semi-definiteness of the covariance kernel implies

$$\left(H(w_k - w_j) \cdot e^{\pi i \Im(w_k \overline{w_j})}\right)_{j,k=1,\dots,n} \geq 0 \qquad \text{for all } w_1, \dots, w_n \in \mathbb{C}$$

- Since H is up to a constant of modulus one an ambiguity function, we have $|H(0)| = \|h\|^2 > 0$ and normalize $H(0) = 1$
- For the same reason $|H(w)| < |H(0)|$
- Furthermore, we assume $H \in C^2$ and that almost every realization of F is $C^2(\mathbb{R}^2)$

Point Processes

- The zero set of a GWHF is a random set of points, a so-called point process
- The most elegant way to describe point processes is as *random measures*, in our case

$$\mathcal{Z}_F := \sum_{w \in \mathbb{C},\, F(w)=0} \delta_w \,,$$

  where δ_w denotes the Dirac measure at w
- Benefits over viewing point processes as random sets:
  - No problem to have the same point twice
  - Additive structure $\sum_{w \in \mathbb{C},\, F(w)=0} \delta_w$ is easier to work with than $\bigcup_{w \in \mathbb{C},\, F(w)=0} \{w\}$
  - Generalization to weighted point processes is straightforward

Moment Measures

- Similar to moments of a random variable, moment measures give an intuitive characterization of a point process
- First moment measure gives the expected number of points in a domain

$$\mu_1(E) = \mathbb{E}\Bigg[\sum_{w\in\mathbb{C},\, F(w)=0} \delta_w(E)\Bigg] \\ = \mathbb{E}\big[\#\{w \in E : F(w) = 0\}\big]$$

- Radon-Nikodym derivative $\rho_1(w)$ w.r.t. Lebesgue measure is called *first intensity*
- Higher moment measures describe interactions between points (e.g., repulsion or attraction) and deviations from the first moment measure

Overview

Zeros of GWHF

Theorem (First intensity of zero sets)

Let F be a GWHF with twisted kernel H satisfying the standing assumptions. Then $\mathcal{Z}_F$ is a stationary random measure with first intensity:

$$\rho_1 = \frac{1}{2\pi} \frac{\Delta_H + 2\pi^2}{\sqrt{\Delta_H + \pi^2}},$$

where

$$\Delta_H := \det \begin{pmatrix} -H^{(2,0)}(0) + \left(H^{(1,0)}(0)\right)^2 & -H^{(1,1)}(0) - \pi i + H^{(1,0)}(0) H^{(0,1)}(0) \\ -H^{(1,1)}(0) + \pi i + H^{(1,0)}(0) H^{(0,1)}(0) & -H^{(0,2)}(0) + \left(H^{(0,1)}(0)\right)^2 \end{pmatrix}$$

In addition, $\Delta_H \geq 0$, and therefore $\rho_1 \geq 1$.

Kac-Rice Formula

- Starting point: Kac-Rice formula

$$\rho_1(w) = \mathbb{E}\left[|\mathrm{Jac}\, F(w)| \,\middle|\, F(w) = 0 \right] p_{F(w)}(0)$$

where $p_{F(w)}$ is the probability density function of $F(w)$
- Conditional expectation can be well-defined because $\big(F(w), F^{(1,0)}(w), F^{(0,1)}(w)\big)$ is a Gaussian vector
- $\mathrm{Jac}\, F(w) = -\Im\big[F^{(1,0)}(w) \cdot \overline{F^{(0,1)}(w)}\big]$
- We have to calculate expectations $\mathbb{E}\big[F^{(1,0)}(w_1)\overline{F(w_2)}\big]$, etc.

Conditional Expectation

- (X, Y) circularly symmetric Gaussian random vector in $\mathbb{C}^{n+m}$ with covariance matrix

$$\mathrm{Cov}[(X, Y)] = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

- For any locally bounded $h\colon \mathbb{C}^n \to \mathbb{R}$

$$\mathbb{E}\big[h(X) \,\big|\, Y = 0\big] = \mathbb{E}\big[h(Z)\big]$$

- Here,

$$\mathrm{Cov}[Z] = A - BC^{-1}B^*.$$

First Intensity Calculation

- First intensity simplifies to

$$\rho_1(w) = \frac{1}{\pi}\mathbb{E}\left[\left|\Im\left[Z_1\overline{Z_2}\right]\right|\right]$$

- Covariance of Z is

$$\Omega = \begin{pmatrix} -H^{(2,0)}(0)+\left(H^{(1,0)}(0)\right)^2 & -H^{(1,1)}(0)-\pi i+H^{(1,0)}(0)H^{(0,1)}(0) \\ -H^{(1,1)}(0)+\pi i+H^{(1,0)}(0)H^{(0,1)}(0) & -H^{(0,2)}(0)+\left(H^{(0,1)}(0)\right)^2 \end{pmatrix}$$

- Note independence of w
- Absolute mixed moments of Gaussians are more difficult than expected

First Intensity Calculation

- Have to calculate

$$\pi\rho_1 = \frac{1}{\pi^2\Delta_H}\int_{\mathbb{C}^2} |\Im(w_1\overline{w_2})| e^{-(w_1,w_2)^*\Omega^{-1}(w_1,w_2)} dA(w_1)dA(w_2)$$

- Use the integration trick

$$|x| = \frac{1}{\pi}\int_{-\infty}^{+\infty}(1-\cos(xt))\frac{dt}{t^2} = \frac{1}{\pi}\Re\left(\int_{-\infty}^{+\infty}\left(1-e^{itx}\right)\frac{dt}{t^2}\right),$$

- And generalized Gaussian normalization

$$\frac{1}{\pi^2}\int_{\mathbb{C}^2} e^{-(w_1,w_2)^*\,\Omega^{-1}\,(w_1,w_2)} e^{it\Im(w_1\bar{w}_2)} dA(w_1)\,dA(w_2)$$
$$= \frac{1}{\det\left(\Omega^{-1} + \frac{t}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)}$$

Theorem (First intensity of zero sets)

Let F be a GWHF with twisted kernel H satisfying the standing assumptions. Then $\mathcal{Z}_F$ is a stationary random measure with first intensity:

$$\rho_1 = \frac{1}{2\pi} \frac{\Delta_H + 2\pi^2}{\sqrt{\Delta_H + \pi^2}},$$

where

$$\Delta_H := \det \begin{pmatrix} -H^{(2,0)}(0) + \left(H^{(1,0)}(0)\right)^2 & -H^{(1,1)}(0) - \pi i + H^{(1,0)}(0) H^{(0,1)}(0) \\ -H^{(1,1)}(0) + \pi i + H^{(1,0)}(0) H^{(0,1)}(0) & -H^{(0,2)}(0) + \left(H^{(0,1)}(0)\right)^2 \end{pmatrix}$$

In addition, $\Delta_H \geq 0$, and therefore $\rho_1 \geq 1$.

Back to STFT

Recall that $H(w) = e^{-\pi i u v} F_h^h(u, -v)$

Theorem

Zero set of STFT of complex white noise with window h, $\|h\|_2 = 1$, *has first intensity*

$$\rho_{1,h} = \frac{4(c_2 - c_1^2)c_3 - 4c_2c_4^2 - 4c_5^2 - 8c_1c_4c_5 + 1}{4\sqrt{(c_2 - c_1^2)c_3 - c_2c_4^2 - c_5^2 - 2c_1c_4c_5}}$$

where

$$c_1 := \int_{\mathbb{R}} t\,|h(t)|^2 dt, \qquad c_2 := \int_{\mathbb{R}} t^2 |h(t)|^2 dt, \qquad c_3 := \int_{\mathbb{R}} |h'(t)|^2 dt,$$

$$c_4 := -i \int_{\mathbb{R}} h(t)\overline{h'(t)} dt, \qquad c_5 := \Im\left(\int_{\mathbb{R}} t h(t) \overline{h'(t)} dt \right)$$

Theorem

*Zero set of STFT of complex white noise with **real** window* h, $\|h\|_2 = 1$, *has first intensity*

$$\rho_{1,h} = \frac{4(c_2 - c_1^2)c_3 + 1}{4\sqrt{(c_2 - c_1^2)c_3}}$$

where

$$c_1 := \int_{\mathbb{R}} t\,|h(t)|^2 dt, \qquad c_2 := \int_{\mathbb{R}} t^2 |h(t)|^2 dt, \qquad c_3 := \int_{\mathbb{R}} |h'(t)|^2 dt$$

Uncertainty Principle

Theorem

The minimal value of $\rho_{1,h}$ *is* 1*, and it is attained exactly when* $h = g$ *is a generalized Gaussian, i.e.,*

$$g(t) = \frac{\lambda}{\sqrt{\sigma}} e^{-\frac{\pi}{\sigma^2}\left[(t-x_0)^2 + i(\xi_0 \cdot t + \xi_1 \cdot t^2)\right]}$$

with $\sigma > 0$, $\lambda \in \mathbb{C}$, $|\lambda| = 2^{1/4}$, x_0, ξ_0, $\xi_1 \in \mathbb{R}$.

Overview

Second Moment Measure?

- Calculating $\mathbb{E}\big[(\mathcal{Z}_F(E))^2\big]$ is of interest to analyze the expected deviation from ρ_1
- Leads to something like

$$\mathbb{E}\Big[\big|\mathrm{Jac}\, F(z)\ \mathrm{Jac}\, F(w)\big|\ \Big|\ F(z) = F(w) = 0\Big]$$

- Requires the Gaussian vector $\big(F(z), F^{(1,0)}(z), F^{(0,1)}(z), F(w), F^{(1,0)}(w), F^{(0,1)}(w)\big)$
- Managed to derive the conditional distribution
- Calculation of the absolute mixed fourth moment is still an open problem
- Even asymptotic results $E \to \mathbb{C}$ would be interesting

Charged Zeros

- Associate with each zero z the sign $\kappa_z = \operatorname{sgn}(\operatorname{Jac} F(z))$ of the Jacobian at the zero
- Define the charged point process

$$\mathcal{Z}_F^\kappa := \sum_{z\in\mathbb{C},\, F(z)=0} \kappa_z \cdot \delta_z \,.$$

Theorem (First intensity of charged zeros)

Let F be a GWHF with twisted kernel H satisfying the standing assumptions. Then the random signed measure $\mathcal{Z}_F^\kappa$ has first intensity $\rho_1^\kappa = 1$, i.e.,

$$\mathbb{E}\Bigg[\sum_{w\in E,\, F(w)=0} \kappa_w\Bigg] = |E|, \qquad E \subseteq \mathbb{C} \text{ Borel set.}$$

Proof: Charged Intensity

- Generalization of Kac-Rice to include weights yields

$$\rho_1^{\kappa}(w) = \frac{1}{\pi}\mathbb{E}\left[-\Im\left[Z_1\overline{Z_2}\right]\right]$$

- Covariance structure of Z already known
- Imaginary part of a certain mixed (non-absolute!) moment

$$\Omega = \begin{pmatrix} -H^{(2,0)}(0)+\left(H^{(1,0)}(0)\right)^2 & -H^{(1,1)}(0)-\pi i+H^{(1,0)}(0)H^{(0,1)}(0) \\ -H^{(1,1)}(0)+\pi i+H^{(1,0)}(0)H^{(0,1)}(0) & -H^{(0,2)}(0)+\left(H^{(0,1)}(0)\right)^2 \end{pmatrix}$$

Second Moment Measure

- Semi-charged two-point intensity $\tau_2^\kappa \colon \mathbb{C} \to \mathbb{R}$

$$\tau_2^\kappa(z-w) = \frac{\mathbb{E}\Big[\operatorname{Jac} F(w_1) \ \operatorname{Jac} F(w_2) \,\Big|\, F(w_1) = F(w_2) = 0 \Big]}{\pi^2 \big(1 - |H(w_1 - w_2)|^2 \big)}$$

- τ_2^κ is well-defined and serves as density for

$$\mathbb{E}\Big[\big(\mathcal{Z}_F^\kappa(E) \big)^2 - \mathcal{Z}_F(E) \Big] = \int_{E \times E} \tau_2^\kappa(w_1 - w_2) \, dA(w_1) dA(w_2)$$

Second Moment Measure

- Second moments in the charged case require calculation of

$$\mathbb{E}\left[\operatorname{Jac} F(w_1)\ \operatorname{Jac} F(w_2)\,\middle|\, F(w_1) = F(w_2) = 0\right]$$

- In the end, "only" a mixed fourth moment of a Gaussian
- Calculation based on Wick's formula for Gaussian vector v with covariance Ω

$$\begin{aligned}\mathbb{E}\Big[\Im(Z_1\bar{Z}_2)\cdot\Im(Z_3\bar{Z}_4)\Big] \\ = -\frac{1}{2}\Re\Big[\Omega_{1,2}\Omega_{3,4} + \Omega_{1,4}\Omega_{3,2} - \Omega_{2,1}\Omega_{3,4} - \Omega_{2,4}\Omega_{3,1}\Big]\end{aligned}$$

Hyperuniformity of Charge

Theorem

Let F be a GWHF with twisted kernel H satisfying the standing assumptions. Assume further that $H(z) = P\big(|z|^2\big)$ and

$$\sup_{r \geq 0} \big(|P(r)| + |P'(r)| + |P''(r)|\big) r^2 < \infty.$$

Then

$$\operatorname{Var}\big[\mathcal{Z}_F^{\kappa}(B_R(w))\big] \leq CR, \qquad R > 0,$$

and

$$\frac{1}{R}\operatorname{Var}\Big[\mathcal{Z}_F^{\kappa}(B_R(w))\Big] \to \int_0^{\infty} \frac{2r^2 P'(r^2)^2}{1 - P(r^2)^2} dr, \qquad \text{as } R \to \infty$$

STFT with Charges

- Expected charge is independent of the window
- For Gaussian window, there are only positive charges
- In other cases, additional positive and negative charges have to “cancel”