Seven roads to time-frequency

Patrick Flandrin

ENS DE LYON

Lab PHYS ENS de LYON

cnrs

Agenda

Agenda

1 Atomic decompositions

Agenda

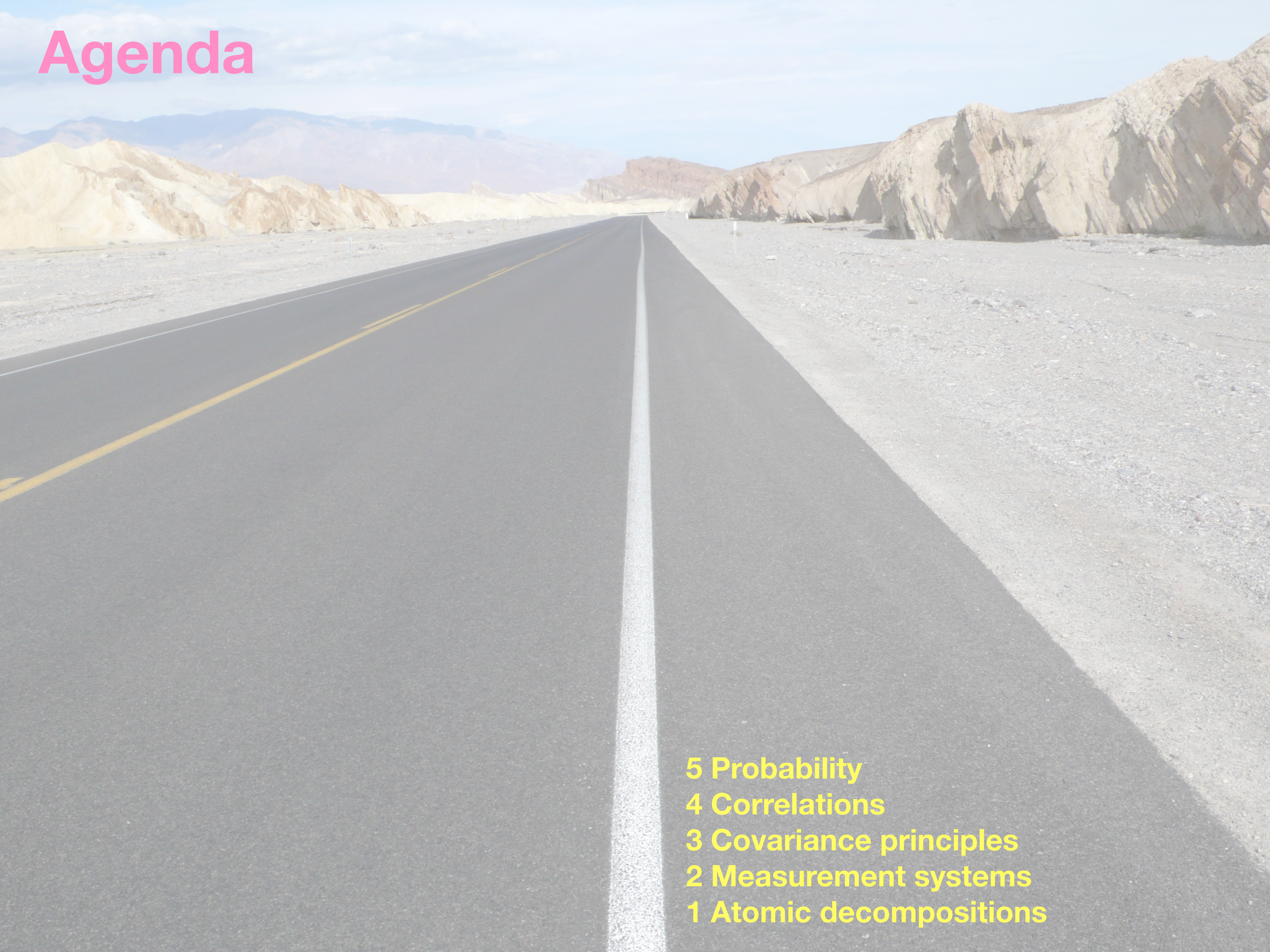
2 Measurement systems
1 Atomic decompositions

Agenda

3 Covariance principles
2 Measurement systems
1 Atomic decompositions

Agenda

4 Correlations
3 Covariance principles
2 Measurement systems
1 Atomic decompositions


Agenda
5 Probability
4 Correlations
3 Covariance principles
2 Measurement systems
1 Atomic decompositions

Agenda

6 Quantum operators
5 Probability
4 Correlations
3 Covariance principles
2 Measurement systems
1 Atomic decompositions

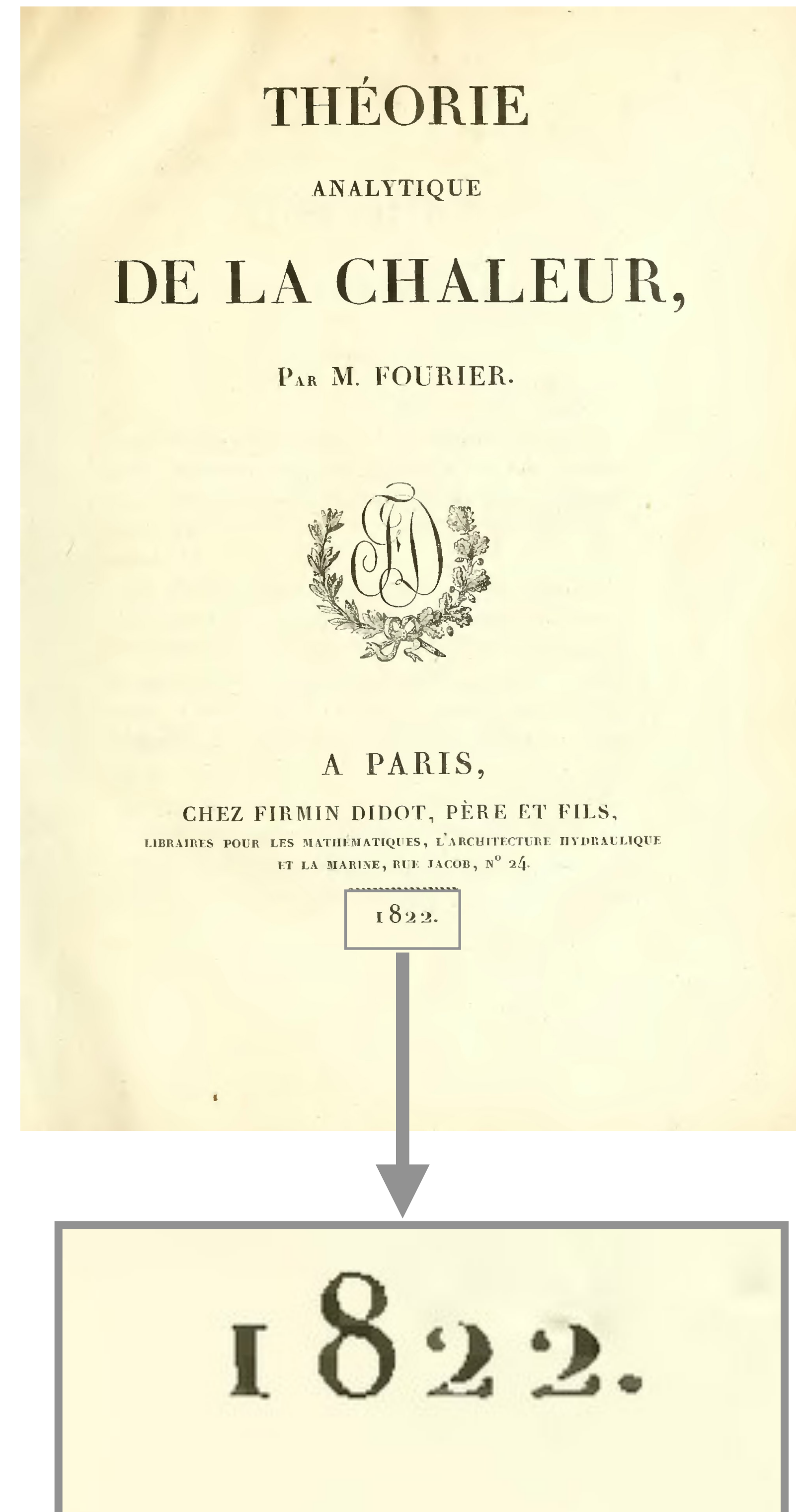
Agenda

7 Geometry
6 Quantum operators
5 Probability
4 Correlations
3 Covariance principles
2 Measurement systems
1 Atomic decompositions

intuition

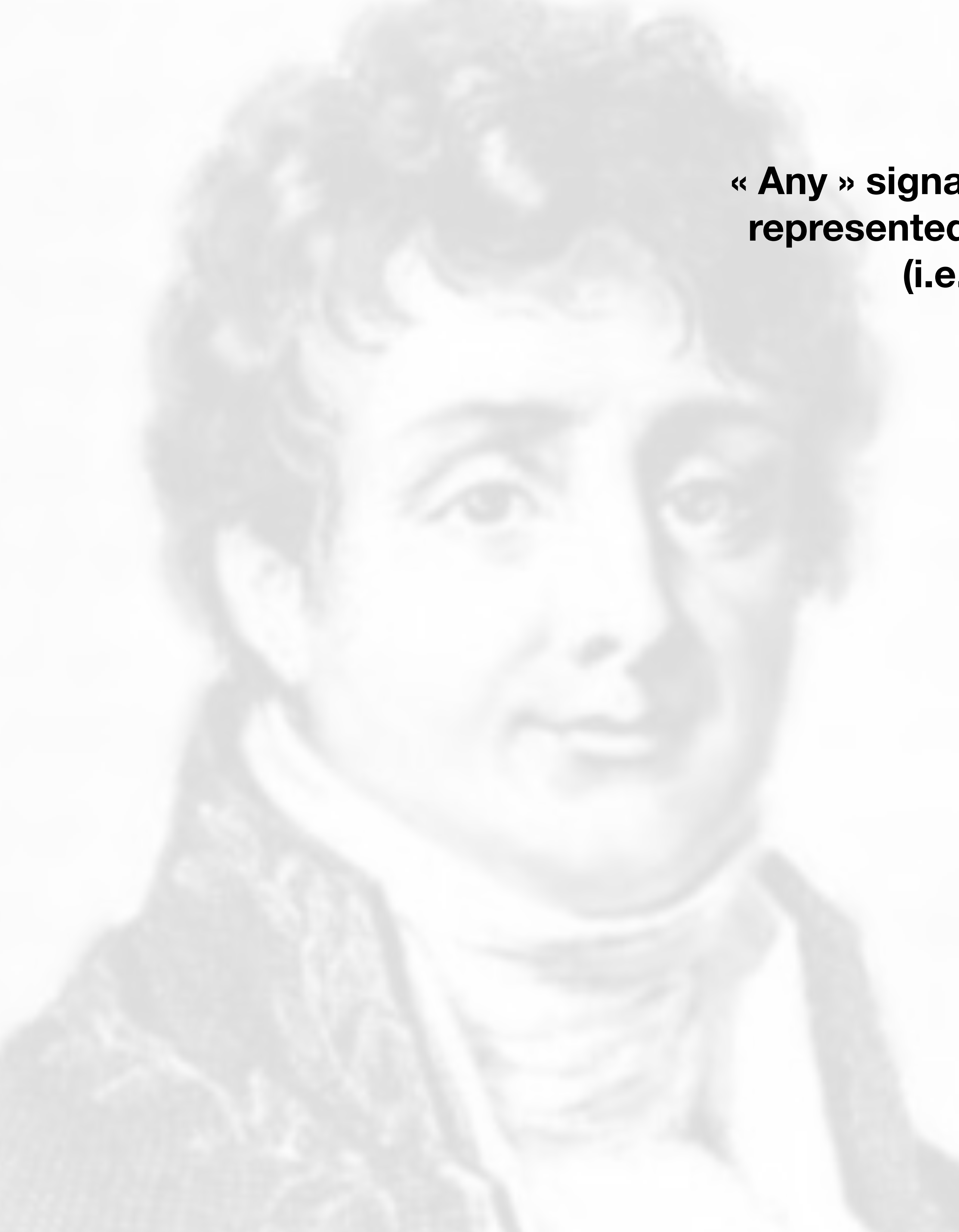
Joseph Fourier
(1768-1830)

Joseph Fourier
(1768-1830)



THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.

A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.



« Any » signal can be decomposed into (or represented with) complex exponentials (i.e., sines and cosines)

$$e_f(t) = e^{i2\pi f t}$$

$$x(t) = \int \langle x, e_f \rangle \, e_f(t) \, df$$

$X(f)$
Fourier transform

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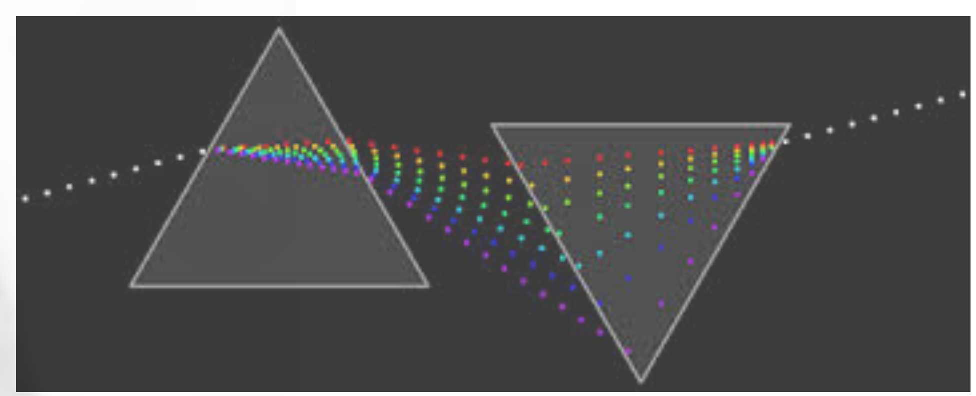
$\langle x, e_f \rangle \rightarrow X(f)$



Any « harmonic » signal is efficiently decomposed into Fourier modes

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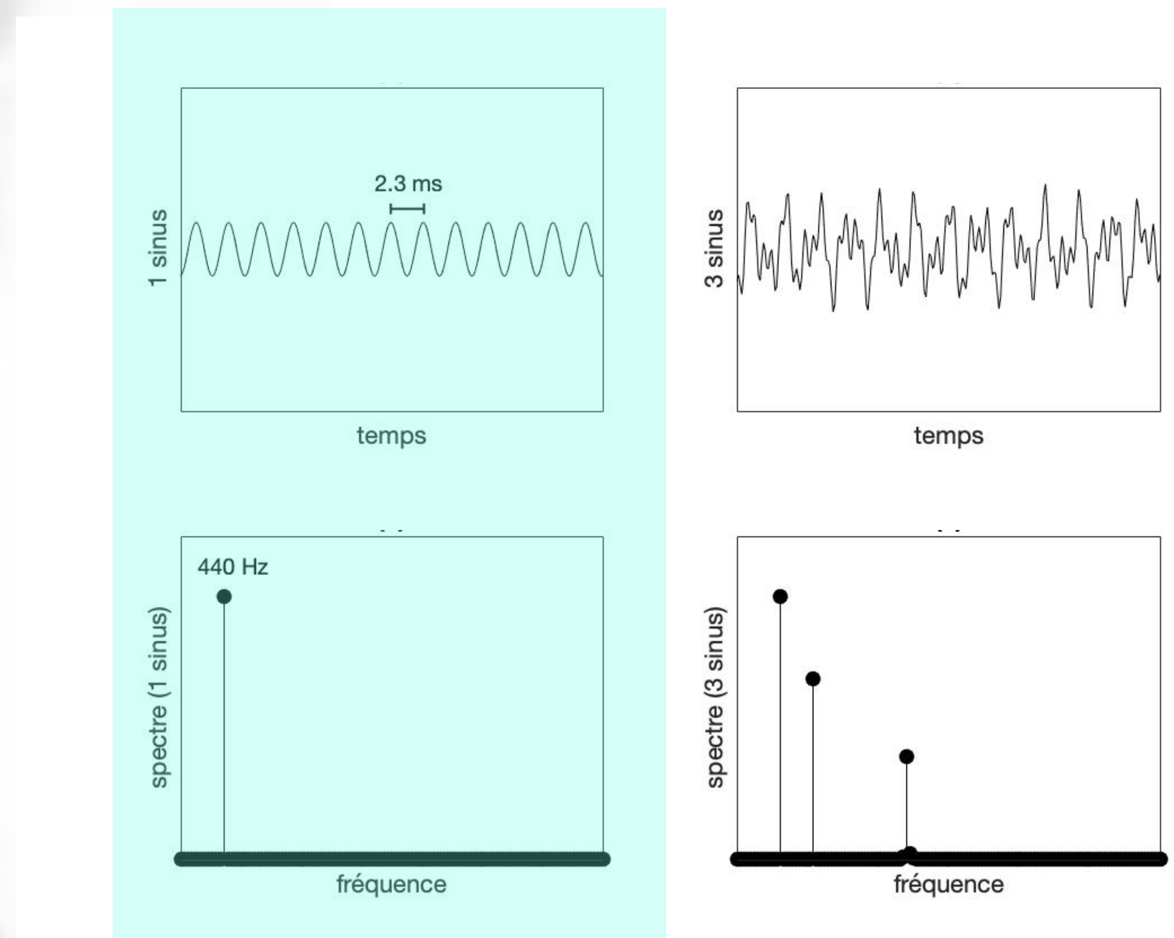

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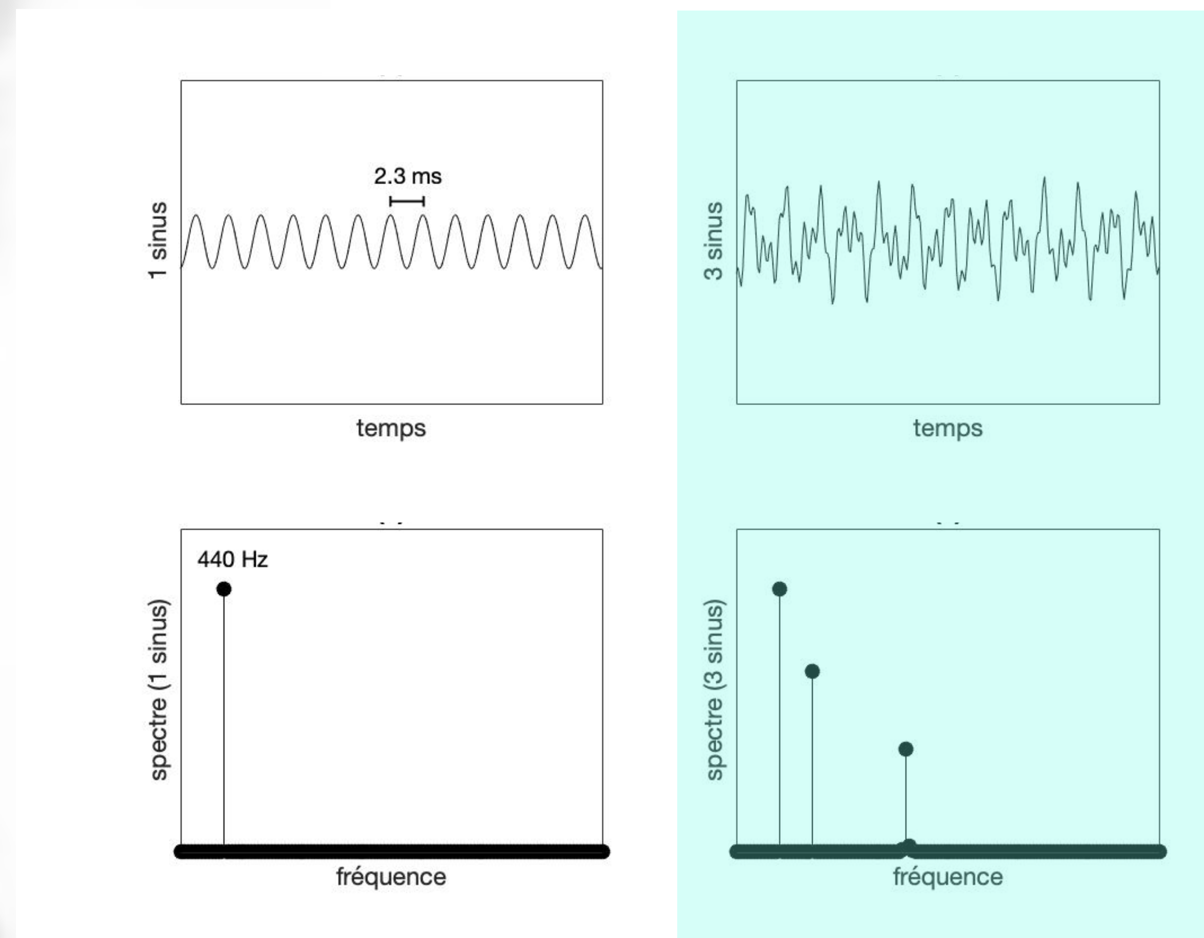
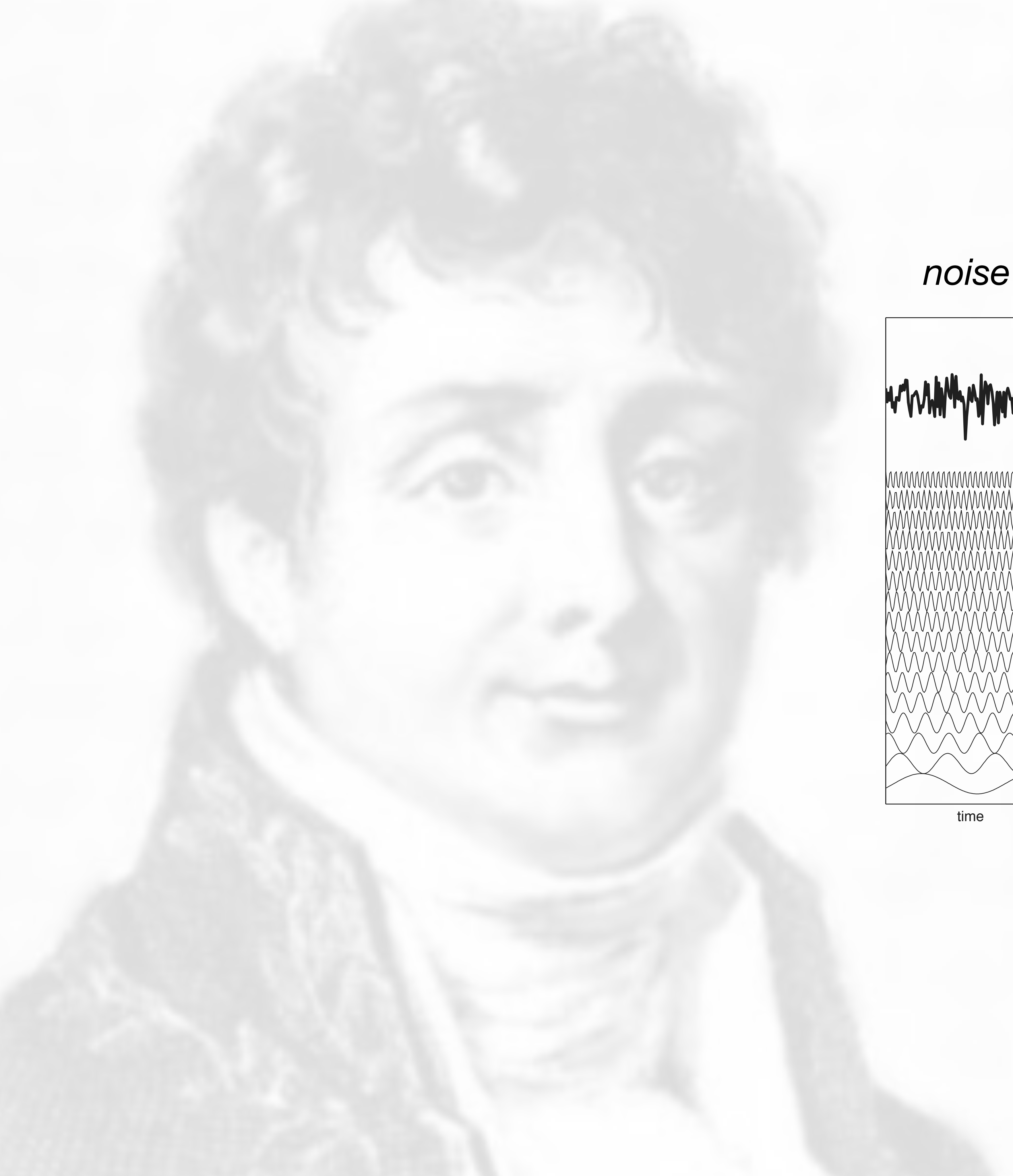
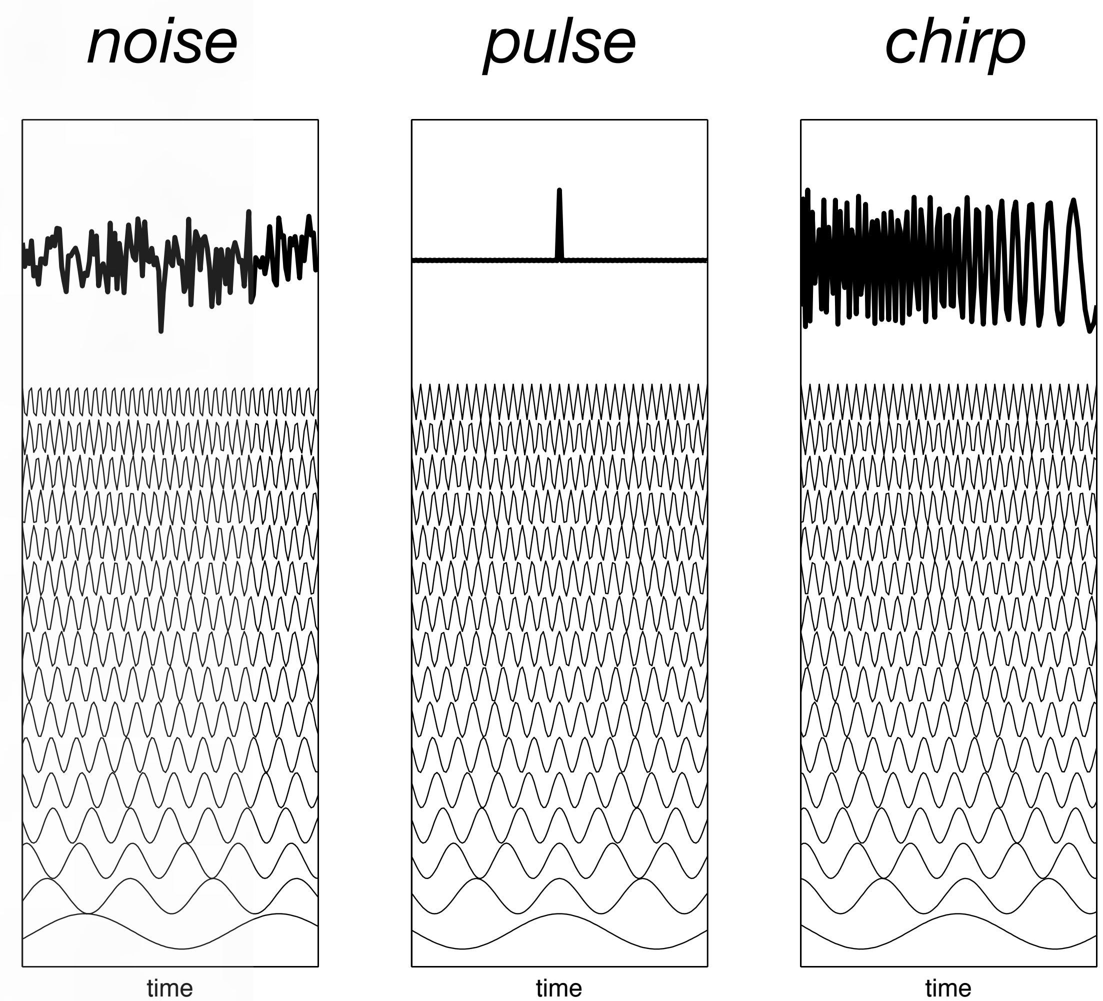
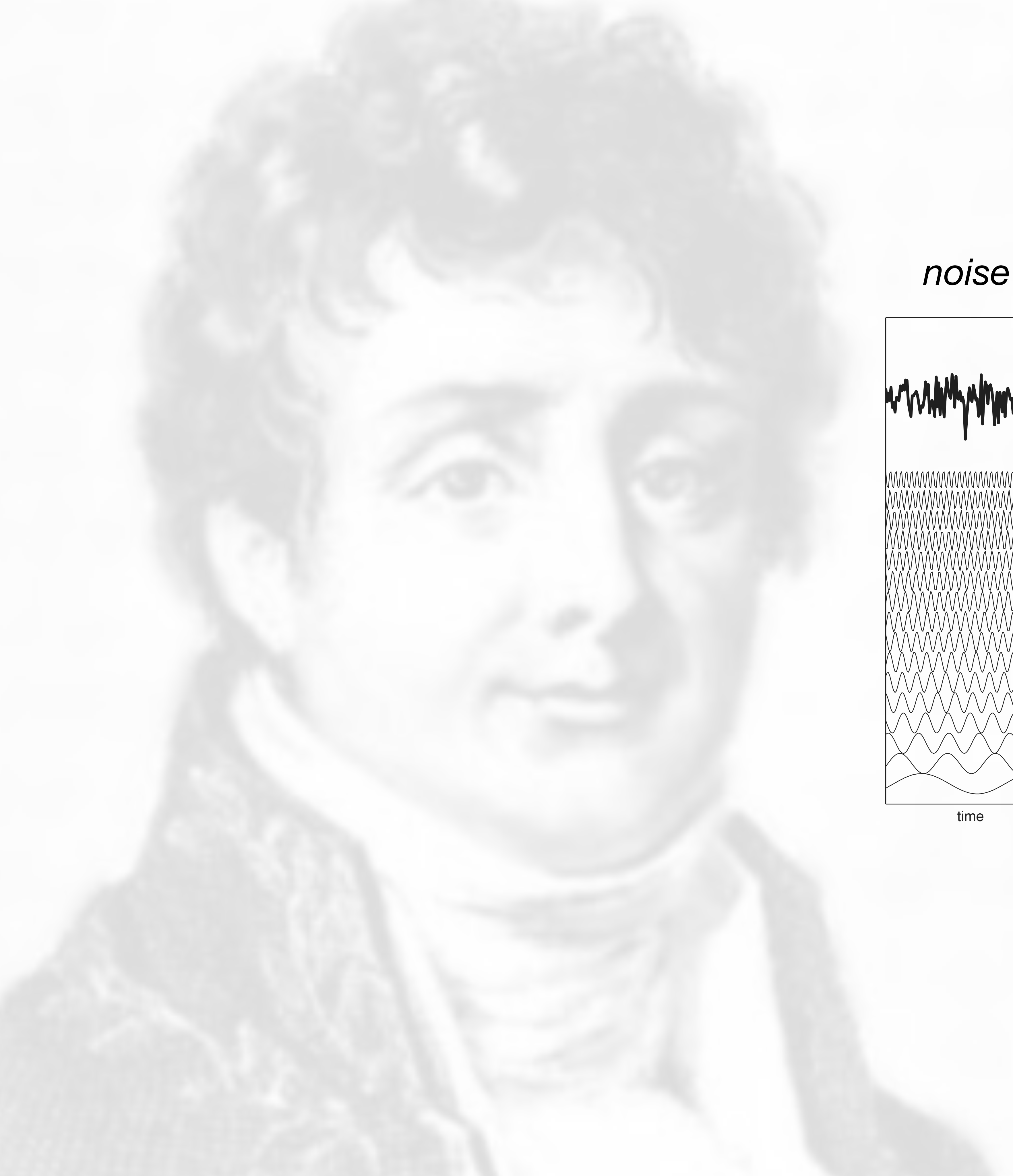


« Any » signal ?







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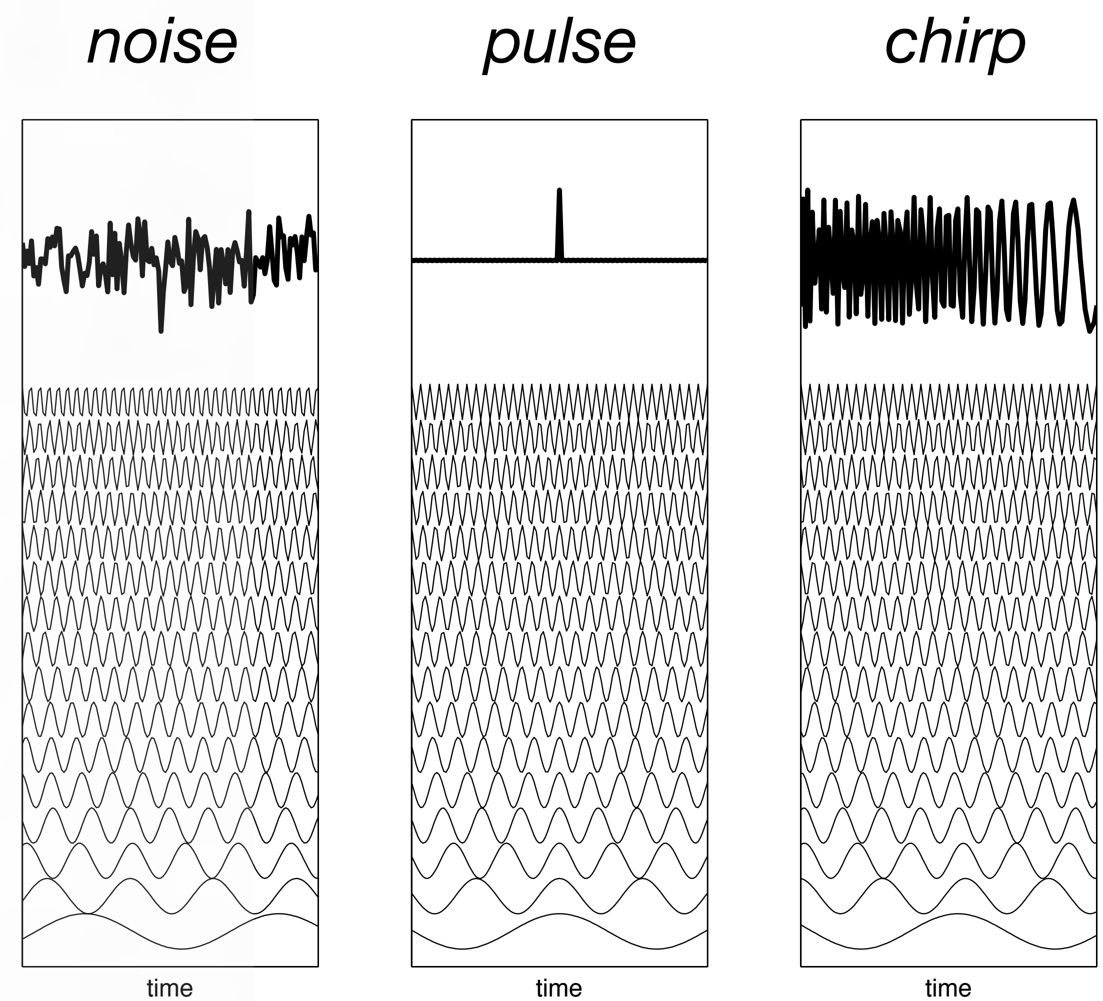


Maths ✓

Physics ??

« If we consider a fragment [of music] containing many components (which is the least that one should ask) and one note, *la* (A) for example, appears once in the fragment, the harmonic analysis will present us with the amplitude and the phase of the corresponding frequency, without locating the time point of the *la*. And yet, it is obvious that in the course of the fragment there will be instants were the *la* will not be heard. Nevertheless, the representation is mathematically correct, because the phase of the notes near the *la* acts to destroy this note by interference when *la* is not heard, and to reinforce it, also by interference, when it is heard; but if there exists in this concept a cleverness which doses justice to mathematical analysis, there is also a distortion of reality; in fact, when *la* is not heard, the true reason is that *la* is not emitted. »

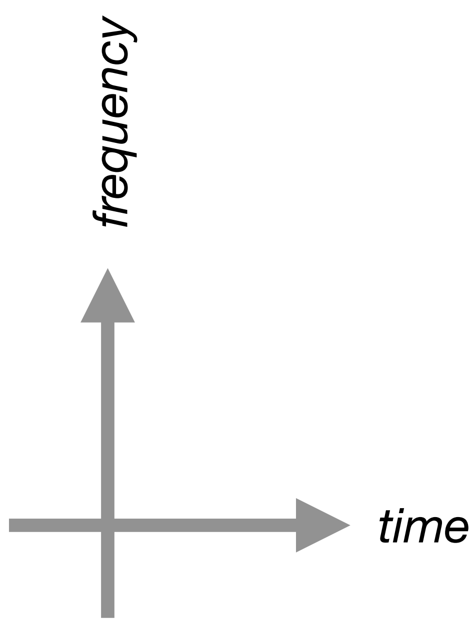
J. Ville (1948)

Fourier 2.0

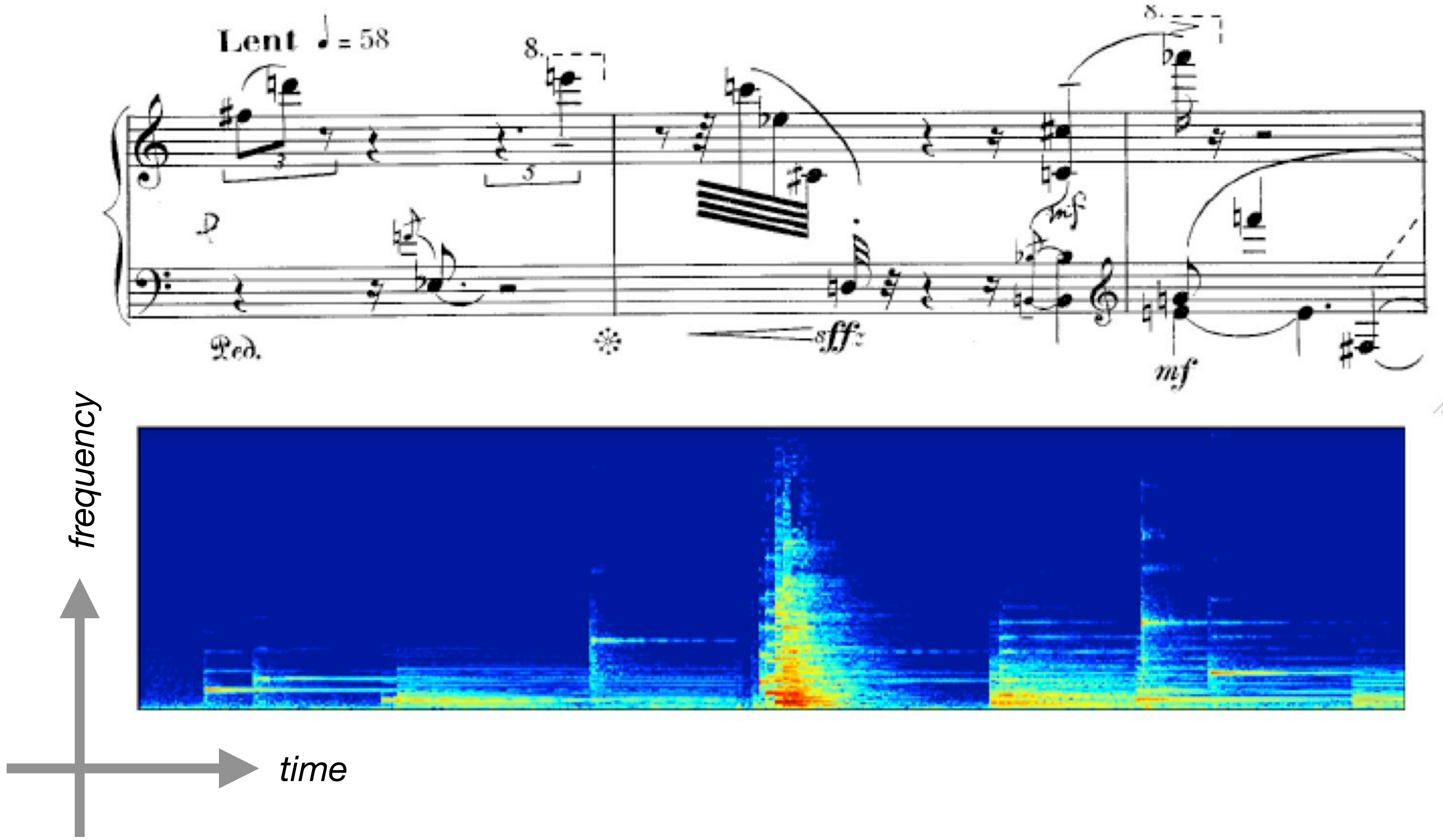
Boulez
1946

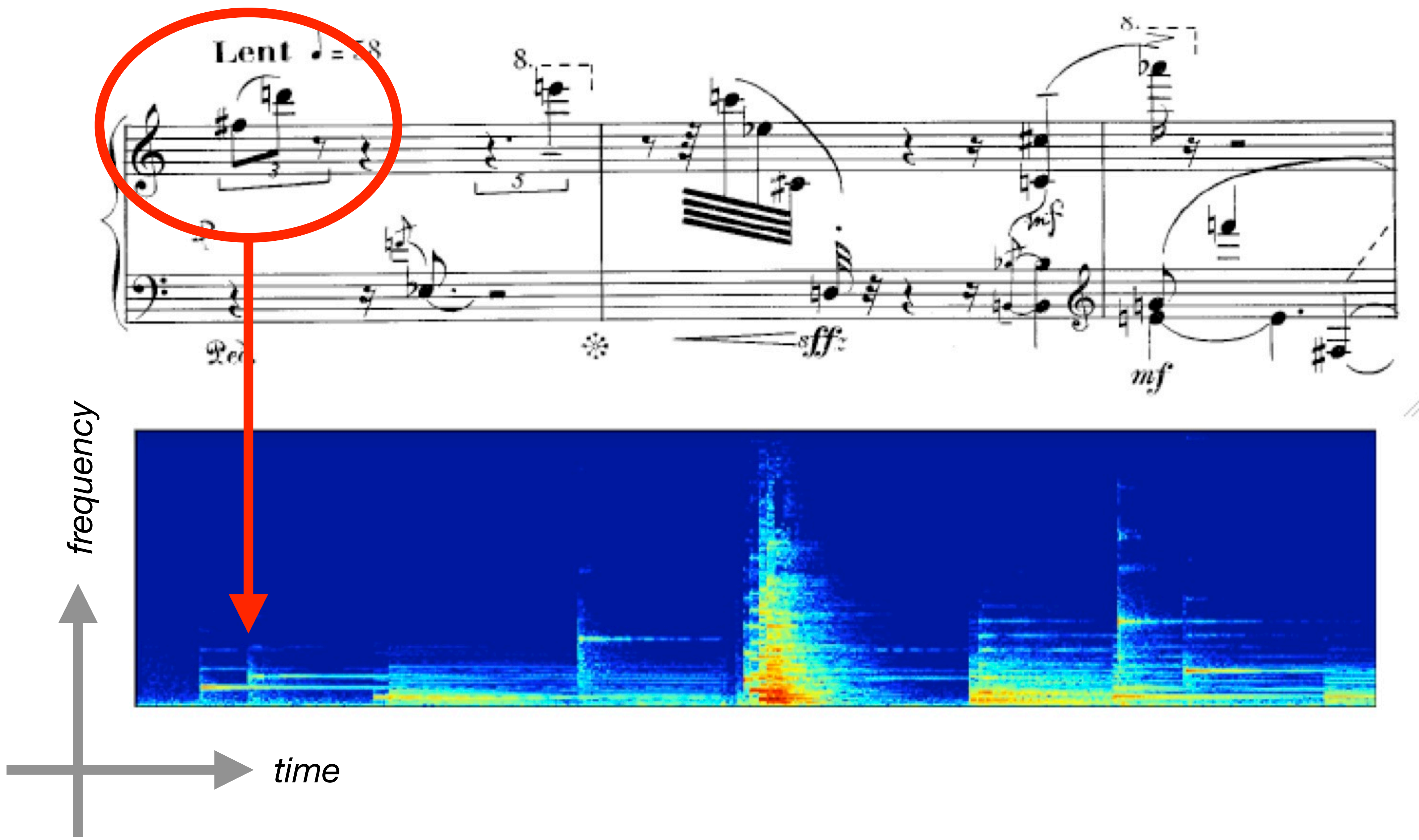
time

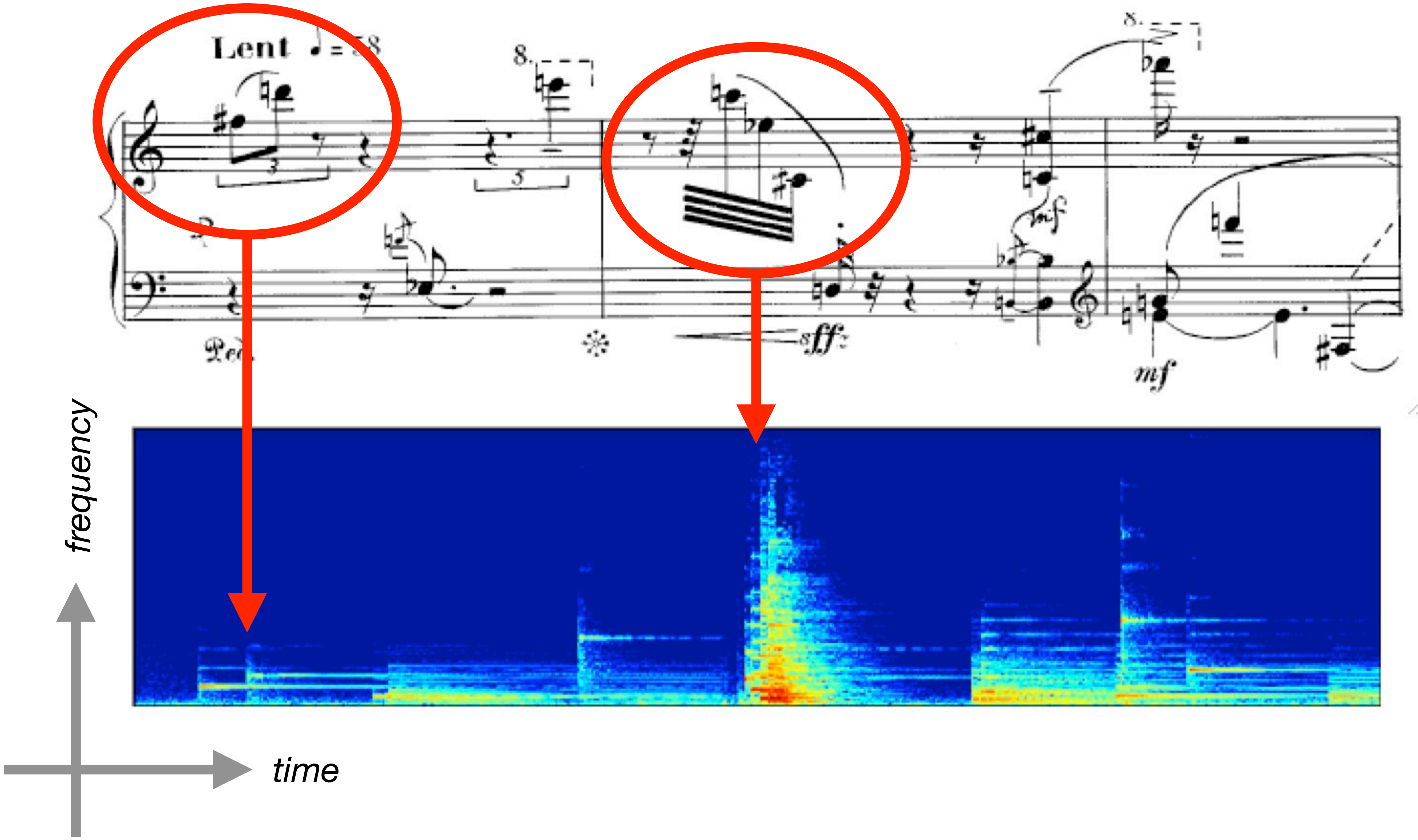

Lent ♩ = 58
8.
Ped.
sff
mf


frequency
time

A mathematical musical score




Lent
frequency
time


Lent
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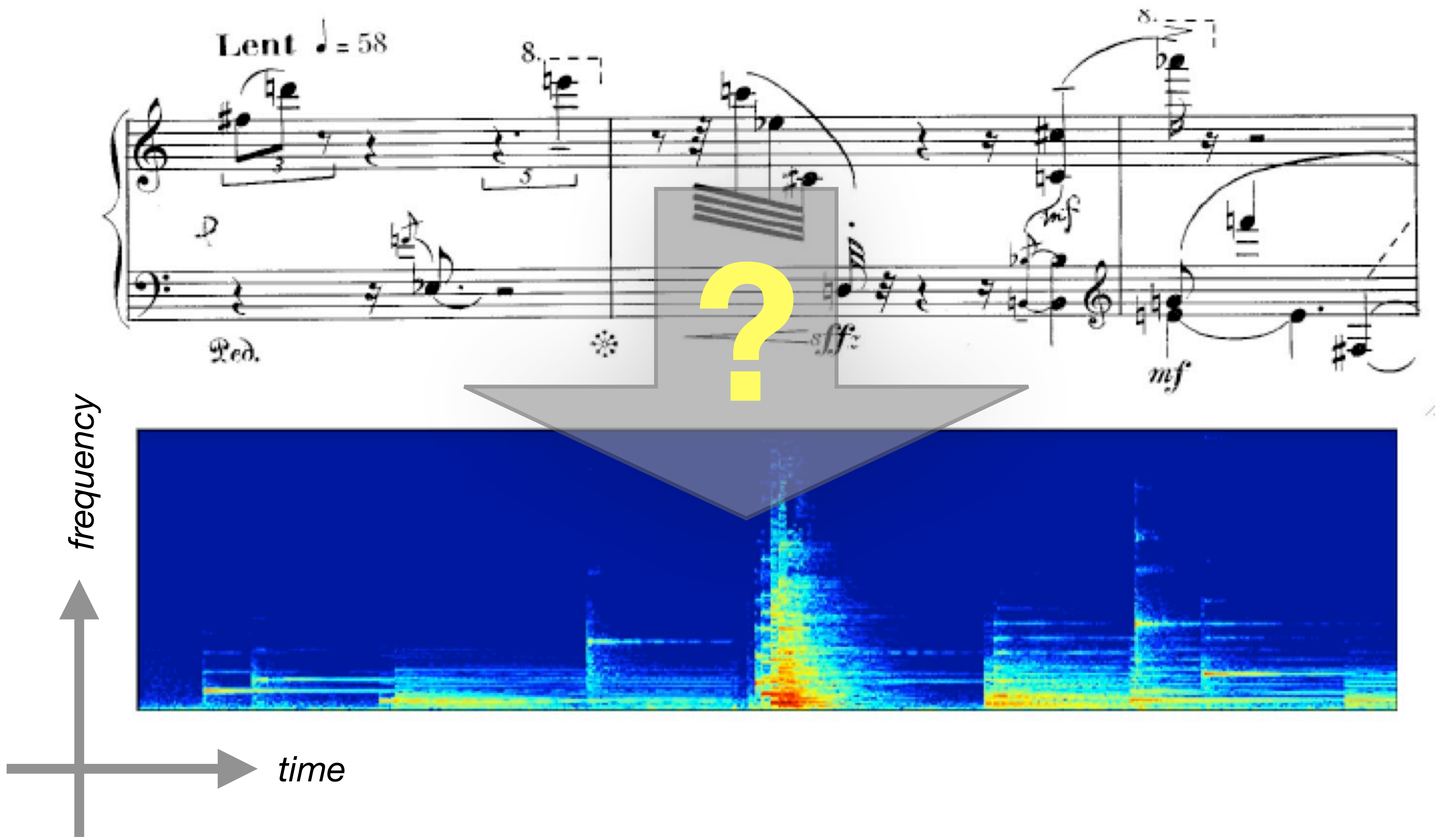

Lent ♩= 58
?
frequency
time

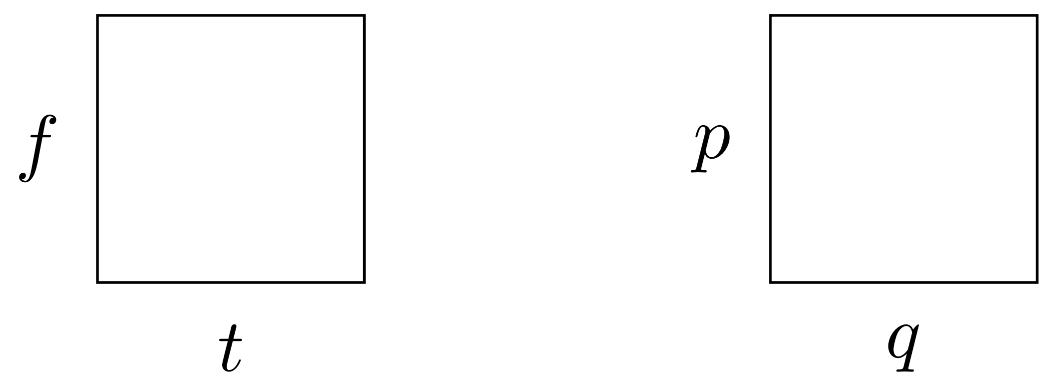
Joint representation based on a Fourier pair of variables

Analogy

$$(t, f) \longleftrightarrow (q, p)$$

time-frequency　　　position-momentum

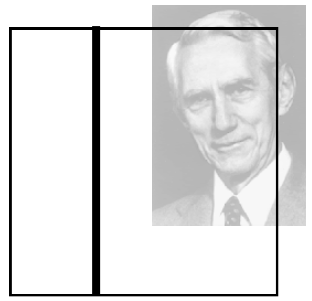
« Phase-space » description



Interplay　　signal theory ⟷ quantum mechanics

1 atomic decompositions

Time, frequency

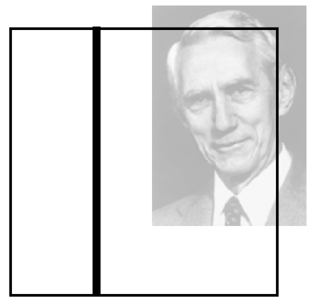


Shannon
1948

$$\int x(t)\,\delta_t(t_0)\,dt = x(t_0) = \int X(f)\,e_f(t_0)\,df$$

Fourier
1822

Time, frequency, and time-frequency



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Fourier
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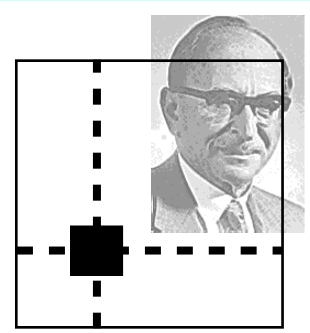
$$\int x(t)\,\delta_t(t_0)\,dt = x(t_0) = \int X(f)\,e_f(t_0)\,df$$

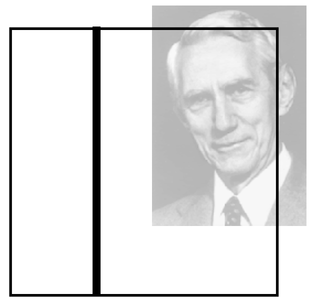
$$\shortparallel$$

$$\int\!\!\int \lambda_x(t,f)\,h_{tf}(t_0)\,dt\,df$$



Gabor
1946

Time, frequency, and time-frequency



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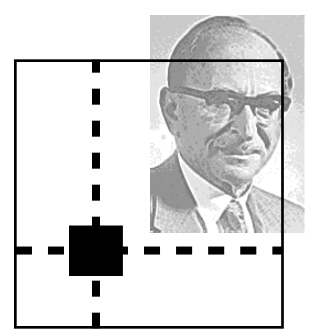
$$\|$$

$$\int\!\!\int \lambda_x(t,f)\,h_{tf}(t_0)\,dt\,df$$

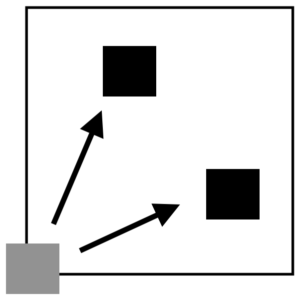
$$X(f) = \langle x, e_f \rangle$$



Gabor
1946

$$\lambda_x(t,f) = \langle x, h_{tf} \rangle = \int x(s)\,h^*_{tf}(s)\,ds$$

Exploring the plane with time-frequency shifts



$$h_{tf}(s) = (\mathbf{T}_{tf}h)(s) = h(s-t)\,e^{i2\pi fs}$$

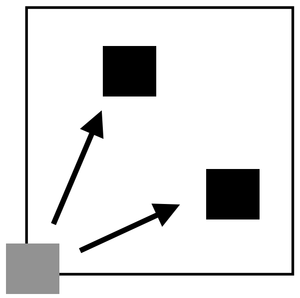
$$\Downarrow$$

$$F_x^h(t,f) = \int x(s)\,h^*(s-t)\,e^{-i2\pi fs}\,ds \longrightarrow S_x^h(t,f) = \left|F_x^h(t,f)\right|^2$$

short-time Fourier transform

spectrogram

Exploring the plane with time-frequency shifts



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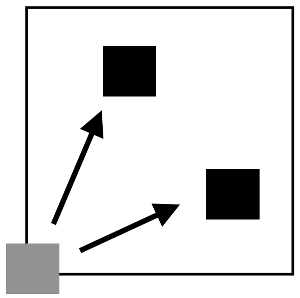
$h = g \quad \Updownarrow$

Q-function

Husimi 1940

Exploring the plane with time-frequency shifts



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short-time Fourier transform

spectrogram

$$h = g \quad \Updownarrow \quad \Delta t_g \Delta f_g = \frac{1}{4\pi}$$

Q-function

No perfect time-frequency localization

$$\Delta t_h \Delta f_h \geq \frac{1}{4\pi}$$

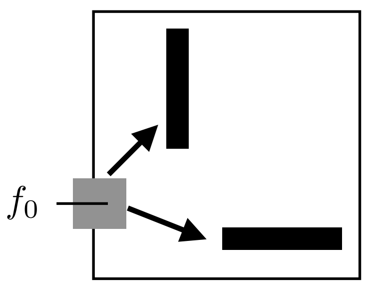
Heisenberg 1925

Husimi 1940

$$F_x^h(t,f) = \iint \left[e^{i2\pi(f'-f)t'}\,F_h^h(t-t',f-f')\right] F_x^h(t',f')\,dt'\,df'$$

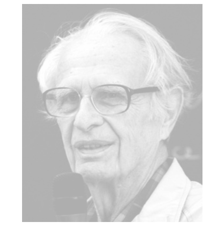
reproducing kernel

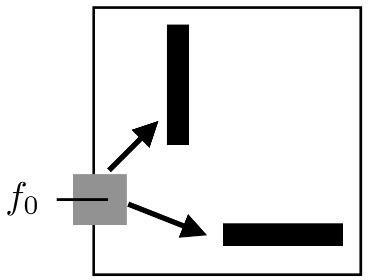
Exploring the plane with time-scale moves



$$\psi_{ta}(s) = (\mathbf{\Lambda}_{ta}\psi)(s) = \frac{1}{\sqrt{a}}\psi\left(\frac{s-t}{a}\right); a = \frac{f_0}{f}$$

Exploring the plane with time-scale moves





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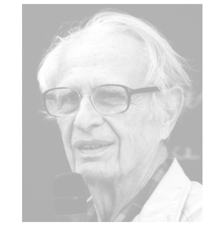
$$\Lambda_x^{\psi}(t,a) = \frac{1}{\sqrt{a}}\int x(s)\,\psi^*\left(\frac{s-t}{a}\right)\,ds \longrightarrow \Theta_x^{\psi}(t,a) = \left|\Lambda_x^{\psi}(t,a)\right|^2$$

wavelet transform

scalogram

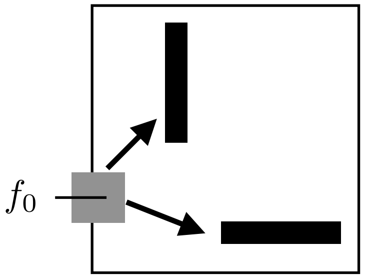


Grossmann

Morlet
1984

Meyer
1985

Exploring the plane with time-scale moves



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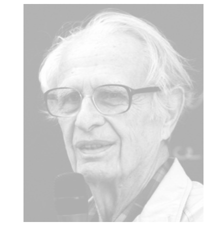
wavelet transform — scalogram

As compared to STFT

- *No perfect time-frequency localization either*
- *Reproducing kernel*
- *Better discretization properties*



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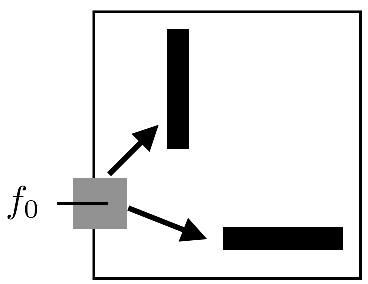

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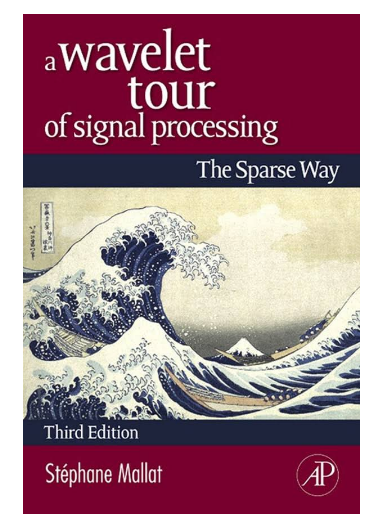
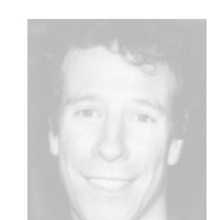




Mallat
1998

2 measurement systems

Effective systems

Koenig
1867

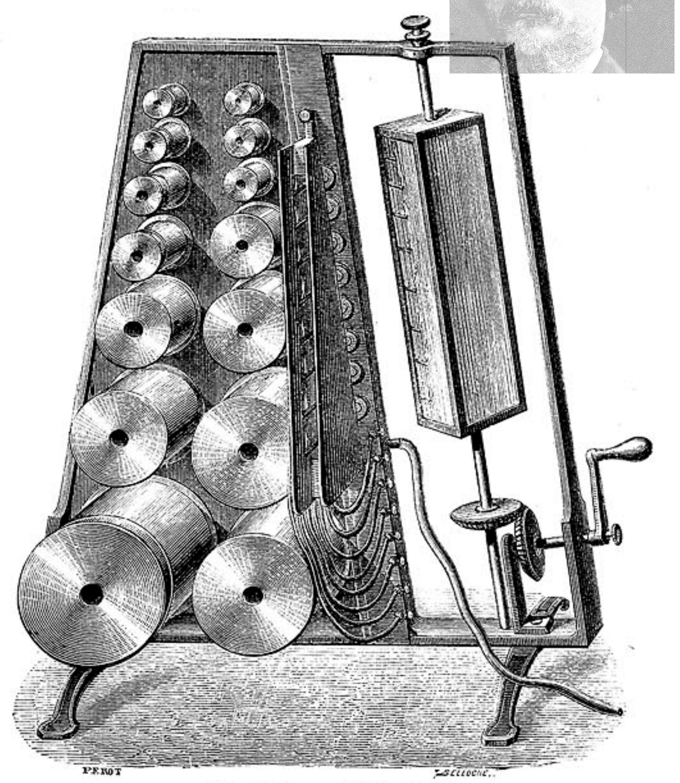
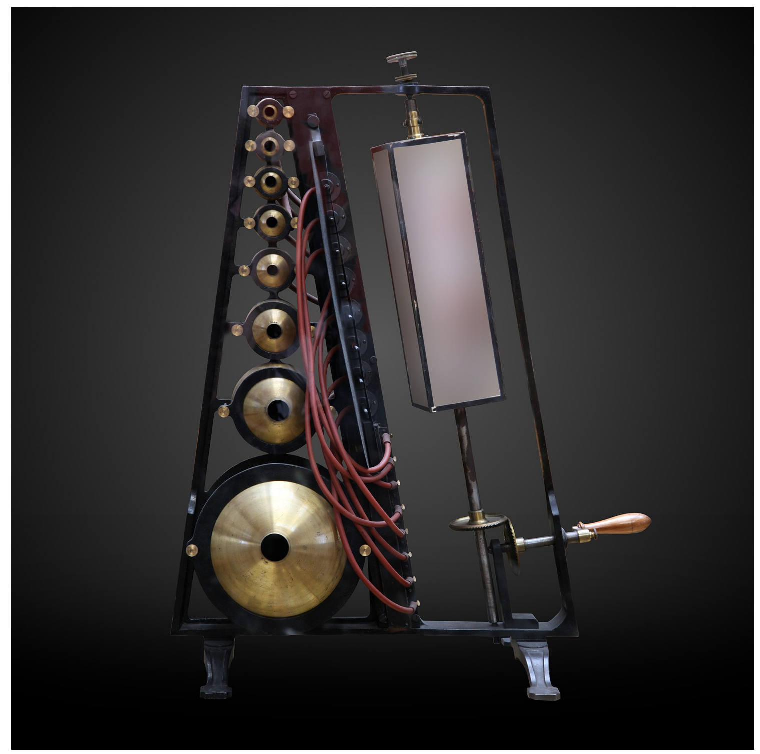


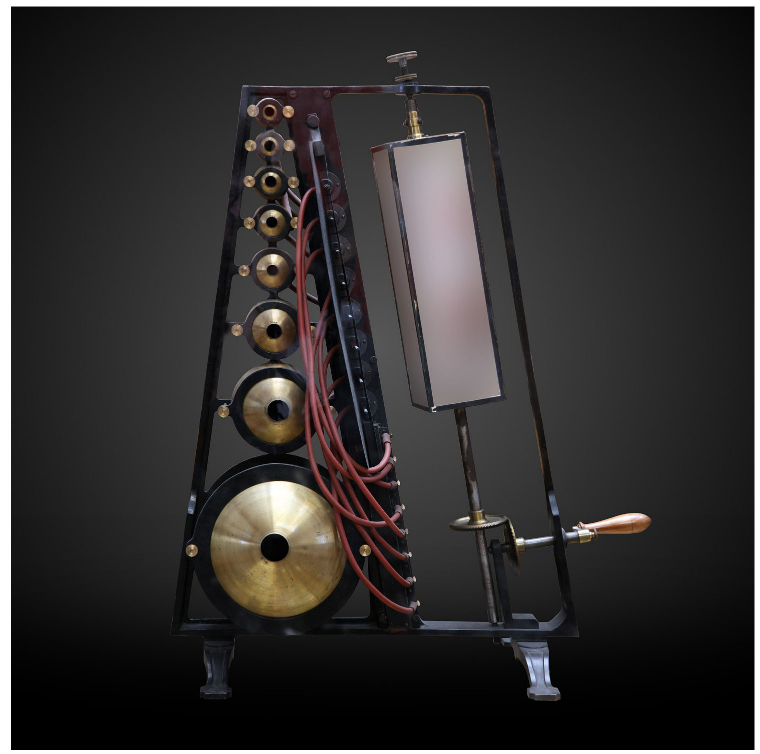
Fig. 116 (h. = 0m,90) (No 242).

242. **Analyseur du timbre des sons** à flammes manométriques, avec 14 résonateurs universels (fig. 116) . . . . **650** fr.
Manometric flame Analyser for the timbre of sounds, with 14 universal resonators.

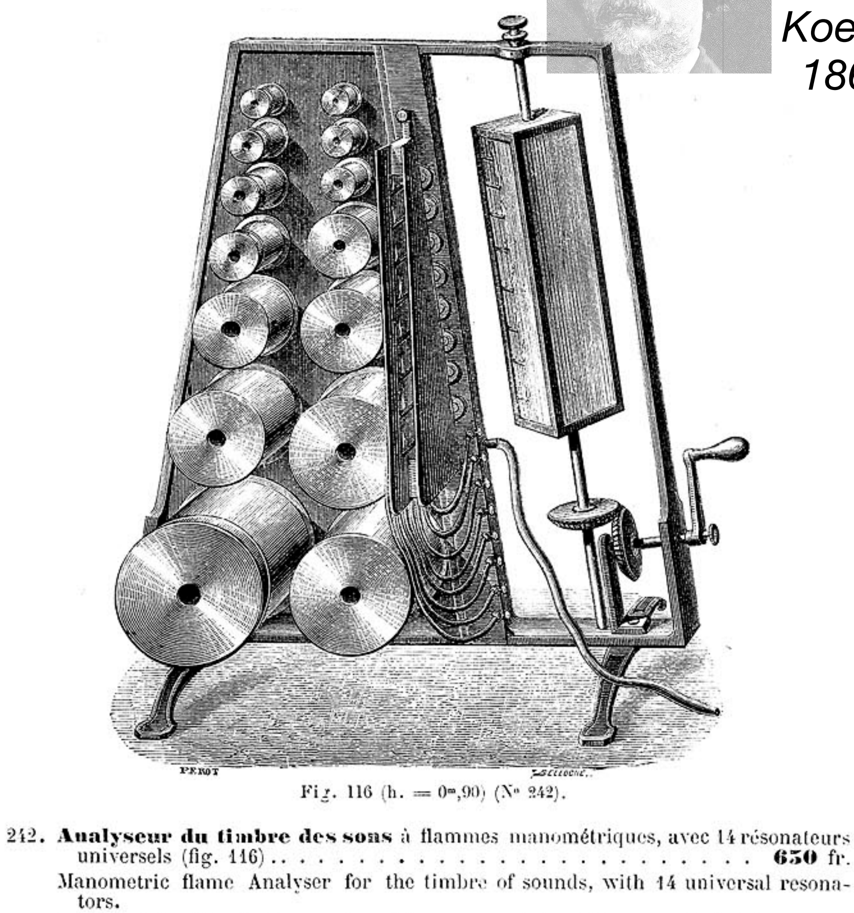


sound analyzer

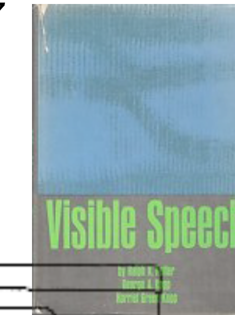
Effective systems

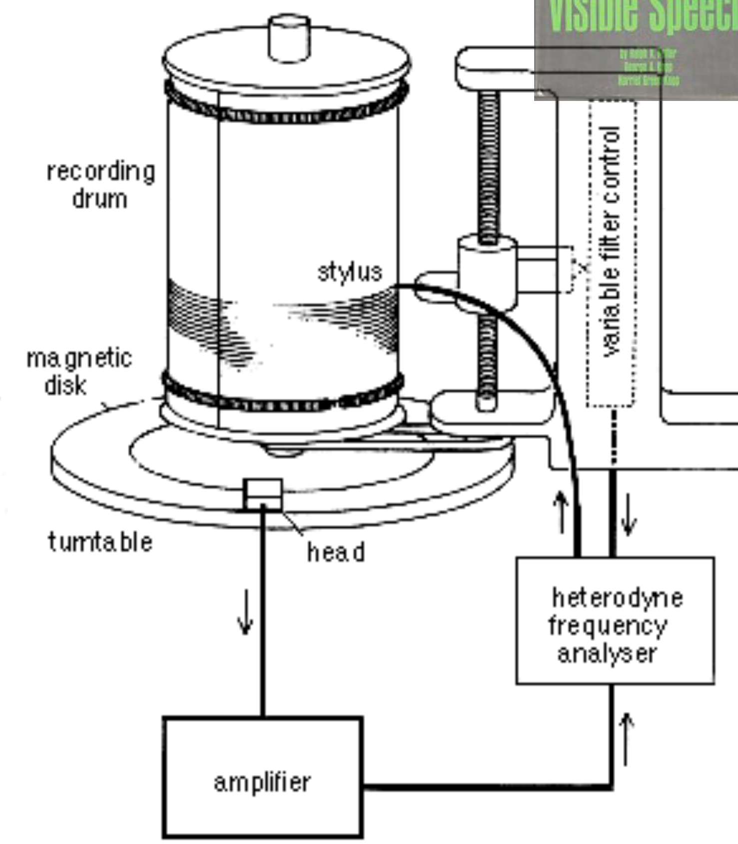


Koenig
1867



Potter et al.
1947

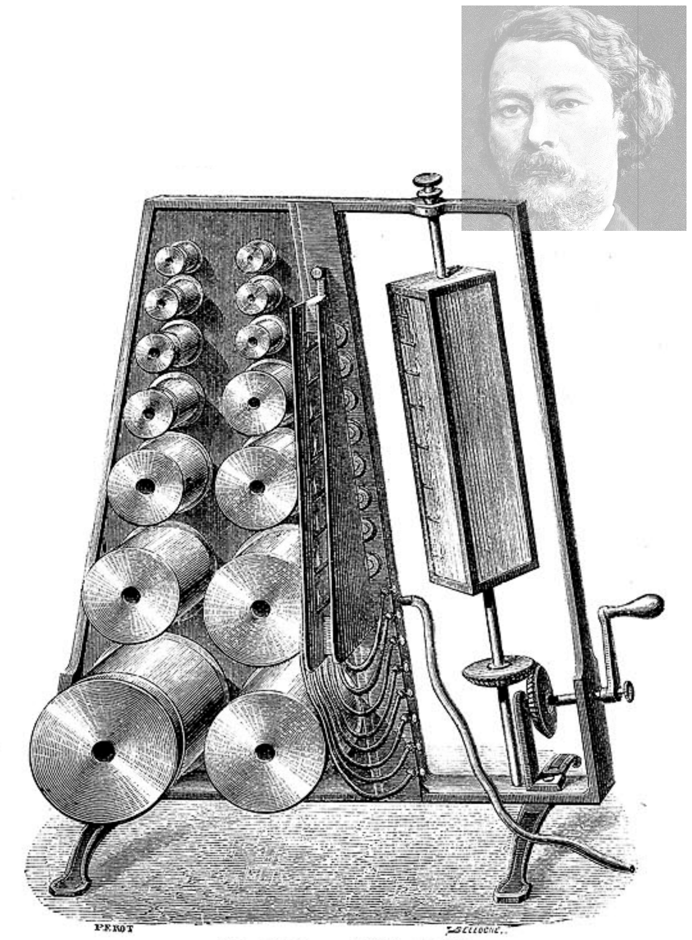




sound analyzer

sonagram

Effective systems



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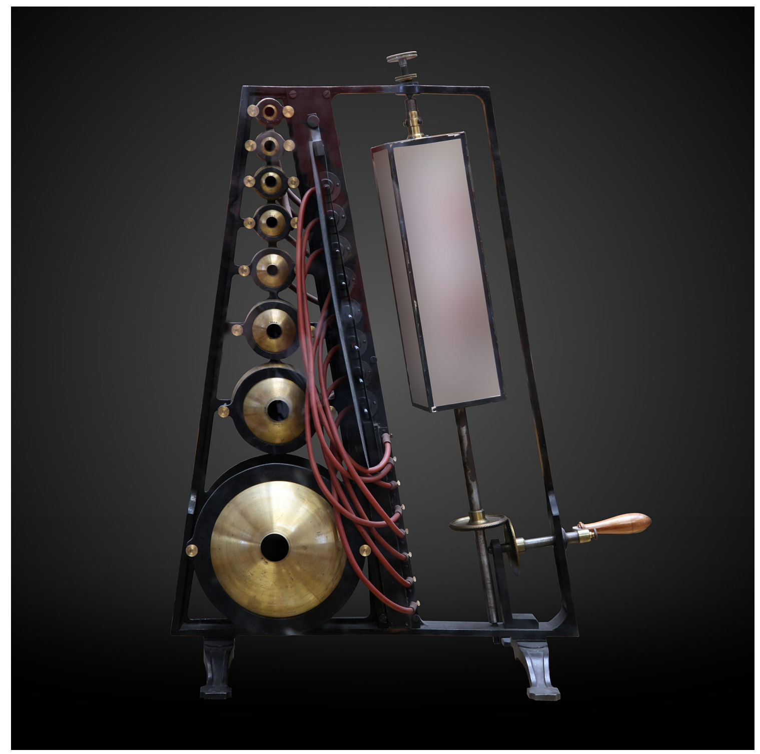
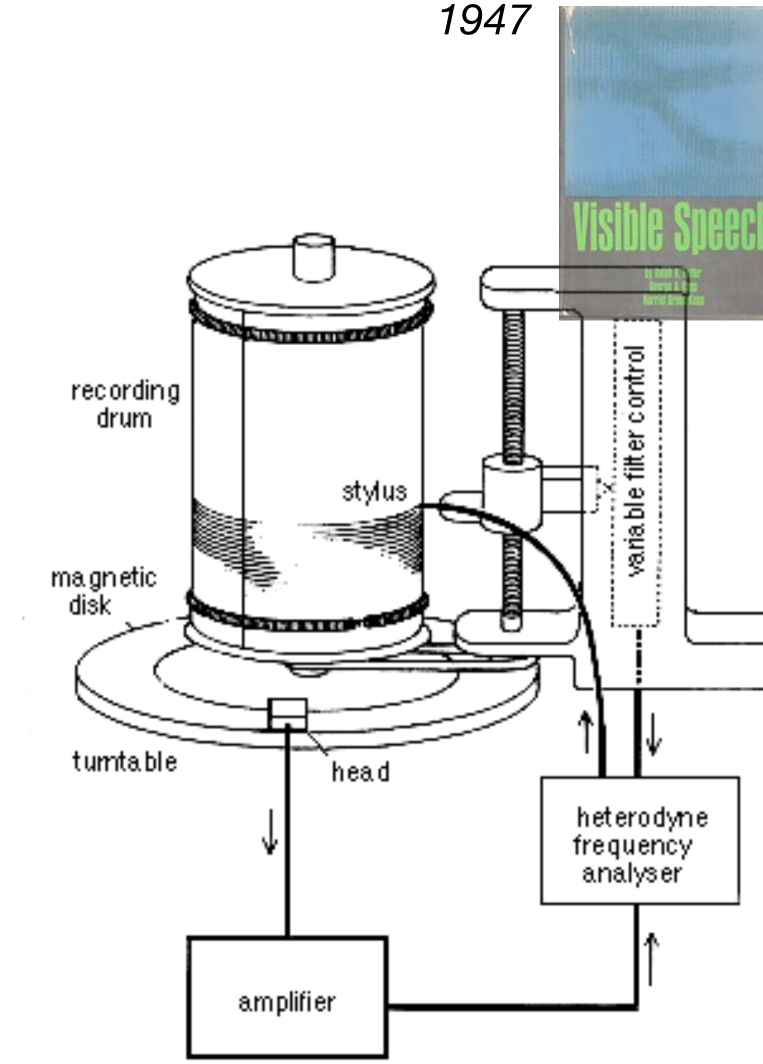


sound analyzer

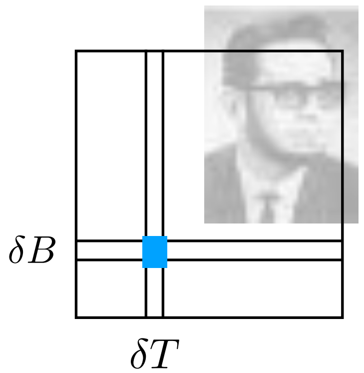
Potter et al.
1947



sonagram

$$\left| \int \left[X(\xi)\, H^*(\xi - f) \right] e^{i 2\pi \xi t}\, d\xi \right|^2 = S_x^h(t, f)$$

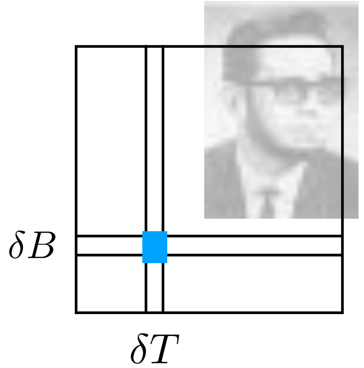
Idealized systems



Rihaczek
1968

$$R_x(t,f) := \lim_{\delta T\,\delta B \to 0} \frac{1}{\delta T\,\delta B} \int_{t-\delta T/2}^{t+\delta T/2} x(s) \left[\int_{f-\delta B/2}^{f+\delta B/2} X(\xi)\, e^{i2\pi\xi s}\, d\xi \right]^* ds$$

Idealized systems



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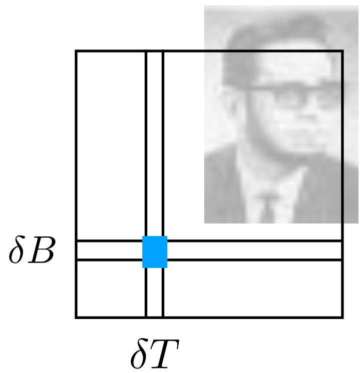
$$R_x(t,f) = x(t)\, X^*(f)\, e^{-i2\pi f t}$$

Rihaczek complex energy density function

Idealized systems



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Rihaczek complex energy density function

(a.k.a. Margenau-Hill distribution)

Margenau
1961

3 covariance principles

Energy distributions

$$\iint \frac{S_x^h(t,f)}{\|h\|^2}\,dt\,df = \|x\|^2$$

$$\iint R_x(t,f)\,dt\,df = \|x\|^2$$

$$\cdots$$

Energy distributions

$$\iint \frac{S_x^h(t,f)}{\|h\|^2}\, dt\, df = \|x\|^2$$

$$\iint R_x(t,f)\, dt\, df = \|x\|^2$$

$$\cdots$$

$$x(t) \xrightarrow{?} \rho_x(t,f) \text{ s.t. } \iint \rho_x(t,f)\, dt\, df = \|x\|^2$$

Energy distributions

$$\iint \frac{S_x^h(t,f)}{\|h\|^2}\,dt\,df = \|x\|^2$$

$$\iint R_x(t,f)\,dt\,df = \|x\|^2$$

$$\cdots$$

$$x(t) \xrightarrow{?} \rho_x(t,f) \text{ s.t. } \iint \rho_x(t,f)\,dt\,df = \|x\|^2$$

Sesquilinearity

$$\rho_x(t,f) = \iint K(s,s';t,f)\,x(s)\,x^*(s')\,ds\,ds'$$

with $\iint K(s,s';t,f)\,dt\,df = \delta(s-s')$ for energy

Going further

$$\begin{array}{ccc} x(t) & \longrightarrow & \rho_x(t,f) \\ \downarrow & & \downarrow \\ (\mathbf{G}x)(t) & \longrightarrow & \rho_{\mathbf{G}x}(t,f) = (\tilde{\mathbf{G}}\rho_x)(t,f) \end{array}$$

Going further

$$\begin{array}{ccc} x(t) & \longrightarrow & \rho_x(t,f) \\ \downarrow & & \downarrow \\ (\mathbf{G}x)(t) & \longrightarrow & \rho_{\mathbf{G}x}(t,f) = (\tilde{\mathbf{G}}\rho_x)(t,f) \end{array}$$

Covariance w.r.t. time-frequency shifts

$$\mathbf{G} = \mathbf{T}_{tf} \Rightarrow \exists K_0 \,|\, K(s,s';t,f) = K_0(s-t,s'-t)\, e^{-i2\pi f(s-s')}$$

$$\rho_x(t,f) = C_x(t,f;\varphi)$$

Cohen's class

Cohen 1966

$$C_x(t,f;\varphi) = \iiint \varphi(\xi,\tau)\, x\left(s+\frac{\tau}{2}\right)\, x^*\left(s-\frac{\tau}{2}\right)\, e^{i2\pi[\xi(s-t)-f\tau]}\, ds\, d\xi\, d\tau$$

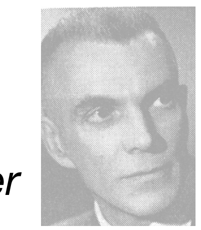
with $\varphi(\xi,\tau) := \int K_0\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-i2\pi\xi t}\, dt$

Special cases

$$\varphi(\xi, \tau) = 1 \longrightarrow W_x(t,f) = \int x\left(t + \frac{\tau}{2}\right)\, x^*\left(t - \frac{\tau}{2}\right)\, e^{-i2\pi f\tau}\, d\tau$$

Wigner(-Ville) distribution

Wigner
1932



Ville
1948

Special cases

$$\varphi(\xi, \tau) = 1 \longrightarrow W_x(t, f) = \int x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-i2\pi f\tau}\, d\tau$$

Wigner(-Ville) distribution

Wigner
1932

« *This expression was found by L. Szilard and the present author some years ago for another purpose*. »

Special cases

$$\varphi(\xi, \tau) = 1 \longrightarrow W_x(t, f) = \int x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-i2\pi f \tau} \, d\tau$$

$$\varphi(\xi, \tau) = e^{i\pi\xi\tau} \longrightarrow R_x(t, f)$$

Special cases

$$\varphi(\xi,\tau) = 1 \longrightarrow W_x(t,f) = \int x\left(t+\frac{\tau}{2}\right)\, x^*\left(t-\frac{\tau}{2}\right)\, e^{-i2\pi f\tau}\, d\tau$$

$$\varphi(\xi,\tau) = e^{i\pi\xi\tau} \longrightarrow R_x(t,f)$$

$$\varphi(\xi,\tau) = \iint W_h(t,f)\, e^{i2\pi(\xi t+\tau f)}\, dt\, df \longrightarrow S_x^h(t,f)$$

$$\ldots$$

Special cases

$$\varphi(\xi,\tau) = 1 \longrightarrow W_x(t,f) = \int x\left(t+\frac{\tau}{2}\right)\, x^*\left(t-\frac{\tau}{2}\right)\, e^{-i2\pi f\tau}\, d\tau$$

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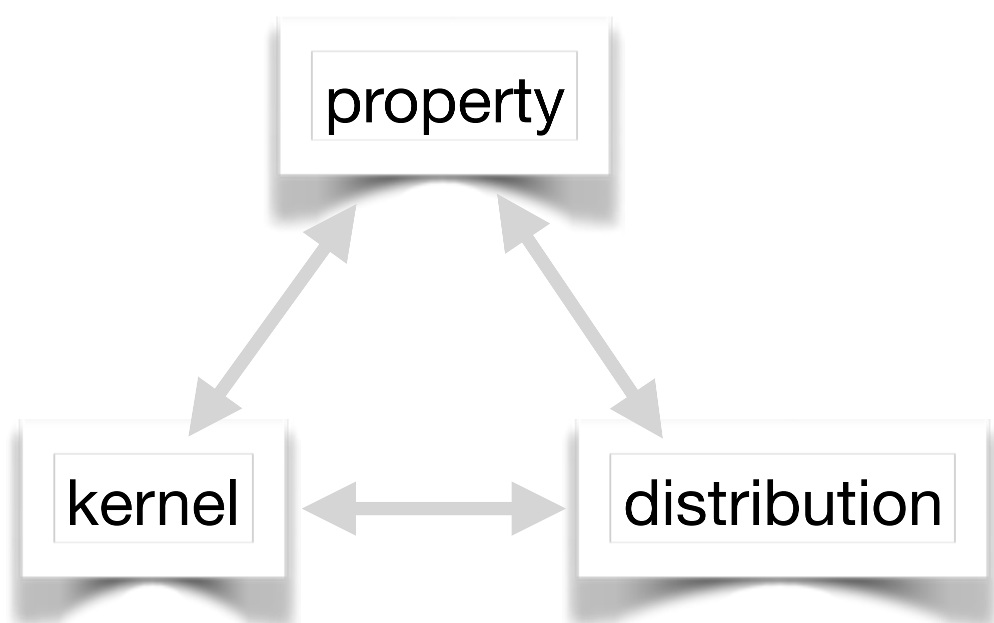
$$\dots$$

Kernel / distribution design from constraints



Special cases

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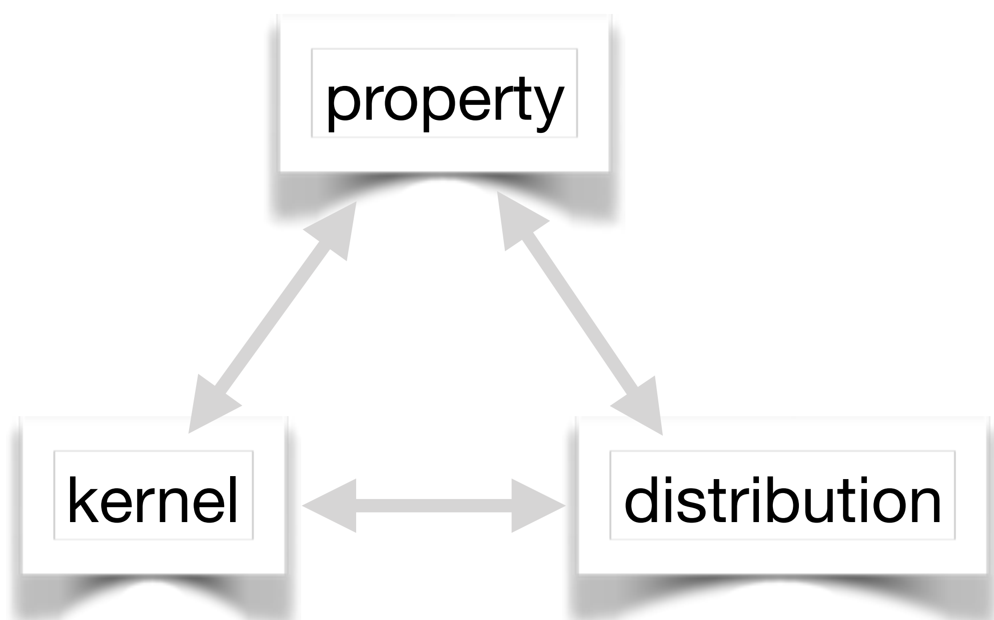
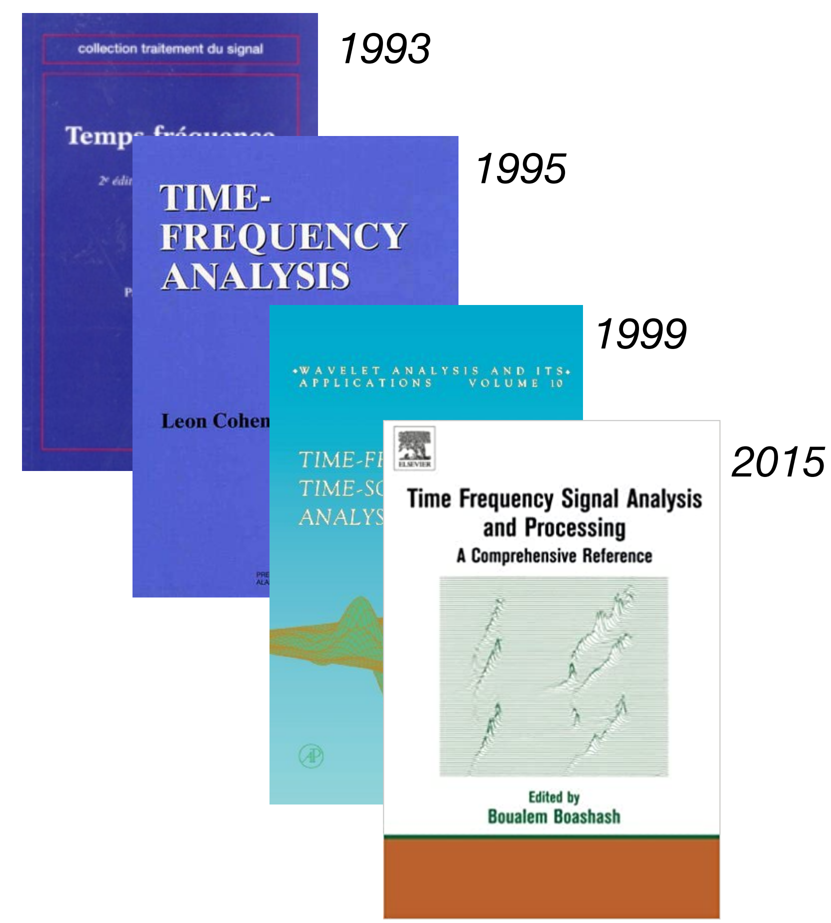
$\ldots$

Kernel / distribution design from constraints





4 correlations

Correlation as inner product in the time domain

$$\gamma_x(\tau) = \mathbb{E}\{x(t)\,(\mathbf{T}_\tau x)^*(t)\}$$

Power spectrum as 1D Fourier transform

stochastic

deterministic

$$\Gamma_x(f) \quad = \quad \int \gamma_x(\tau)\, e^{-i2\pi f\tau}\, d\tau$$

estimation $\downarrow$

$$\hat{\Gamma}_x(f) \quad = \quad \int w(\tau)\, \langle x, \mathbf{T}_\tau x\rangle\, e^{-i2\pi f\tau}\, d\tau$$

Correlation as inner product in the time domain

$$\gamma_x(\tau) = \mathbb{E}\{x(t)\,(\mathbf{T}_\tau x)^*(t)\}$$

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From time to time-frequency

$$\langle x, \mathbf{T}_\tau x\rangle \longrightarrow \langle x, \mathbf{T}_{\tau/2}\mathbf{T}_\xi \mathbf{T}_{\tau/2}\rangle = \int x\left(t+\frac{\tau}{2}\right)\, x^*\left(t-\frac{\tau}{2}\right)\, e^{i2\pi\xi t}\, dt =: A_x(\xi,\tau)$$

ambiguity function

$$\hat{\Gamma}_x(f) \longrightarrow \rho_x(t,f) = \iint \varphi(\xi,\tau)\, A_x(\xi,\tau)\, e^{i2\pi[t\xi+f\tau]}\, d\xi\, d\tau = C_x(t,f;\varphi)$$

Cohen’s class

Correlation as inner product in the time domain

$$\gamma_x(\tau) = \mathbb{E}\{x(t)\,(\mathbf{T}_\tau x)^*(t)\}$$

stochastic

Power spectrum as 1D Fourier transform

$$\Gamma_x(f) \quad = \quad \int \gamma_x(\tau)\, e^{-i2\pi f\tau}\, d\tau$$

deterministic

estimation $\downarrow$

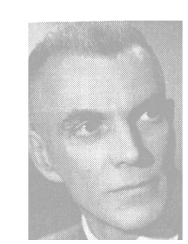
$$\hat{\Gamma}_x(f) \quad = \quad \int w(\tau)\, \langle x, \mathbf{T}_\tau x\rangle\, e^{-i2\pi f\tau}\, d\tau$$

From time to time-frequency

$$\langle x, \mathbf{T}_\tau x\rangle \;\longrightarrow\; \langle x, \mathbf{T}_{\tau/2}\mathbf{T}_\xi\mathbf{T}_{\tau/2}\rangle = \int x\left(t+\frac{\tau}{2}\right)\, x^*\left(t-\frac{\tau}{2}\right)\, e^{i2\pi\xi t}\, dt =: A_x(\xi,\tau)$$

ambiguity function

no weight $\longrightarrow$ $\rho_x(t,f) = \iint A_x(\xi,\tau)\, e^{i2\pi[t\xi + f\tau]}\, d\xi\, d\tau = W_x(t,f)$



Ville 1948

5 probability

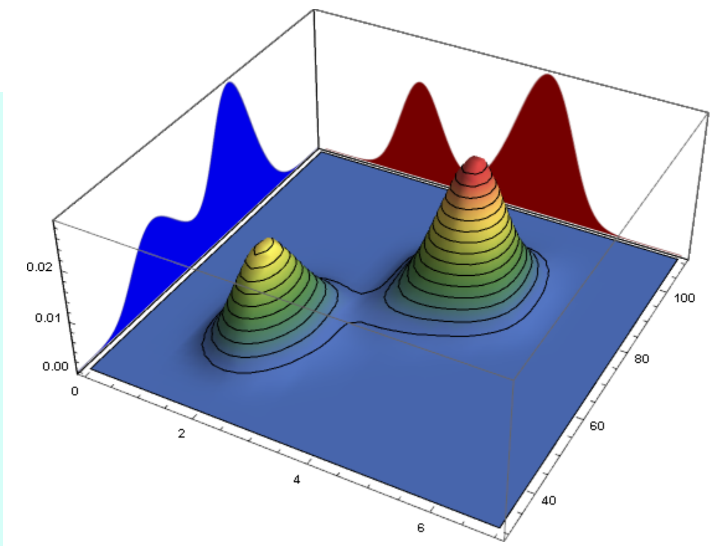
Quasi-probability distribution functions

$$\int |x(t)|^2 \, dt = 1 = \int |X(f)|^2 \, df$$

$$||$$

$$\iint \rho_x(t,f) \, dt \, df$$

$\rho_x(t)$ $\qquad$ $\rho_X(f)$



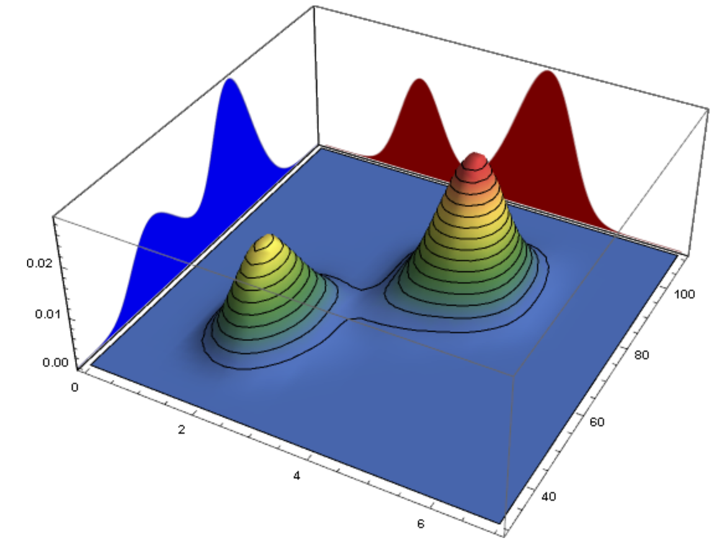
Marginals $\int \rho_x(t,f) \, dt = \rho_X(f) \quad ; \quad \int \rho_x(t,f) \, df = \rho_x(t)$

Quasi-probability distribution functions

$$\int |x(t)|^2 \, dt = 1 = \int |X(f)|^2 \, df$$

$$= \iint \rho_x(t,f) \, dt \, df$$

$$\rho_x(t) \qquad \qquad \rho_X(f)$$



Marginals

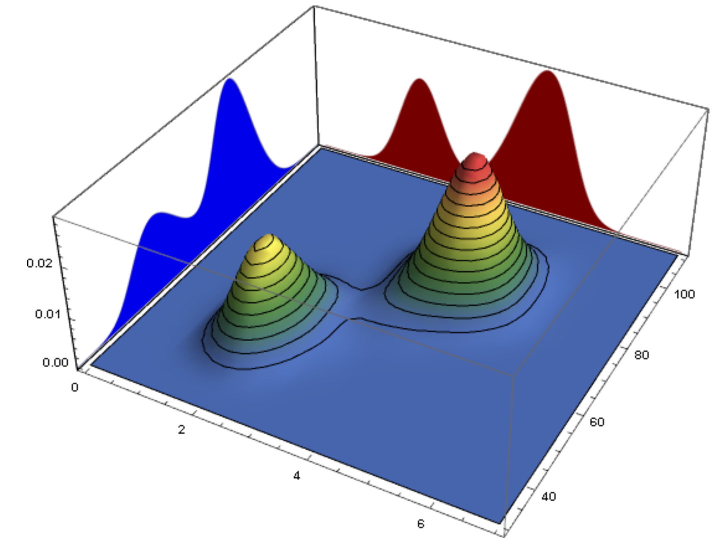
$$\int \rho_x(t,f) \, dt = \rho_X(f) \quad ; \quad \int \rho_x(t,f) \, df = \rho_x(t)$$

Bayes

$$\rho_x(t,f) = \rho_x(t|f) \, \rho_X(f) = \rho_X(f|t) \, \rho_x(t)$$

Bayes 1763

Quasi-probability distribution functions



$$\int |x(t)|^2 \, dt = 1 = \int |X(f)|^2 \, df$$

$$\| $$

$$\iint \rho_x(t,f) \, dt \, df$$

$$\rho_x(t) \qquad \qquad \rho_X(f)$$

Marginals $$\int \rho_x(t,f) \, dt = \rho_X(f) \quad ; \quad \int \rho_x(t,f) \, df = \rho_x(t)$$

Bayes $$\rho_x(t,f) = \rho_x(t|f) \, \rho_X(f) = \rho_X(f|t) \, \rho_x(t)$$

Bayes 1763

1st order local moments

$$\int f \, \rho_X(f|t) \, df = \frac{1}{\rho_x(t)} \int f \, \rho_x(t,f) \, df \to f_x(t) := \frac{1}{2\pi} \frac{d}{dt} \arg x(t)$$

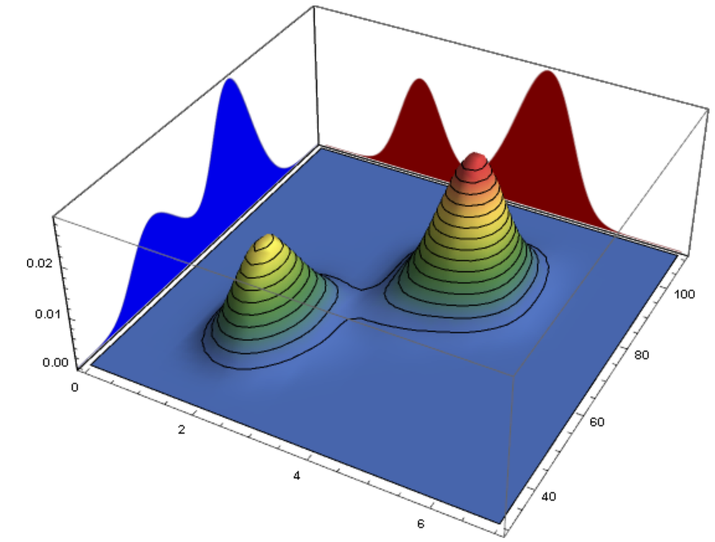
instantaneous frequency

Quasi-probability distribution functions

$$\int |x(t)|^2 \, dt = 1 = \int |X(f)|^2 \, df$$

$$\| \qquad \iint \rho_x(t,f) \, dt \, df$$

$$\rho_x(t) \qquad \qquad \rho_X(f)$$



Marginals $\int \rho_x(t,f) \, dt = \rho_X(f) \quad ; \quad \int \rho_x(t,f) \, df = \rho_x(t)$

Bayes $\rho_x(t,f) = \rho_x(t|f) \, \rho_X(f) = \rho_X(f|t) \, \rho_x(t)$

Bayes 1763

1st order local moments

$$\int f \, \rho_X(f|t) \, df = \frac{1}{\rho_x(t)} \int f \, \rho_x(t,f) \, df \to f_x(t) := \frac{1}{2\pi} \frac{d}{dt} \arg x(t)$$

instantaneous frequency

Cohen's class conditions

$$\varphi(\xi,0) = \varphi(0,\tau) = 1 \, ; \; \frac{\partial \varphi}{\partial \xi}(0,\tau) = \frac{\partial \varphi}{\partial \tau}(\xi,0) = 0$$

Wigner ✓
spectro ✗
Rihaczek ✓

6 quantum operators

Back to analogy

$$(t, f) \longleftrightarrow (q, p)$$

time-frequency position-momentum

Quantization

$$G(t, f) \longrightarrow \mathbf{G}(\mathbf{T}, \mathbf{F})$$

with $(\mathbf{T}x)(t) = t\, x(t) \quad ; \quad (\mathbf{F}x)(t) = \dfrac{1}{i2\pi}\, \dfrac{dx}{dt}(t)$

Back to analogy

$$(t, f) \longleftrightarrow (q, p)$$

time-frequency position-momentum

Quantization

$$G(t, f) \longrightarrow \mathbf{G}(\mathbf{T}, \mathbf{F})$$

with $(\mathbf{T}x)(t) = t\, x(t)$; $(\mathbf{F}x)(t) = \dfrac{1}{i2\pi} \dfrac{dx}{dt}(t)$

$$[\mathbf{T}, \mathbf{F}] := \mathbf{T}\mathbf{F} - \mathbf{F}\mathbf{T} = \frac{i}{2\pi}\, \mathbf{I}$$

non-commuting operators

Back to analogy

$$(t, f) \longleftrightarrow (q, p)$$

time-frequency position-momentum

Quantization

$$G(t, f) \longrightarrow \mathbf{G}(\mathbf{T}, \mathbf{F})$$

with $(\mathbf{T}x)(t) = t\, x(t)$; $(\mathbf{F}x)(t) = \dfrac{1}{i2\pi}\, \dfrac{dx}{dt}(t)$

$$[\mathbf{T}, \mathbf{F}] := \mathbf{TF} - \mathbf{FT} = \tfrac{i}{2\pi}\, \mathbf{I}$$

Observables

$$\langle \mathbf{G} \rangle_x := \int (\mathbf{G}x)(t)\, x^*(t)\, dt = \iint G(t, f)\, \rho_x(t, f)\, dt\, df$$

« quantum » « classical »

Analogy

$$(t, f) \longleftrightarrow (q, p)$$

time-frequency position-momentum

Quantization

$$G(t, f) \longrightarrow \mathbf{G}(\mathbf{T}, \mathbf{F})$$

with $(\mathbf{T}x)(t) = t\, x(t)$; $(\mathbf{F}x)(t) = \dfrac{1}{i2\pi} \dfrac{dx}{dt}(t)$

$$[\mathbf{T}, \mathbf{F}] := \mathbf{TF} - \mathbf{FT} = \tfrac{i}{2\pi}\, \mathbf{I}$$

Observables

$$\langle \mathbf{G} \rangle_x := \int (\mathbf{G}x)(t)\, x^*(t)\, dt = \iint G(t, f)\, \rho_x(t, f)\, dt\, df$$

« quantum » « classical »

no unicity

« correspondence rule » ⟷ « quasi-probability distribution »

Correspondence rules within Cohen's class

$$G(t,f) \longrightarrow \mathbf{G}_{\varphi}(\mathbf{T},\mathbf{F}) = \iint \varphi(\xi,\tau)\, g(\xi,\tau)\, e^{i2\pi(\xi\mathbf{T}+\tau\mathbf{F})}\, dt\, df$$

$$\iint G(t,f)\, e^{-i2\pi(\xi t+\tau f)}\, dt\, df$$

Correspondence rules within Cohen's class

$$G(t,f) \longrightarrow \mathbf{G}_\varphi(\mathbf{T},\mathbf{F}) = \iint \varphi(\xi,\tau)\, g(\xi,\tau)\, e^{i2\pi(\xi\mathbf{T}+\tau\mathbf{F})}\, dt\, df$$

$$\iint G(t,f)\, e^{-i2\pi(\xi t+\tau f)}\, dt\, df$$

Moments

$$t^k f^l \longrightarrow \begin{cases} 2^{-k} \sum_{m=0}^{k} \binom{k}{m} \mathbf{T}^{k-m}\mathbf{F}^l\mathbf{T}^m & \text{if } \varphi(\xi,\tau) = 1 \quad \text{Wigner} \\ \frac{1}{2}\left(\mathbf{T}^k\mathbf{F}^l + \mathbf{F}^l\mathbf{T}^k\right) & \text{if } \varphi(\xi,\tau) = \cos\pi\xi\tau \quad \text{Re\{Rihaczek\}} \\ -\frac{i2\pi}{(k+1)(l+1)}\left[\mathbf{T}^{k+1},\mathbf{F}^{l+1}\right] & \text{if } \varphi(\xi,\tau) = \frac{\sin\pi\xi\tau}{\pi\xi\tau} \quad \text{Born-Jordan} \end{cases}$$

Correspondence rules within Cohen's class

$$G(t,f) \longrightarrow \mathbf{G}_\varphi(\mathbf{T},\mathbf{F}) = \iint \varphi(\xi,\tau)\, g(\xi,\tau)\, e^{i2\pi(\xi\mathbf{T}+\tau\mathbf{F})}\, dt\, df$$

$$\iint G(t,f)\, e^{-i2\pi(\xi t+\tau f)}\, dt\, df$$

Moments

$$t^k f^l \longrightarrow \left| \begin{array}{lll} 2^{-k} \displaystyle\sum_{m=0}^{k} \binom{k}{m} \mathbf{T}^{k-m}\mathbf{F}^l\mathbf{T}^m & \text{if } \varphi(\xi,\tau)=1 & \text{Wigner} \\ \dfrac{1}{2}\left(\mathbf{T}^k\mathbf{F}^l + \mathbf{F}^l\mathbf{T}^k\right) & \text{if } \varphi(\xi,\tau)=\cos\pi\xi\tau & \text{Re\{Rihaczek\}} \\ -\dfrac{i2\pi}{(k+1)(l+1)}\left[\mathbf{T}^{k+1},\mathbf{F}^{l+1}\right] & \text{if } \varphi(\xi,\tau)=\dfrac{\sin\pi\xi\tau}{\pi\xi\tau} & \text{Born-Jordan} \end{array} \right.$$

Kernels

$$(\mathbf{G}_\varphi x)(t) = \int \gamma_\varphi(t,s)\, x(s)\, ds$$

$$\varphi(\xi,\tau)=1 \Rightarrow \gamma_\varphi(t,s) = \gamma\left(\frac{t+s}{2}, t-s\right)$$ with $$\gamma(t,\tau) = \int G(t,f)e^{-i2\pi f\tau} df$$

Weyl quantization

Weyl 1928

7
geometry

Unitarity

$$\iint \rho_x(t,f)\,\rho_y^*(t,f)\,dt\,df = \left|\int x(t)\,y^*(t)\,dt\right|^2$$

Moyal
1949

$$\Downarrow$$

$|\varphi(\tau,\xi)| = 1$ within Cohen's class

Unitarity

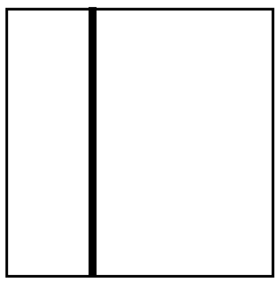
$$\iint \rho_x(t,f)\, \rho_y^*(t,f)\, dt\, df = \left| \int x(t)\, y^*(t)\, dt \right|^2$$

Moyal
1949

$$\Downarrow$$

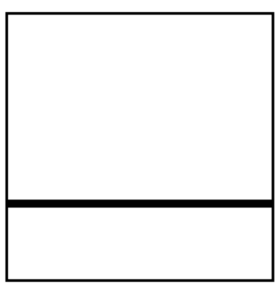
$|\varphi(\tau,\xi)| = 1$ within Cohen's class

Generalized marginals



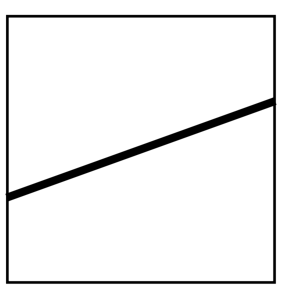
$$\iint \rho_x(t,f)\, \delta(t-t_0)\, dt\, df = \left| \int x(t)\, \delta(t-t_0)\, dt \right|^2 = |x(t_0)|^2$$

pulse



$$\iint \rho_x(t,f)\, \delta(f-f_0)\, dt\, df = \left| \int x(t)\, e^{-i2\pi f_0 t}\, dt \right|^2 = |X(f_0)|^2$$

tone



$$\iint \rho_x(t,f)\, \delta(f-(f_0+\beta t))\, dt\, df = \left| \int x(t)\, e^{-i2\pi(f_0 t + \beta t^2/2)}\, dt \right|^2$$

linear « chirp »

Unitarity

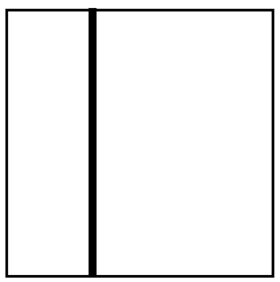
$$\iint \rho_x(t,f)\,\rho_y^*(t,f)\,dt\,df = \left|\int x(t)\,y^*(t)\,dt\right|^2$$

Moyal 1949

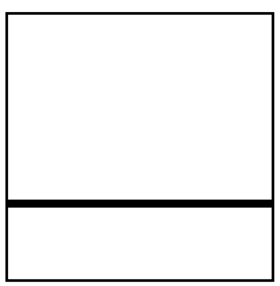
$$\Downarrow$$

$$|\varphi(\tau,\xi)| = 1 \quad \text{within Cohen's class}$$

Generalized marginals



$$\iint \rho_x(t,f)\,\delta(t-t_0)\,dt\,df = \left|\int x(t)\,\delta(t-t_0)\,dt\right|^2 = |x(t_0)|^2$$



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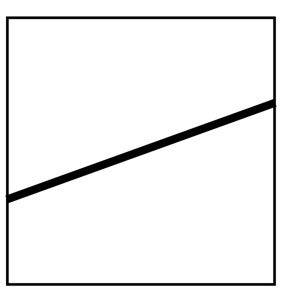
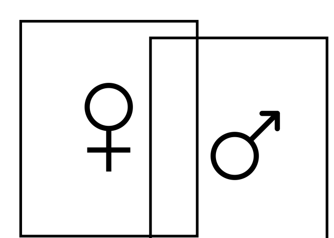


$$\iint \rho_x(t,f)\,\delta(f-(f_0+\beta t))\,dt\,df = \left|\int x(t)\,e^{-i2\pi(f_0 t+\beta t^2/2)}\,dt\right|^2$$

Radon transform

$$\Downarrow$$

Wigner



Bertrand's 1983

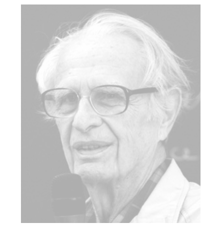
Displacement $\mathbf{D}_{t,f} = e^{i2\pi(f\mathbf{T} - t\mathbf{F})}$

Parity $(\mathbf{\Pi}x)(t) = x(-t) \quad ; \quad (\mathbf{\Pi}X)(f) = X(-f)$

$$\downarrow$$

Symmetry

$$W_x(t,f) = 2\langle \mathbf{D}_{t,f}\,\mathbf{\Pi}\,\mathbf{D}_{-t,-f}\rangle_x$$



Grossmann 1976

Royer 1977

Displacement $\mathbf{D}_{t,f} = e^{i2\pi(f\mathbf{T} - t\mathbf{F})}$

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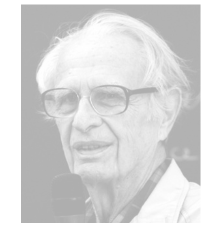
Symmetry $W_x(t,f) = 2\langle \mathbf{D}_{t,f}\,\mathbf{\Pi}\,\mathbf{D}_{-t,-f}\rangle_x$



Grossmann 1976

Royer 1977

A companion perspective

$$|W_x(t,f)|^2 = \iint W_x\left(t + \frac{\tau}{2}, f + \frac{\xi}{2}\right) W_x\left(t - \frac{\tau}{2}, f - \frac{\xi}{2}\right) d\tau\, d\xi$$

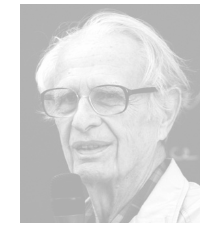
Janssen 1981

Displacement $\mathbf{D}_{t,f} = e^{i2\pi(f\mathbf{T}-t\mathbf{F})}$

Parity $(\mathbf{\Pi}x)(t) = x(-t) \quad ; \quad (\mathbf{\Pi}X)(f) = X(-f)$

$$\downarrow$$

Symmetry $W_x(t,f) = 2\langle \mathbf{D}_{t,f}\,\mathbf{\Pi}\,\mathbf{D}_{-t,-f}\rangle_x$



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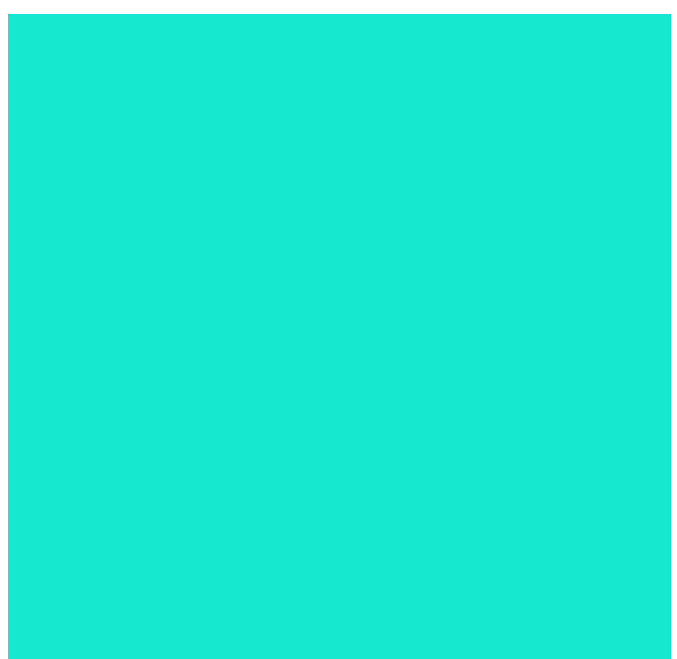


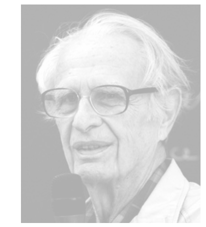
Janssen 1981

Displacement $\mathbf{D}_{t,f} = e^{i2\pi(f\mathbf{T} - t\mathbf{F})}$

Parity $(\mathbf{\Pi}x)(t) = x(-t) \quad ; \quad (\mathbf{\Pi}X)(f) = X(-f)$

$$\downarrow$$

Symmetry $W_x(t,f) = 2\langle \mathbf{D}_{t,f}\, \mathbf{\Pi}\, \mathbf{D}_{-t,-f} \rangle_x$



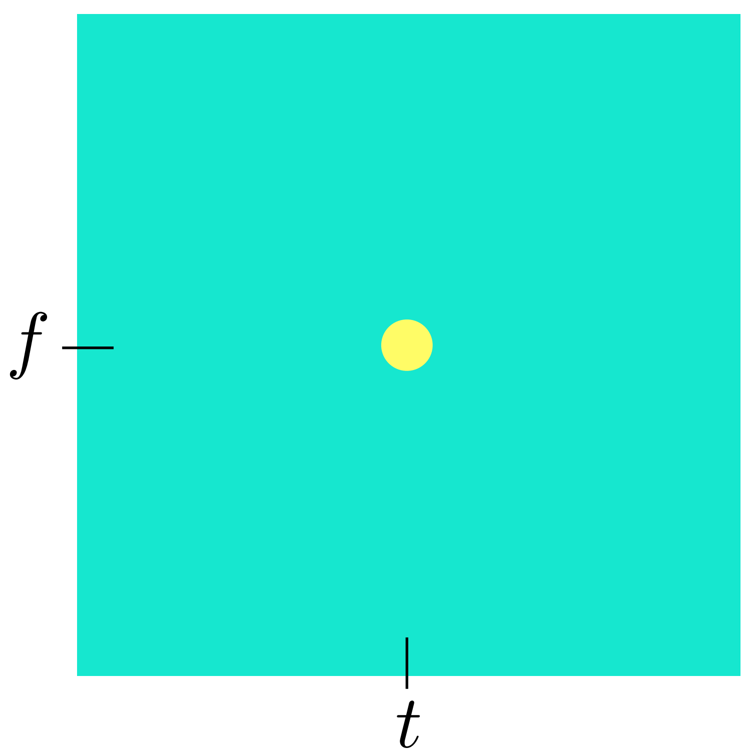
Grossmann 1976

Royer 1977

A companion perspective

$$|W_x(t,f)|^2 = \iint W_x\left(t + \frac{\tau}{2}, f + \frac{\xi}{2}\right) W_x\left(t - \frac{\tau}{2}, f - \frac{\xi}{2}\right) d\tau\, d\xi$$

Janssen 1981



Displacement $\mathbf{D}_{t,f} = e^{i2\pi(f\mathbf{T} - t\mathbf{F})}$

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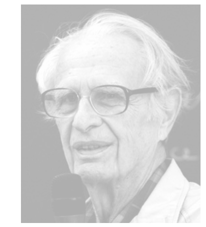

Grossmann 1976

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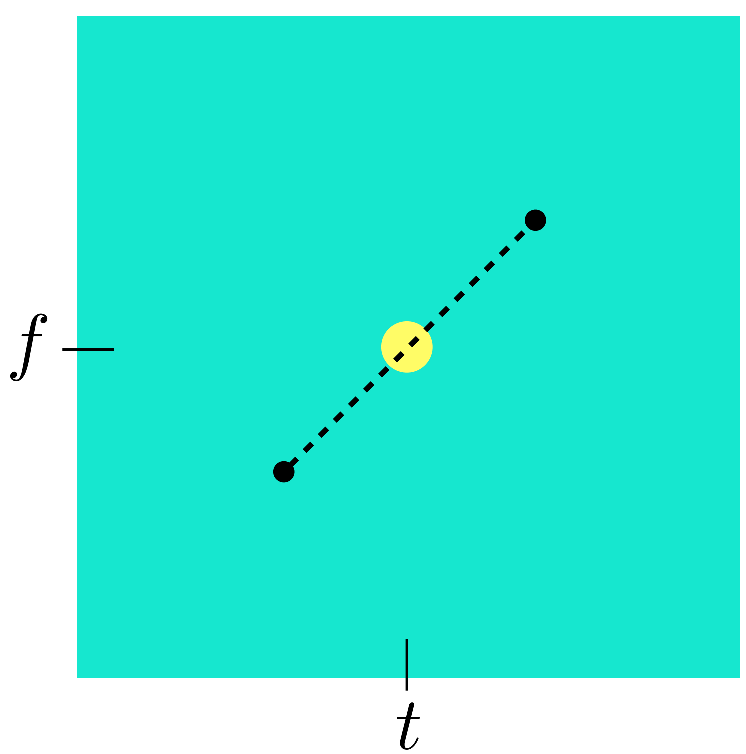

Janssen 1981

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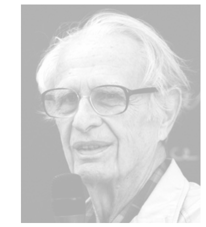

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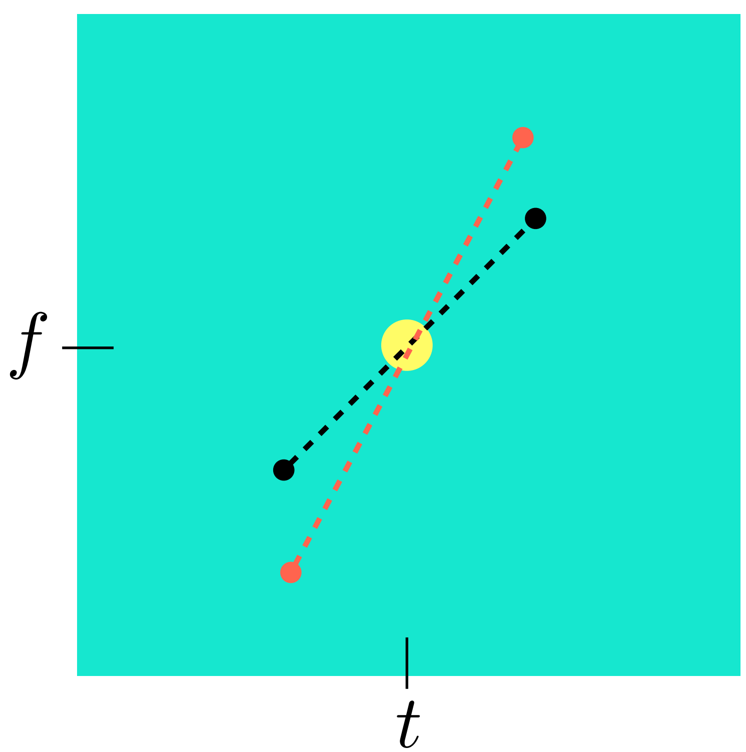

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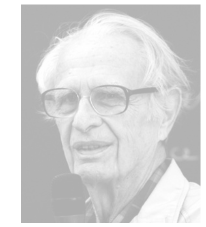

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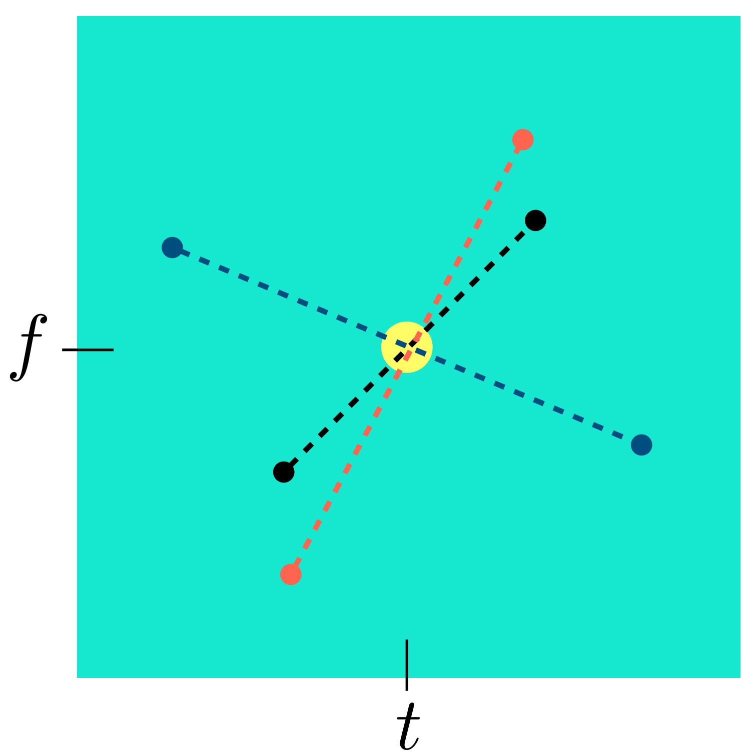

Janssen 1981

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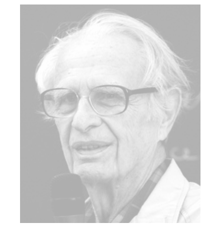

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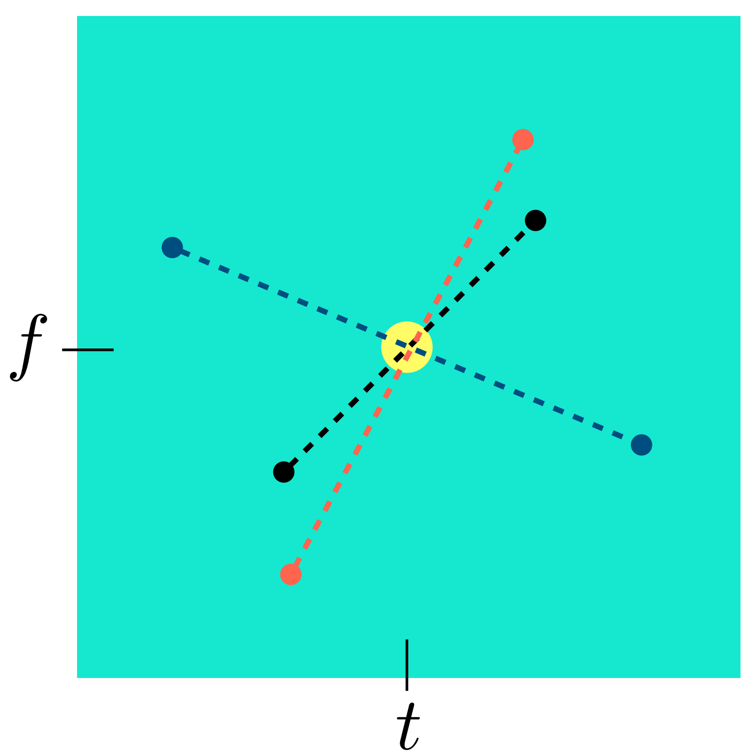

Janssen 1981

- central symmetry
- « mid-point rule »
- localization on straight lines

Back to sesquilinearity

$$\text{“}\,(a+b)^2 = a^2 + b^2 + 2ab\,\text{”}$$

Back to sesquilinearity “ $(a+b)^2 = a^2 + b^2 + 2ab$ ”

$$W_{ax+by}(t,f) = |a|^2\, W_x(t,f) + |b|^2\, W_y(t,f) + 2\,\mathrm{Re}\{W_{x,y}(t,f)\}$$

with $$W_{x,y}(t,f) = \int x\left(t+\frac{\tau}{2}\right)\, y^*\left(t-\frac{\tau}{2}\right)\, e^{-i2\pi f\tau}\, d\tau$$

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« non local » cross-term

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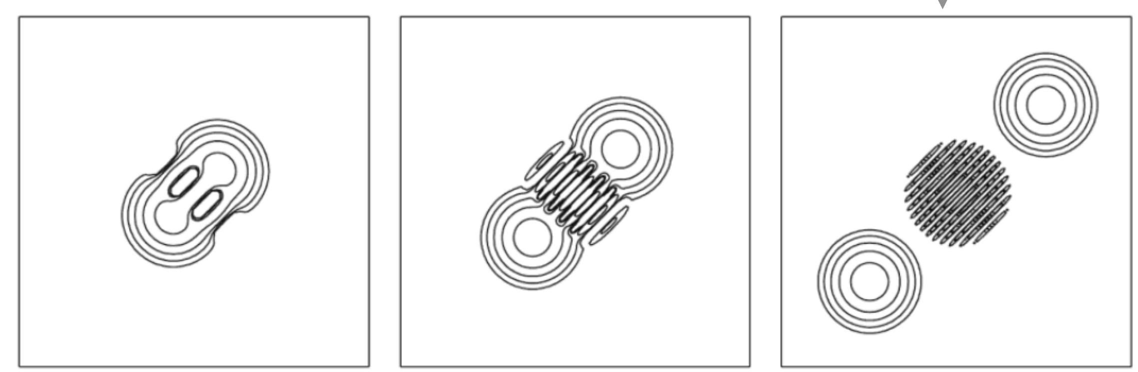

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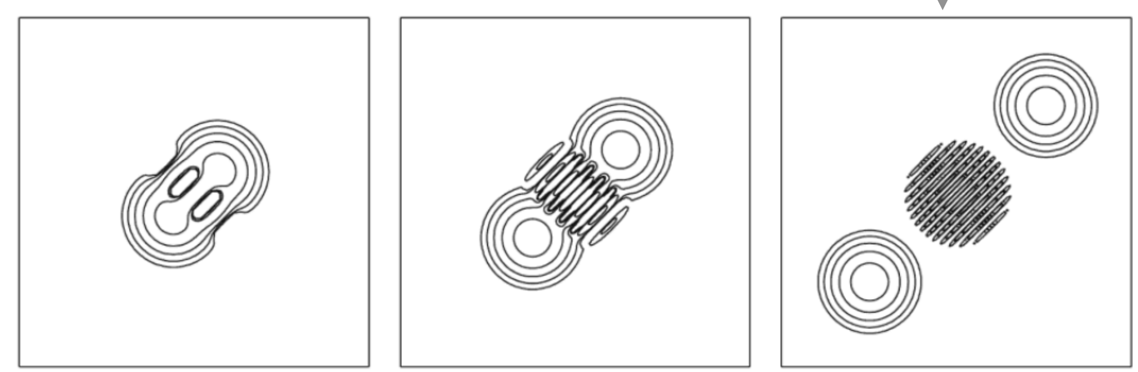

From Wigner to Cohen’s class

- similar superposition principle
- « local » cross-terms in the spectrogram case

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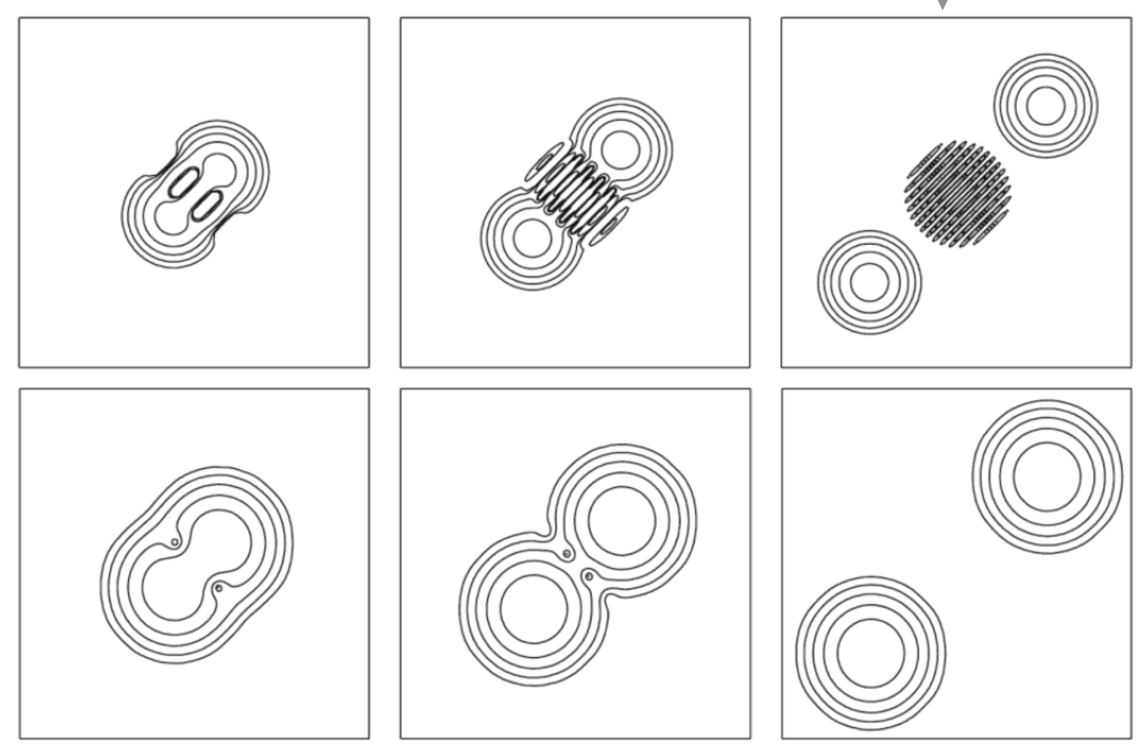

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$$S_{ax+by}(t,f) = |a|^2\, S_x(t,f) + |b|^2\, S_y(t,f) + 2\,\mathrm{Re}\{F_x(t,f)F_y^*(t,f)\}$$

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$$W_{ax+by}(t,f) = |a|^2\, W_x(t,f) + |b|^2\, W_y(t,f) + 2\,\mathrm{Re}\{W_{x,y}(t,f)\}$$

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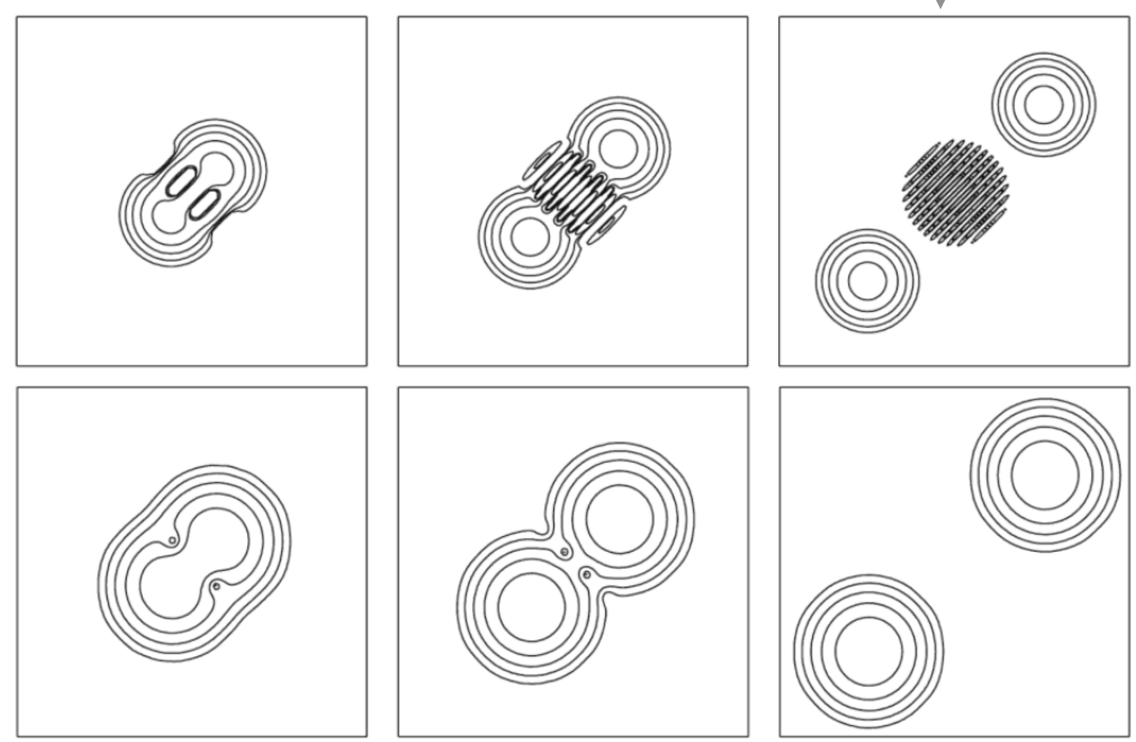
Cross-terms pros and cons

- impair readability
- reveal coherences



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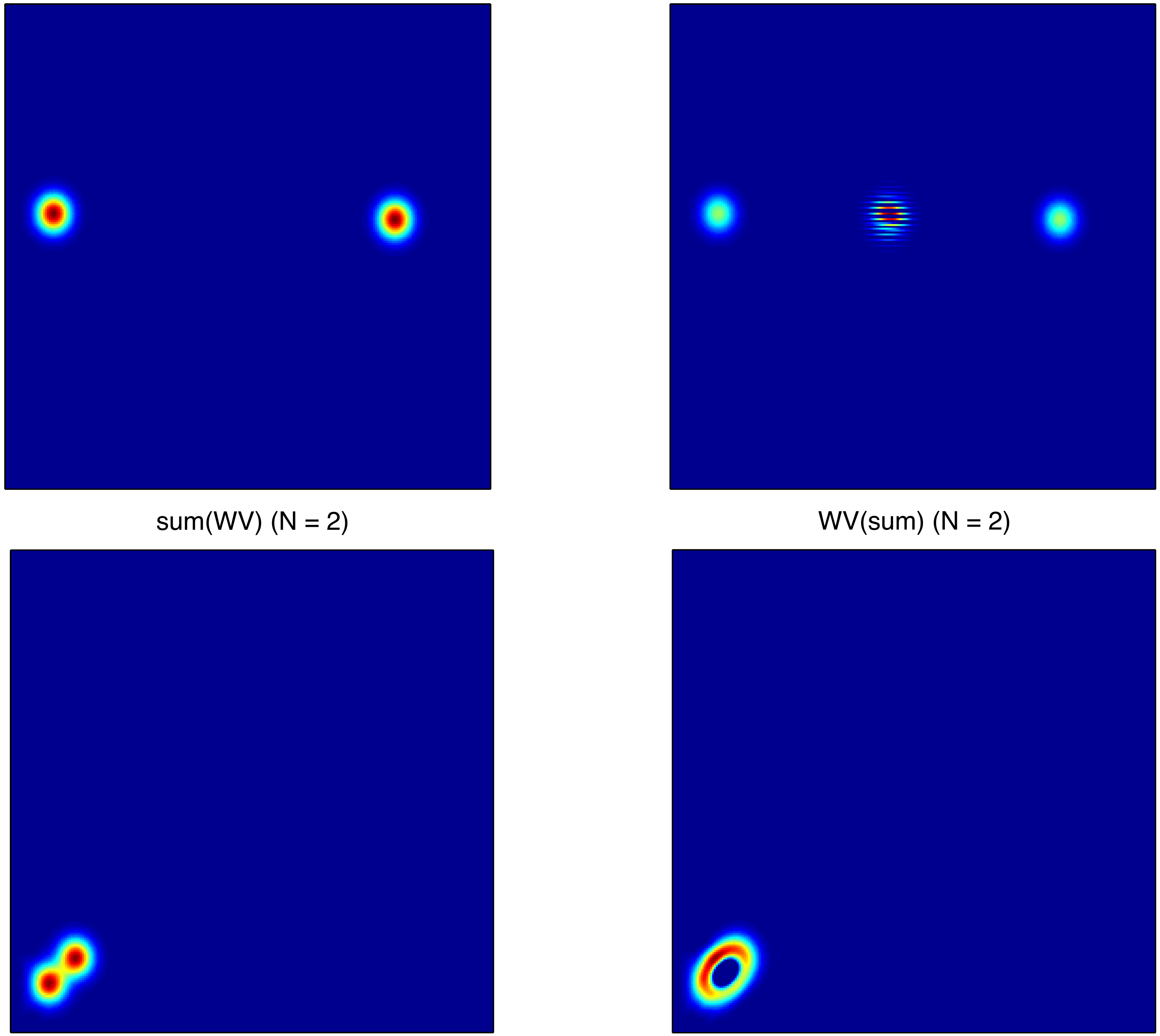
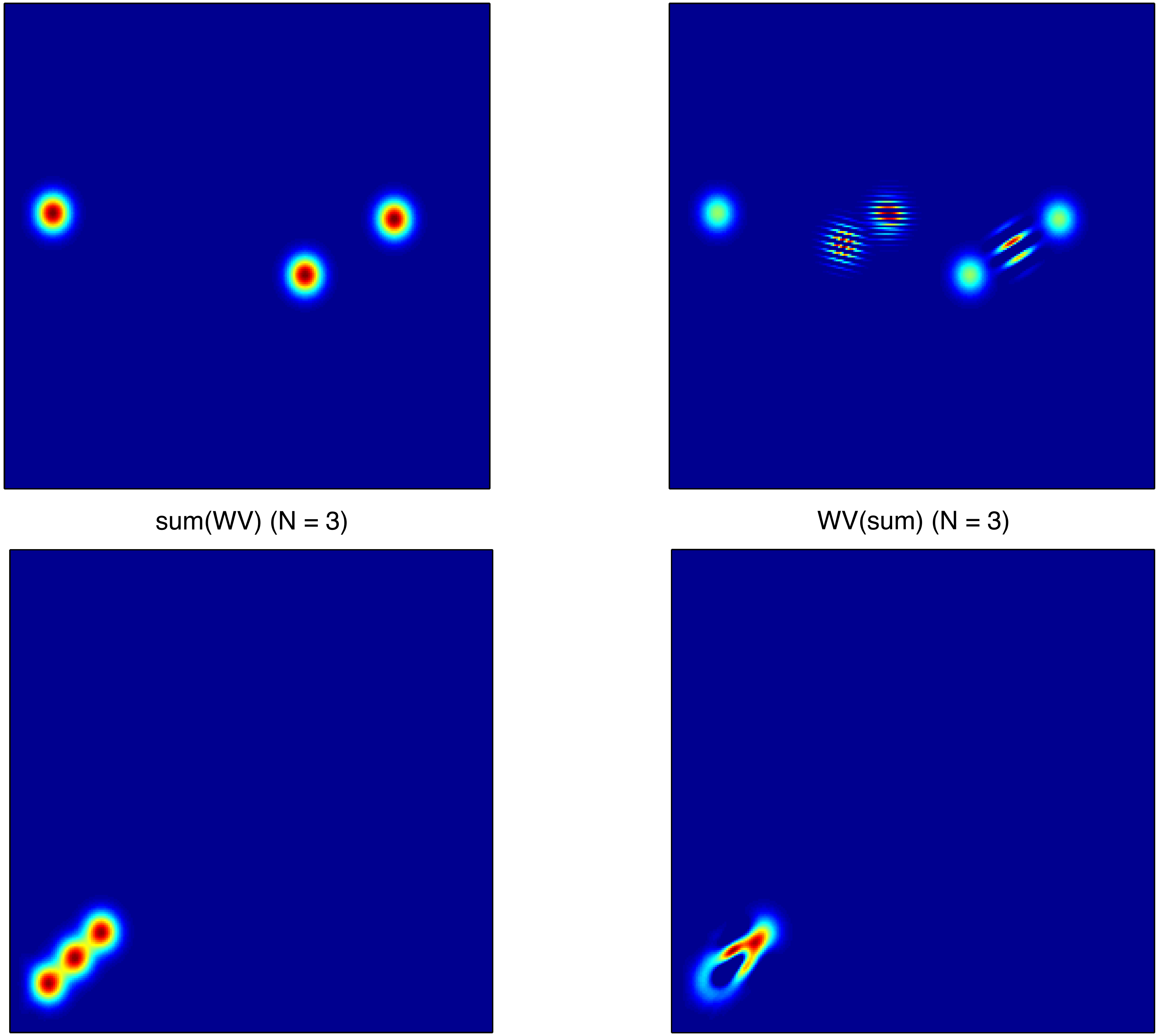
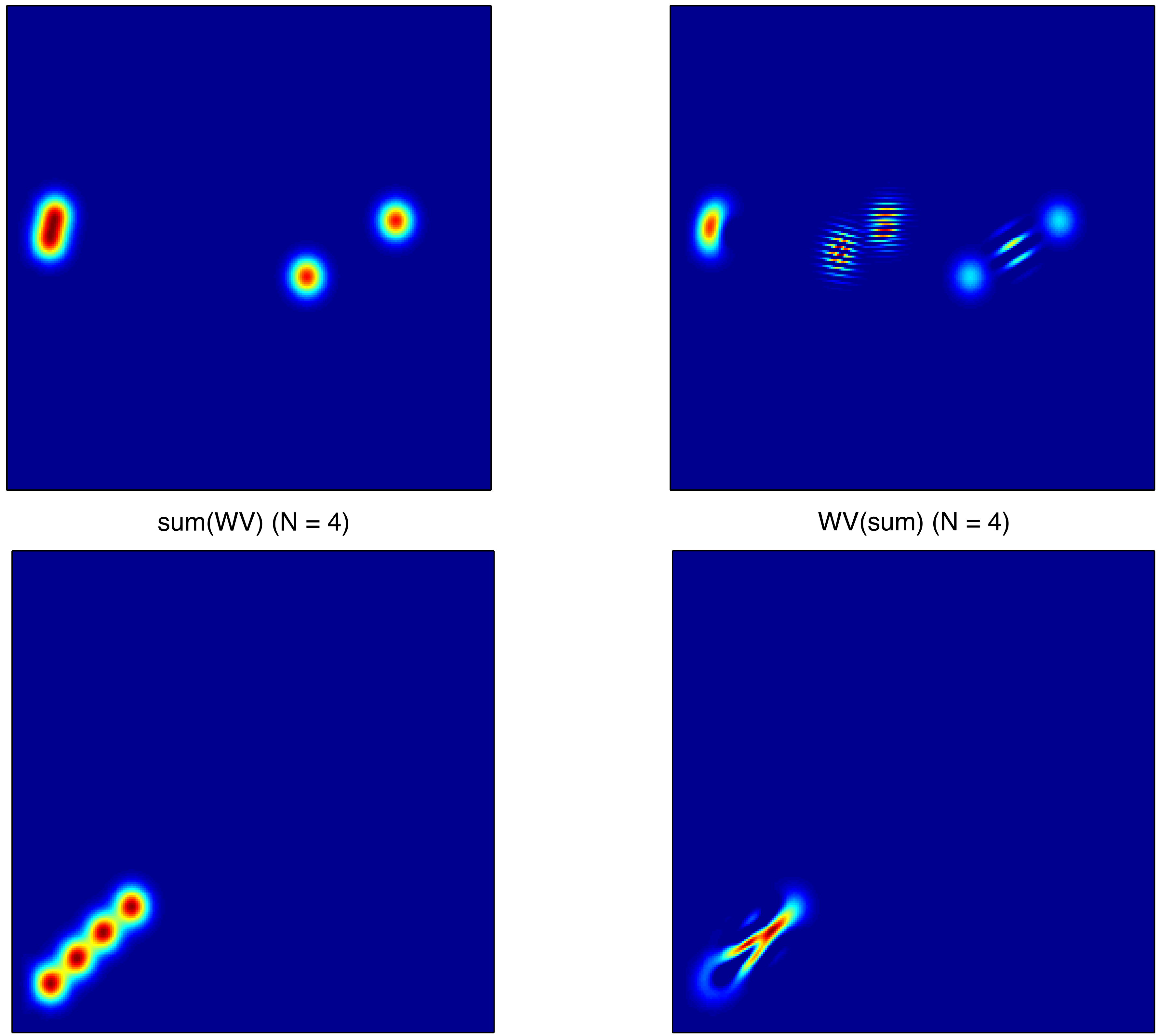
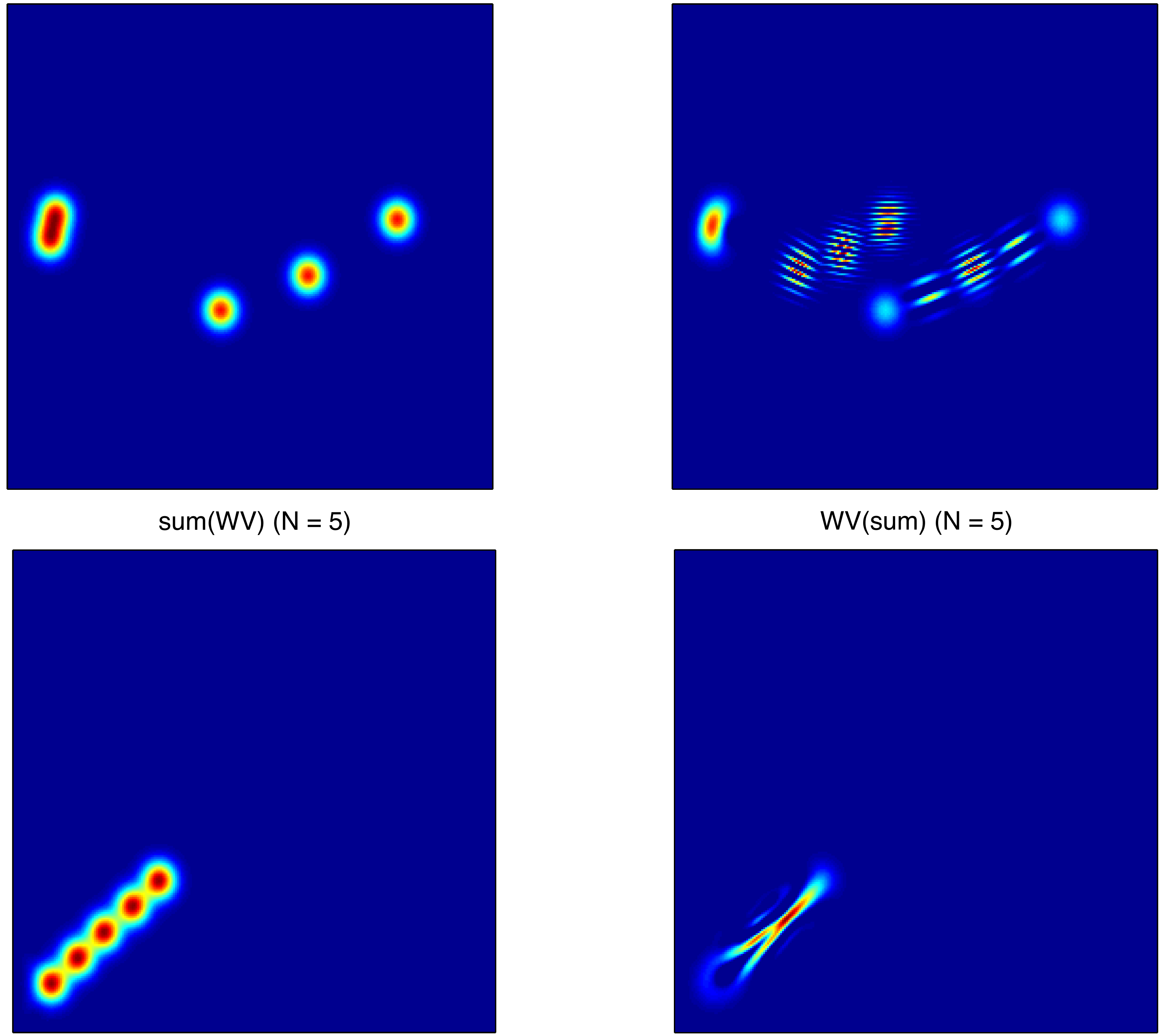
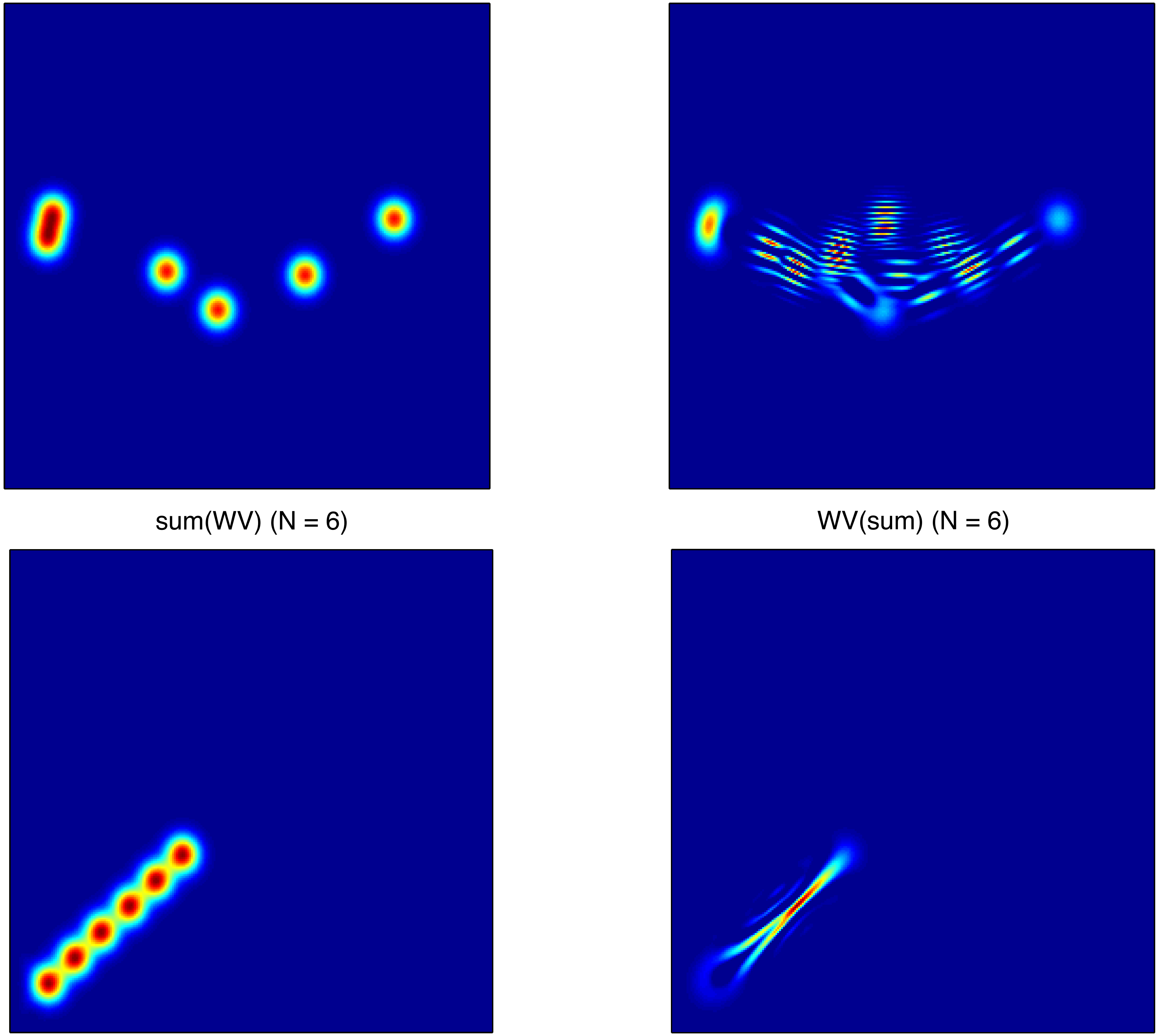
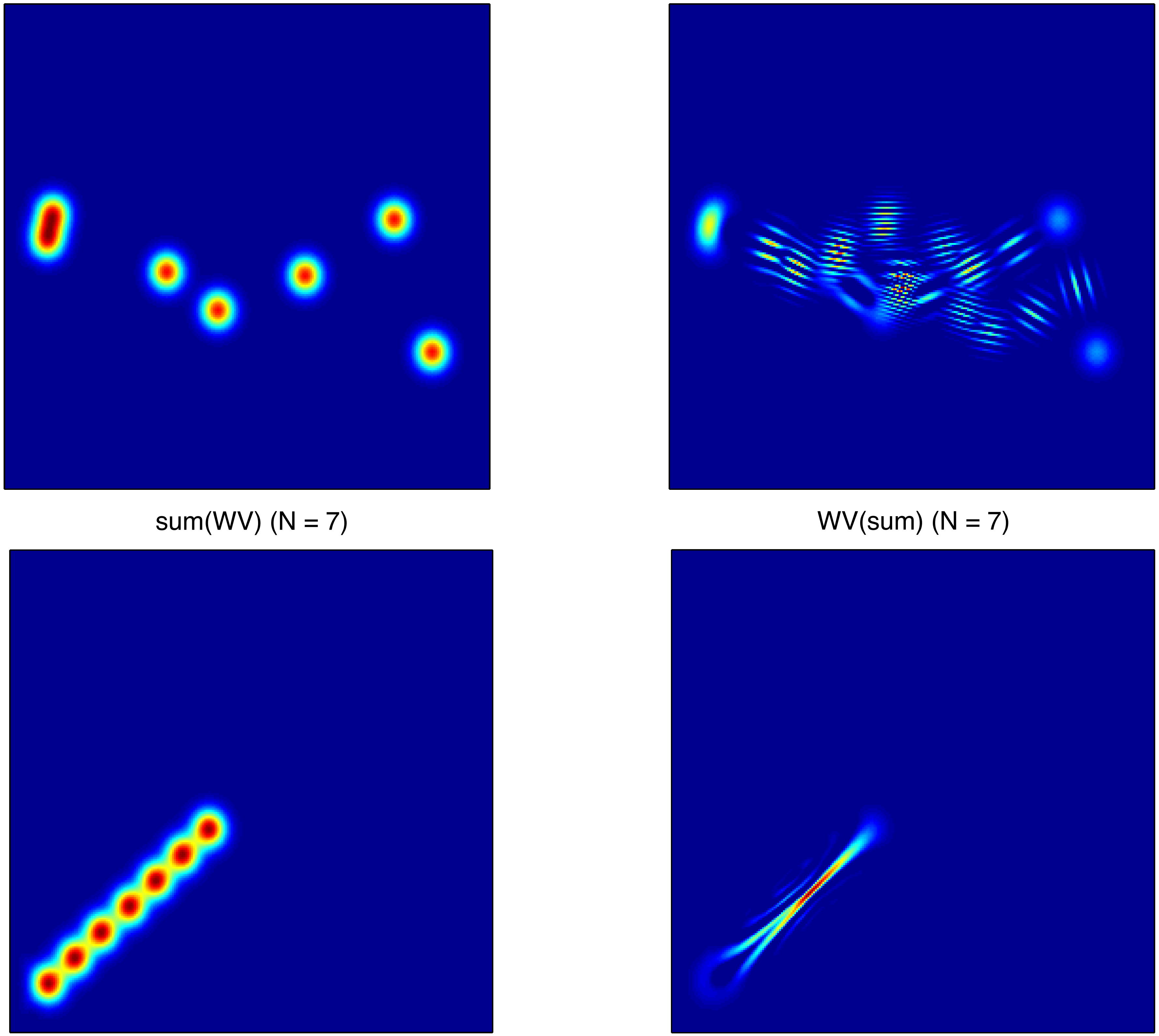

sum(WV) (N = 2)
WV(sum) (N = 2)

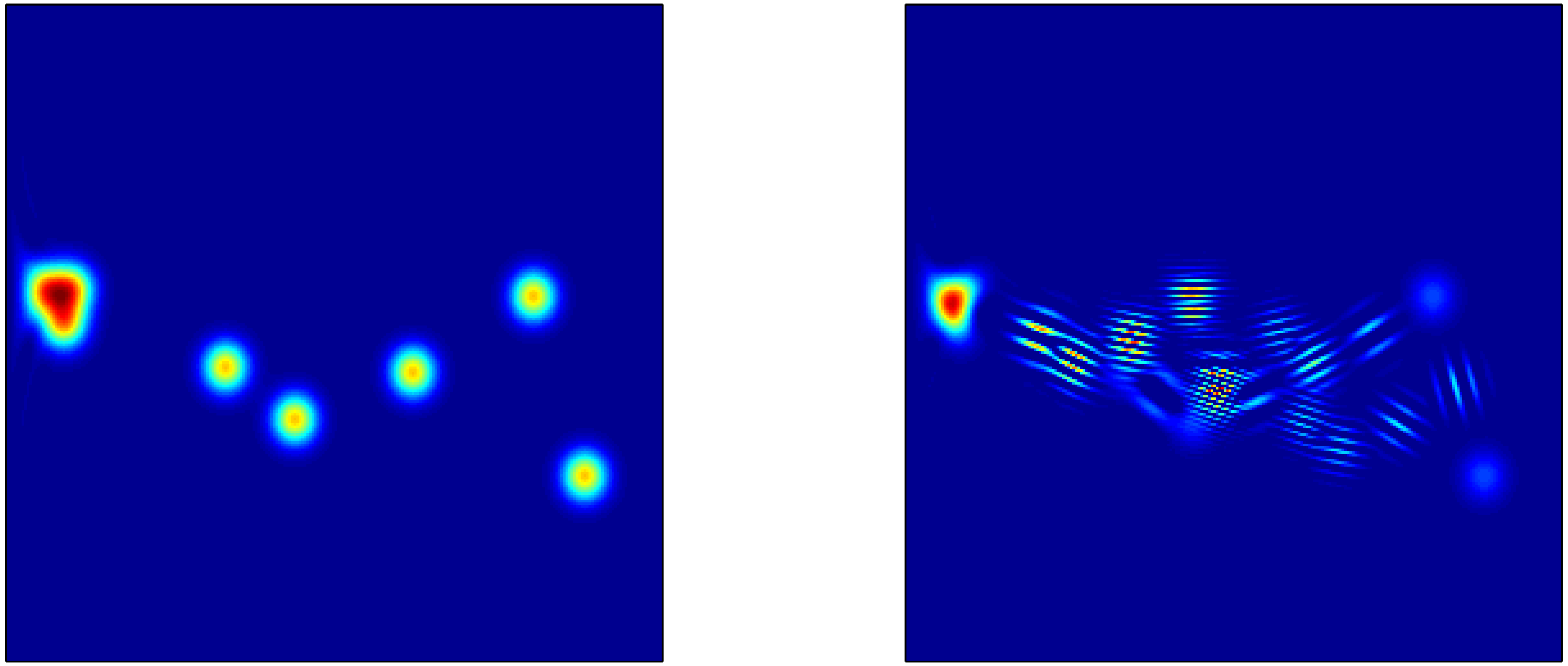

sum(WV) (N = 3)
WV(sum) (N = 3)


sum(WV) (N = 4)
WV(sum) (N = 4)

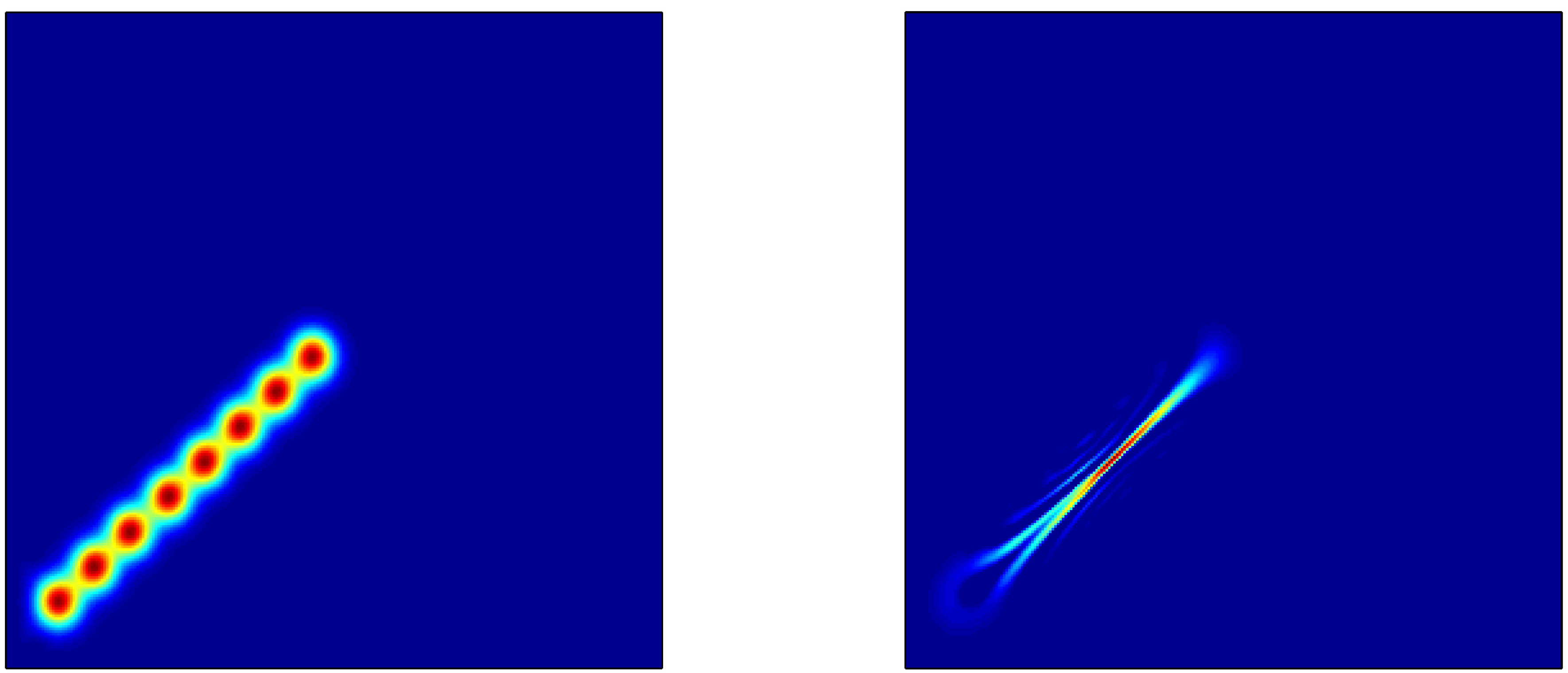

sum(WV) (N = 5)
WV(sum) (N = 5)

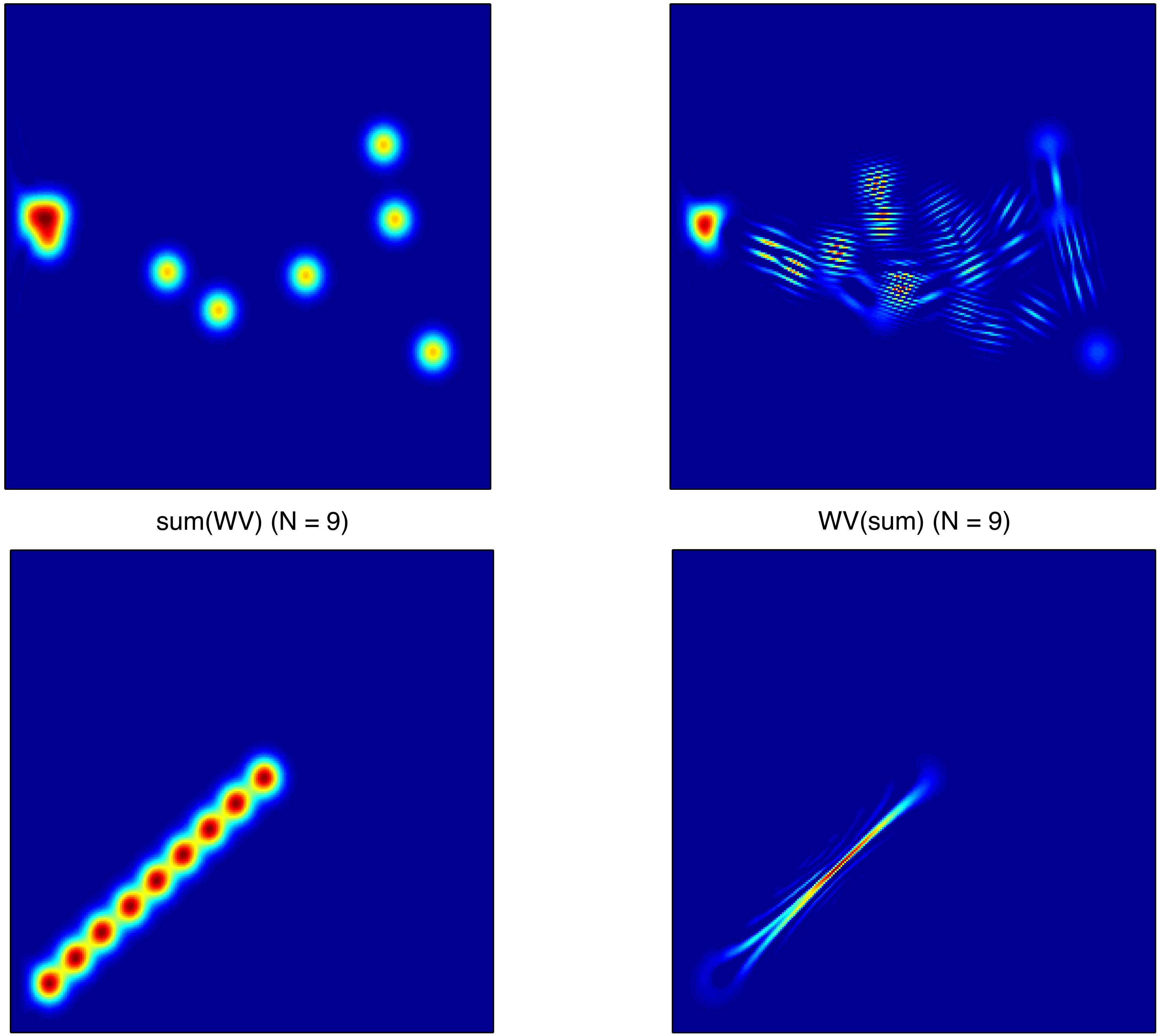

sum(WV) (N = 6)
WV(sum) (N = 6)

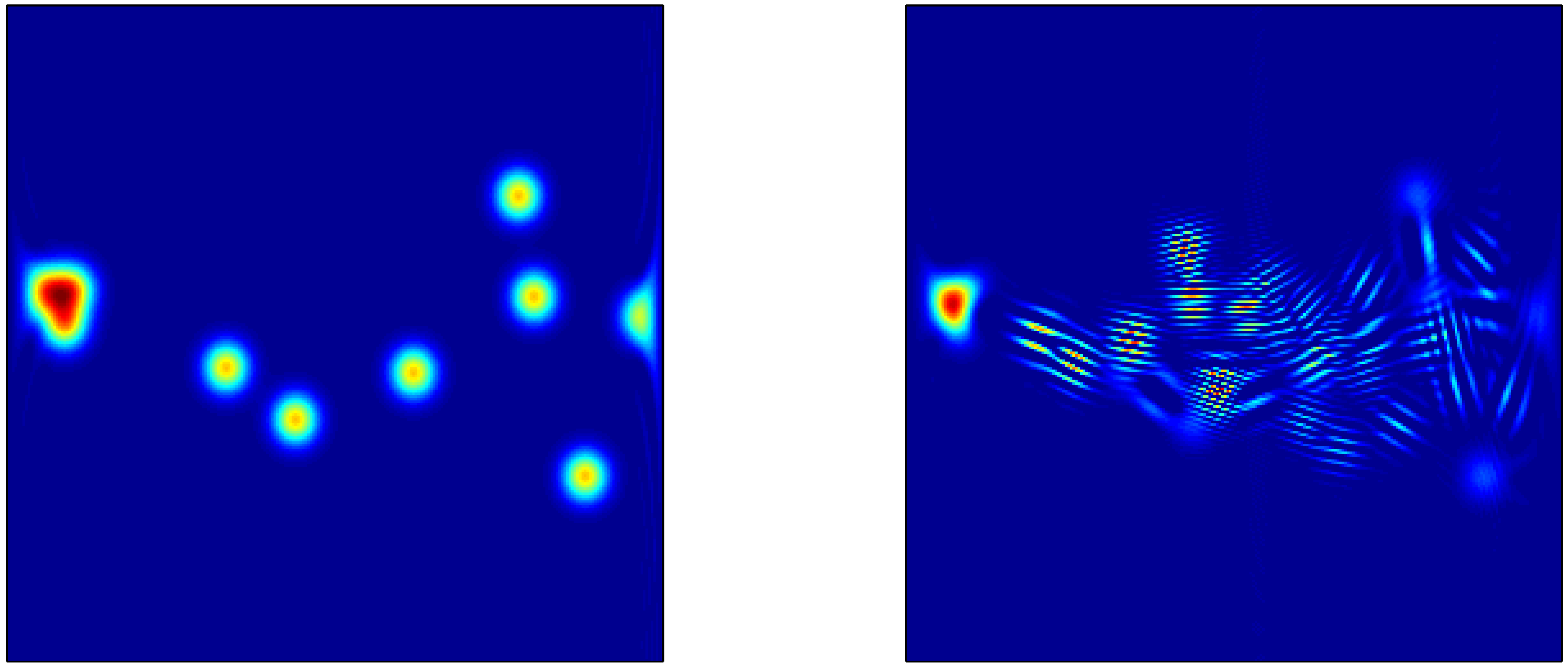

sum(WV) (N = 7)
WV(sum) (N = 7)



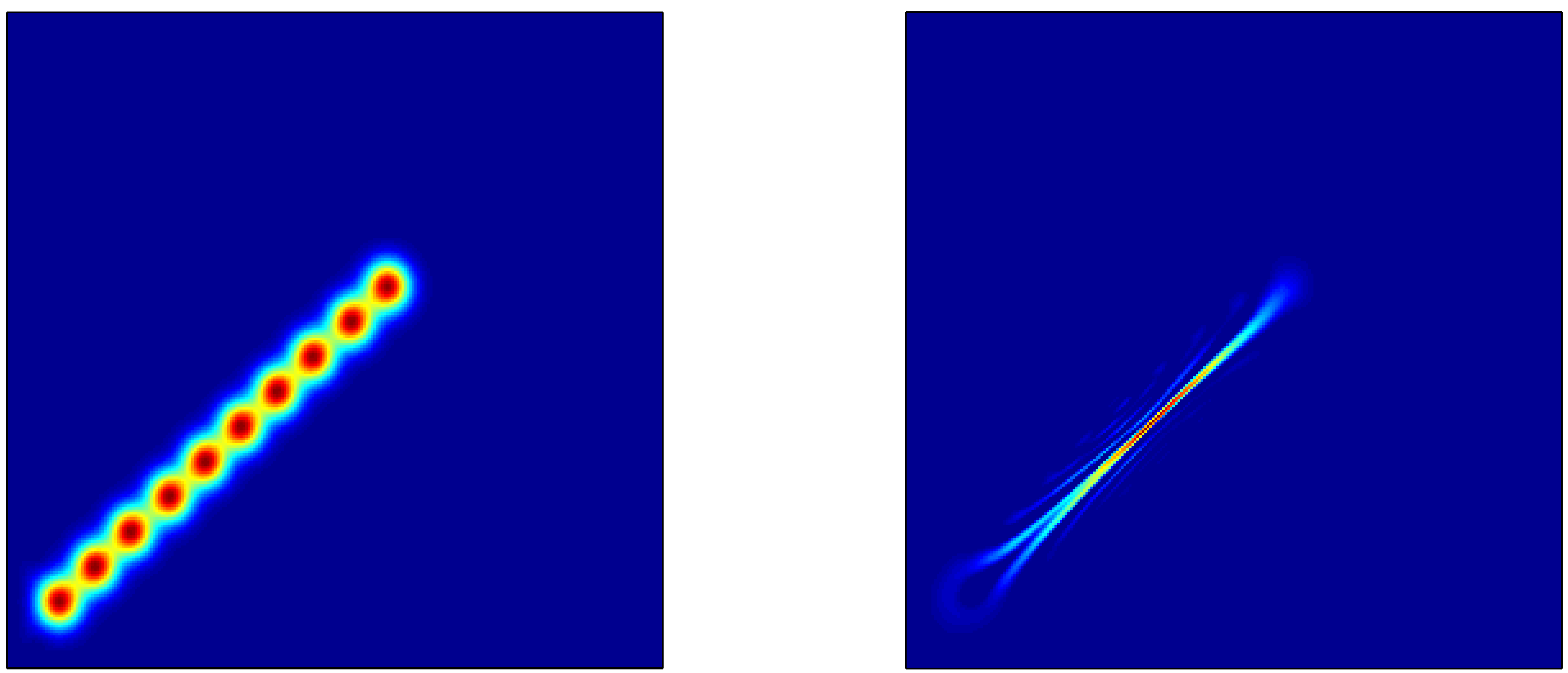
sum(WV) (N = 8) WV(sum) (N = 8)

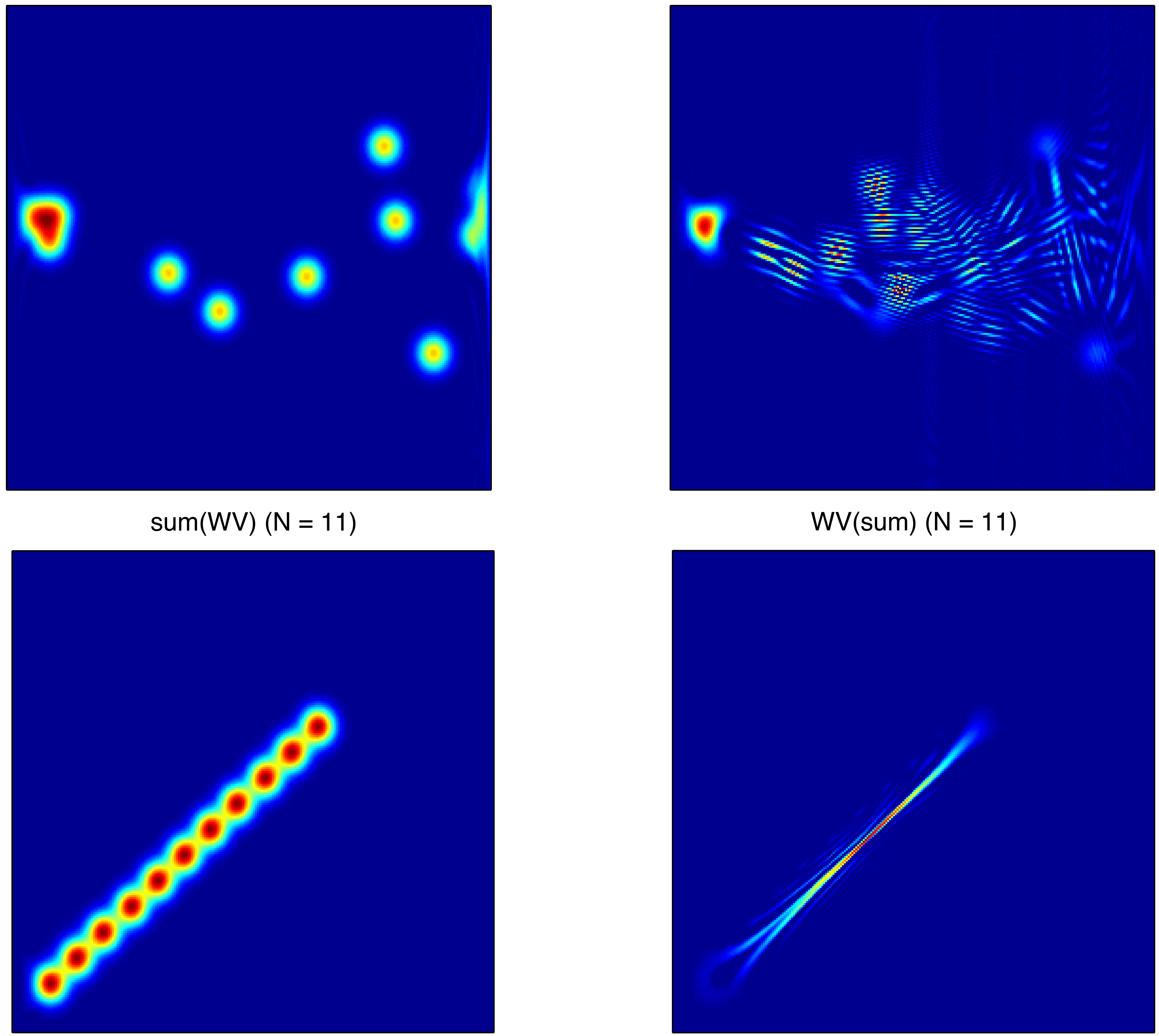


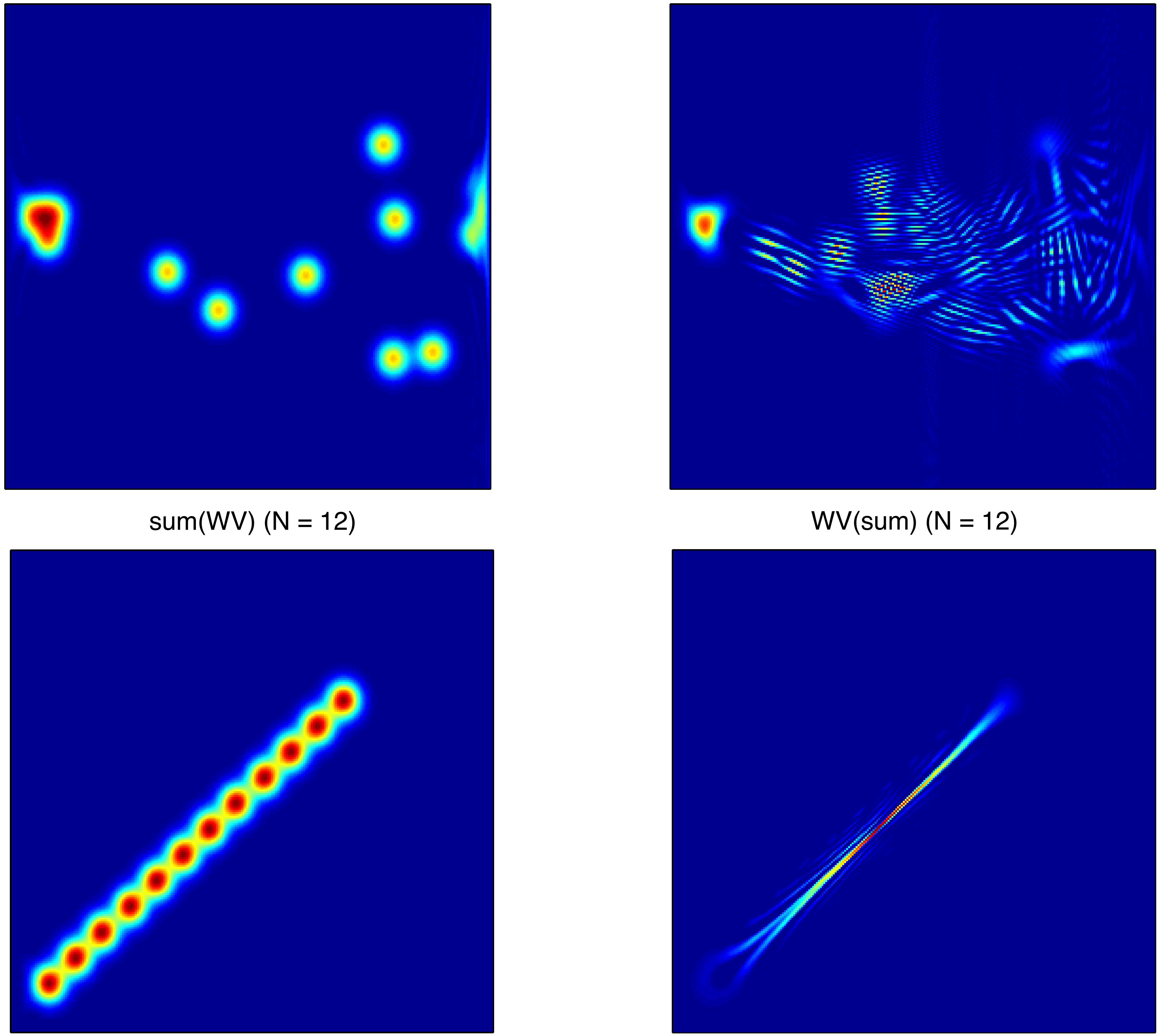

sum(WV) (N = 9)
WV(sum) (N = 9)

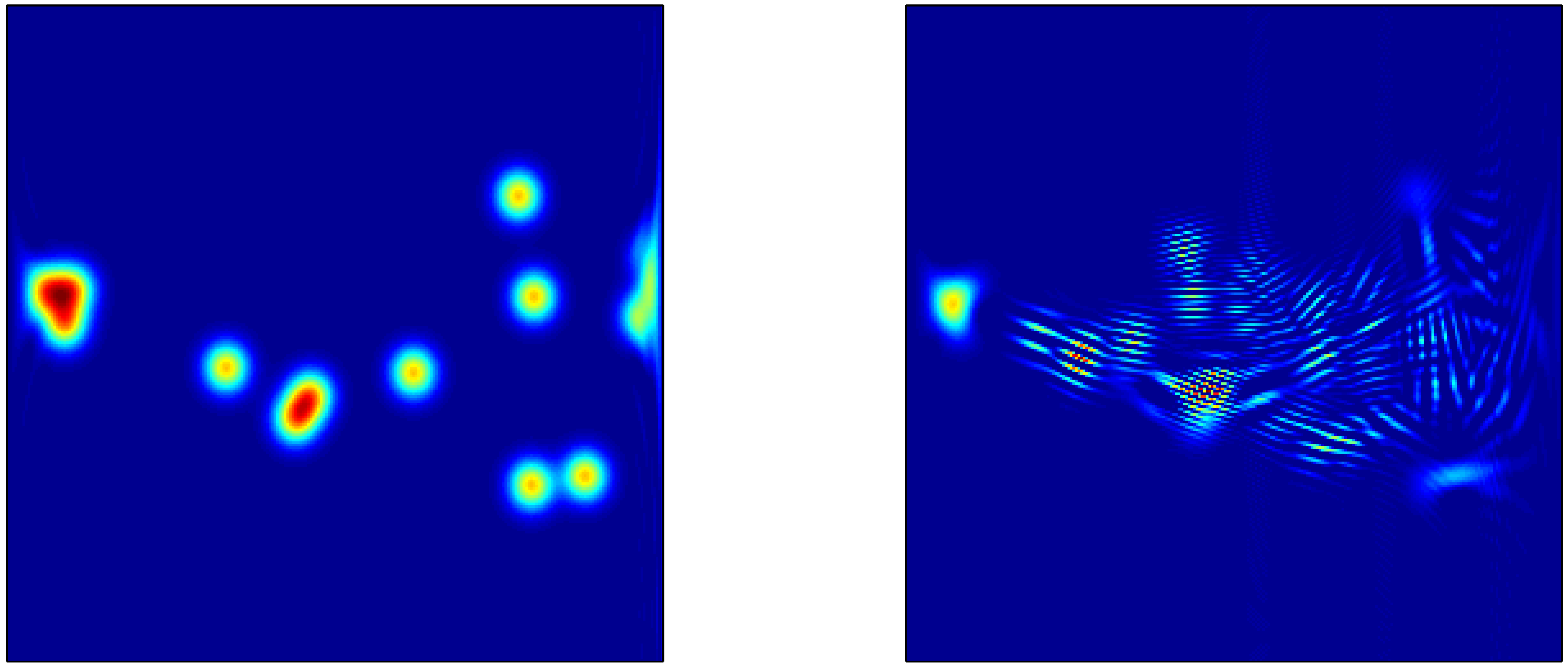


sum(WV) (N = 10) WV(sum) (N = 10)



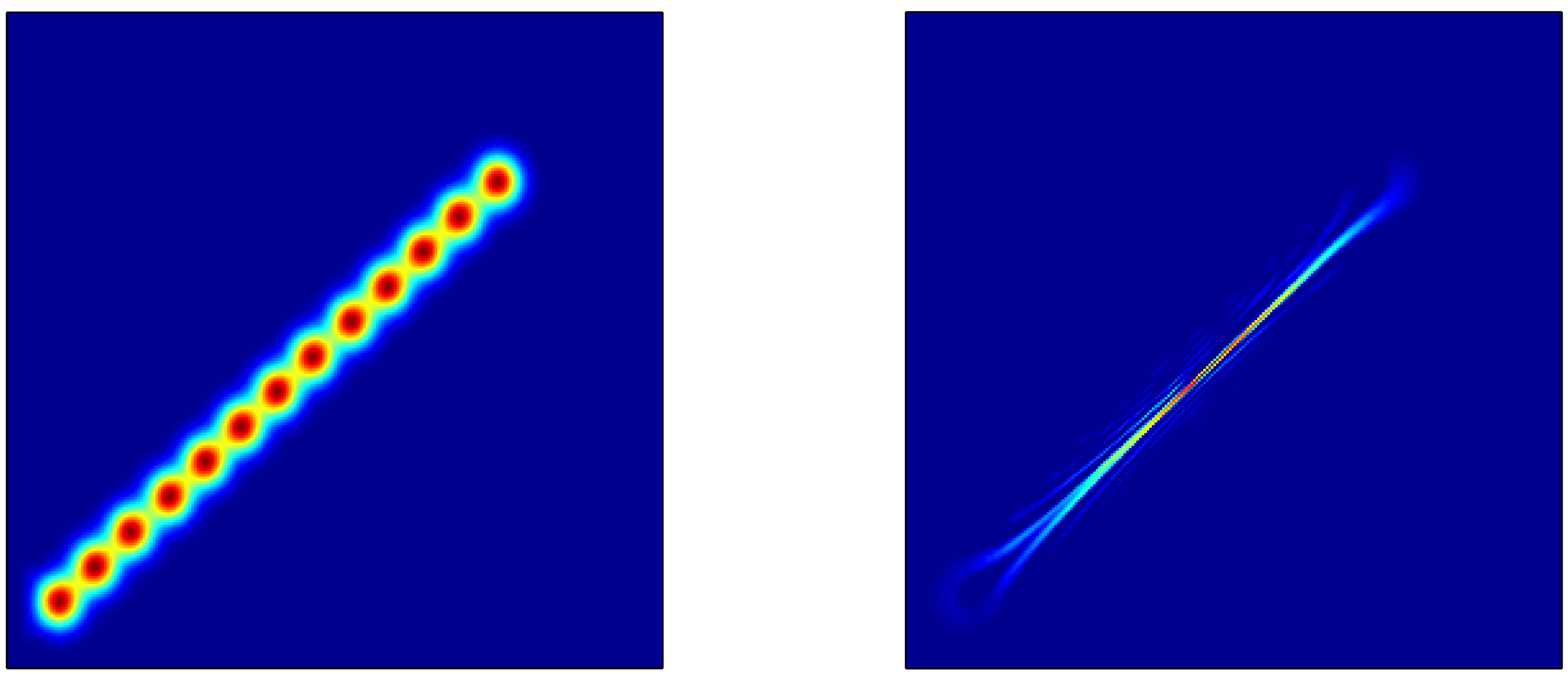

sum(WV) (N = 11)
WV(sum) (N = 11)

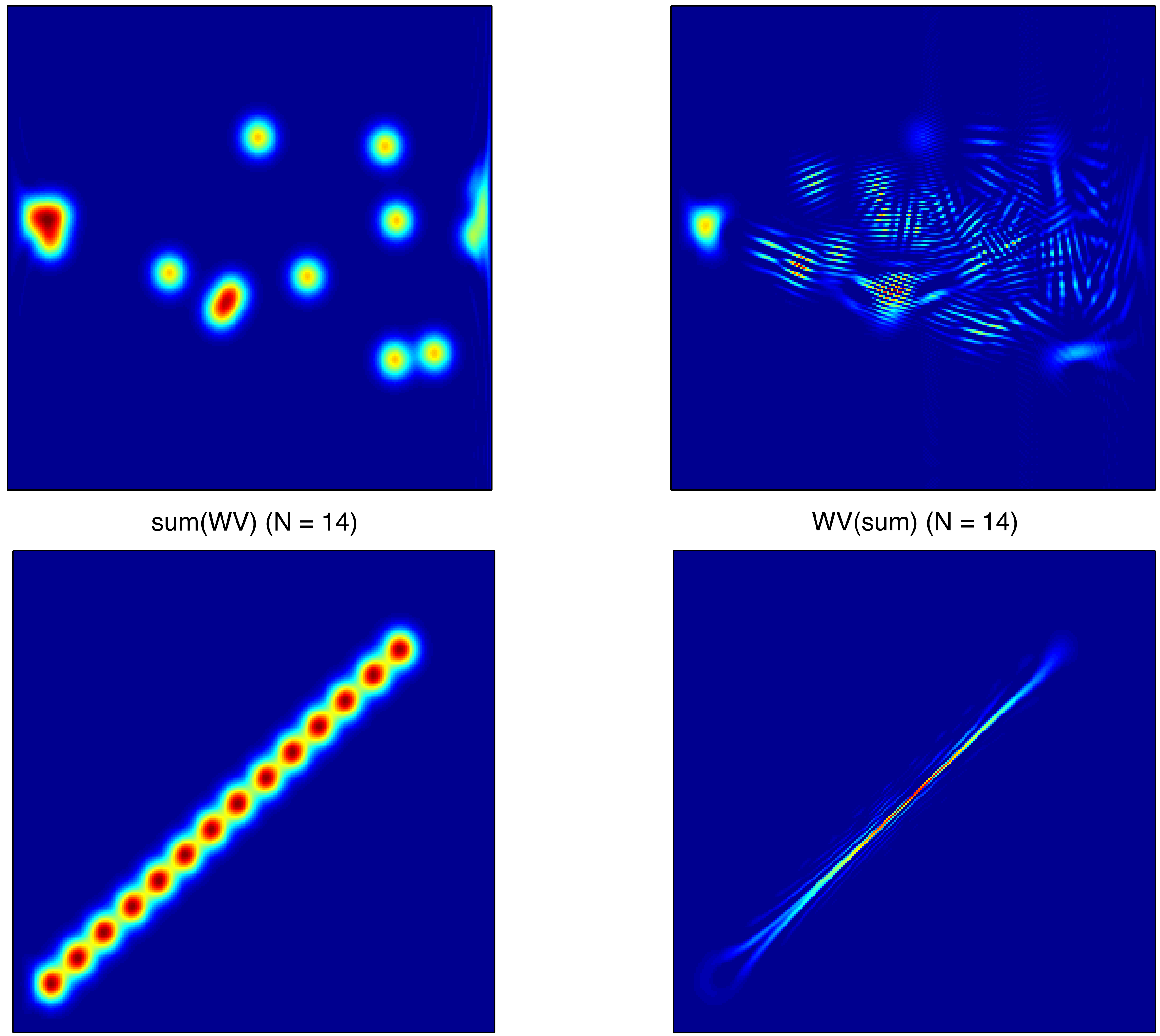

sum(WV) (N = 12)
WV(sum) (N = 12)

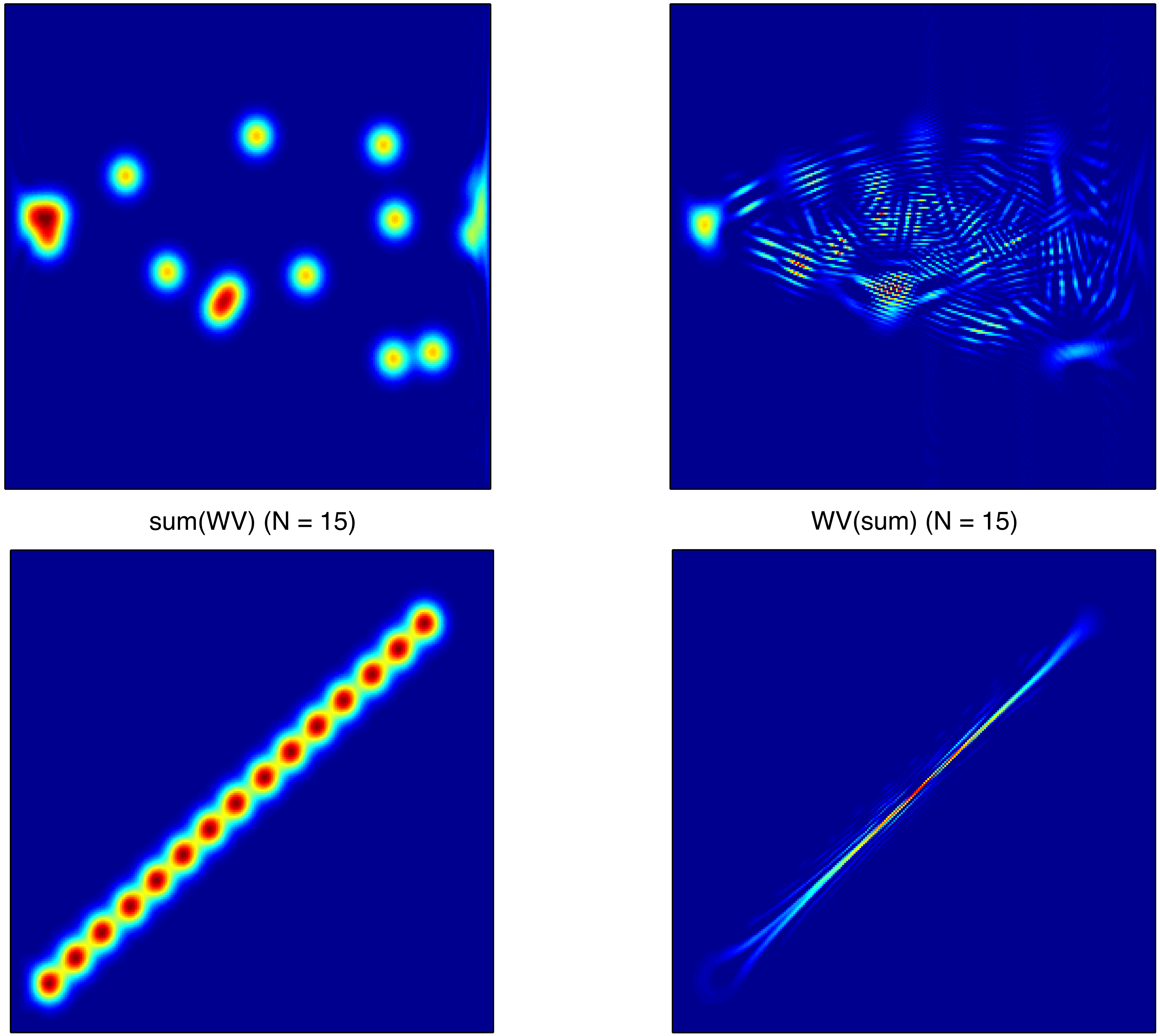


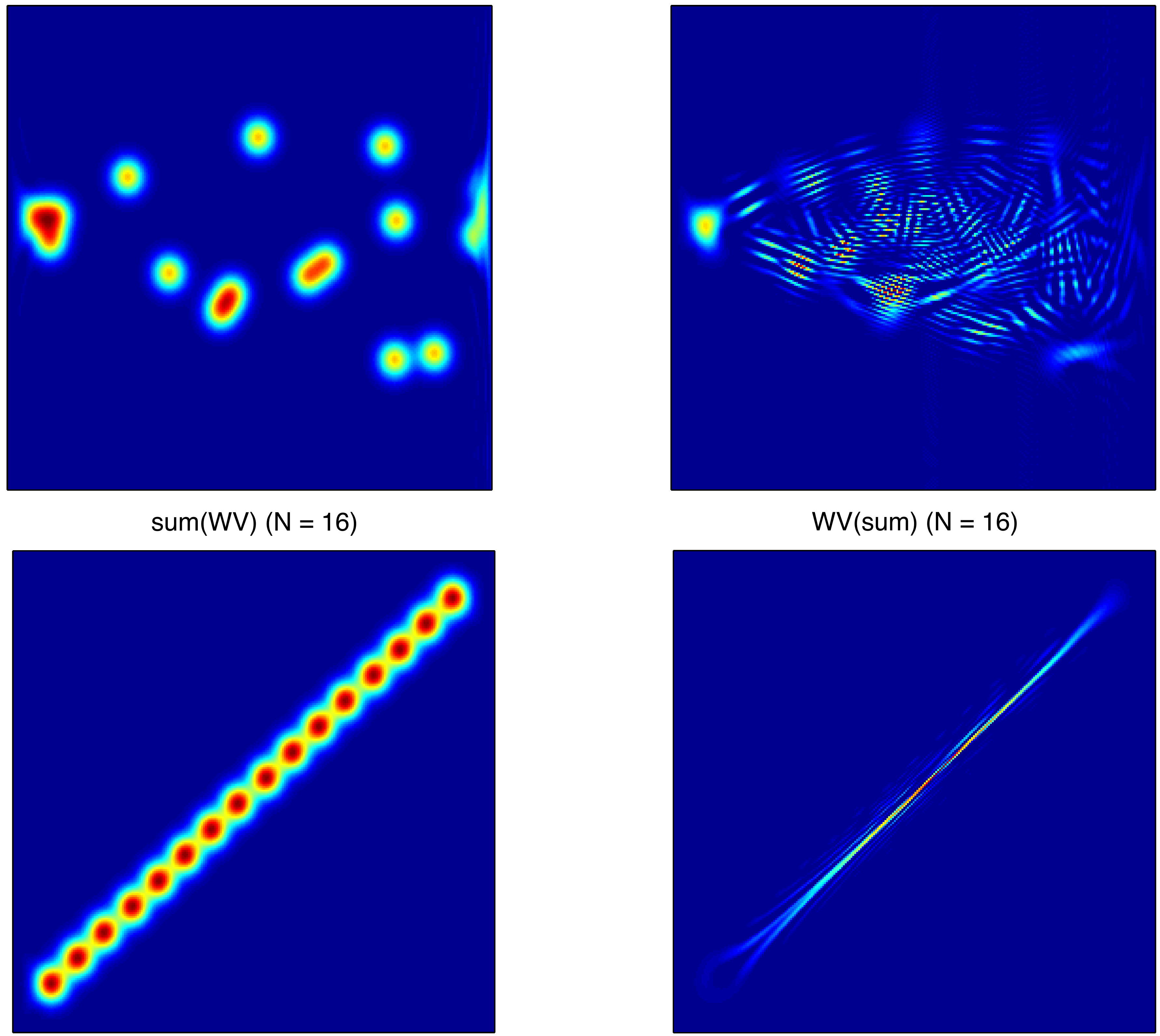
sum(WV) (N = 13)

WV(sum) (N = 13)




sum(WV) (N = 14)
WV(sum) (N = 14)


sum(WV) (N = 15)
WV(sum) (N = 15)


sum(WV) (N = 16)
WV(sum) (N = 16)

Revisiting spectrograms with Wigner

$$S_x^h(t, f) = \iint W_x(s, \xi)\, W_h(s - t, \xi - f)\, ds\, d\xi$$

smoothing kernel

Revisiting spectrograms with Wigner

$$S_x^h(t,f) = \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi$$

smoothing kernel

Reassignment

Kodera

Gendrin

de Villedary
1976

Auger

F
1995

Revisiting spectrograms with Wigner

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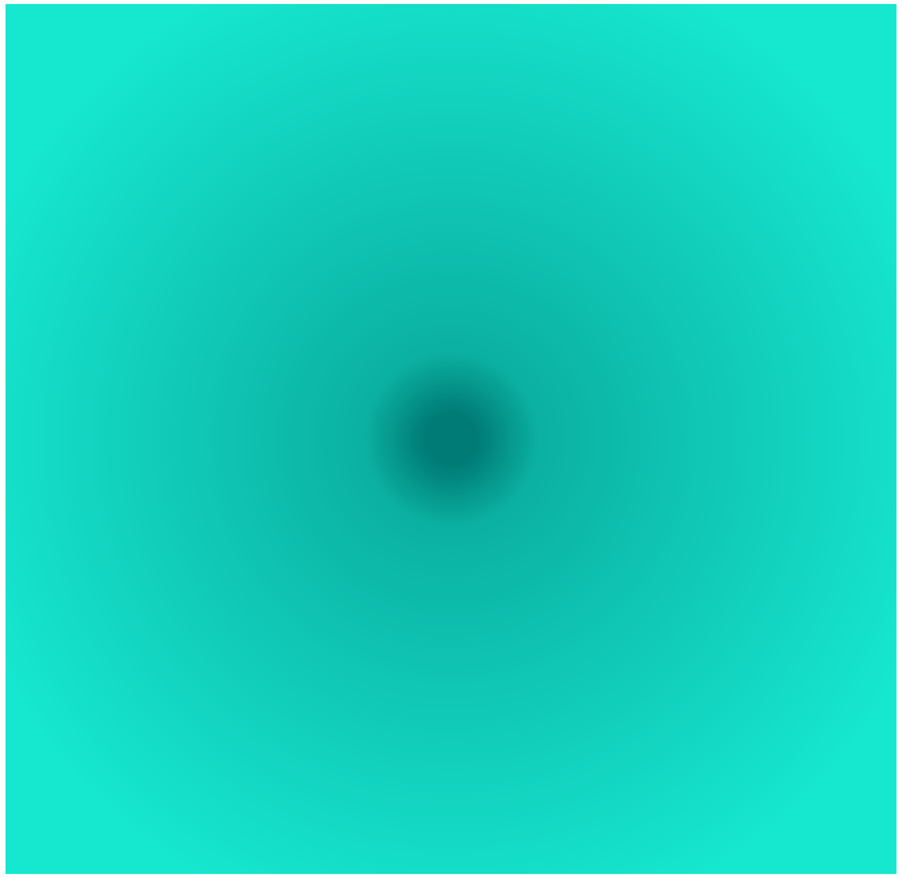
smoothing kernel

Reassignment — A mechanical analogy



Revisiting spectrograms with Wigner

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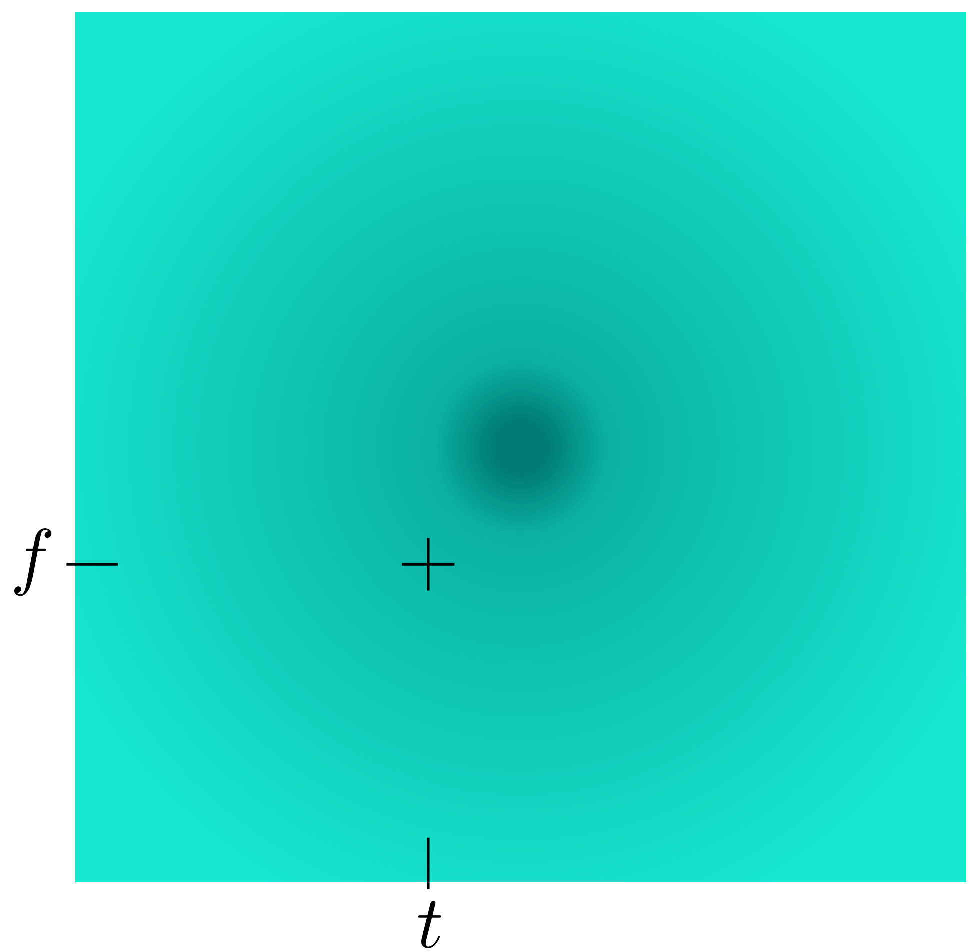
smoothing kernel

Reassignment — A mechanical analogy



Revisiting spectrograms with Wigner

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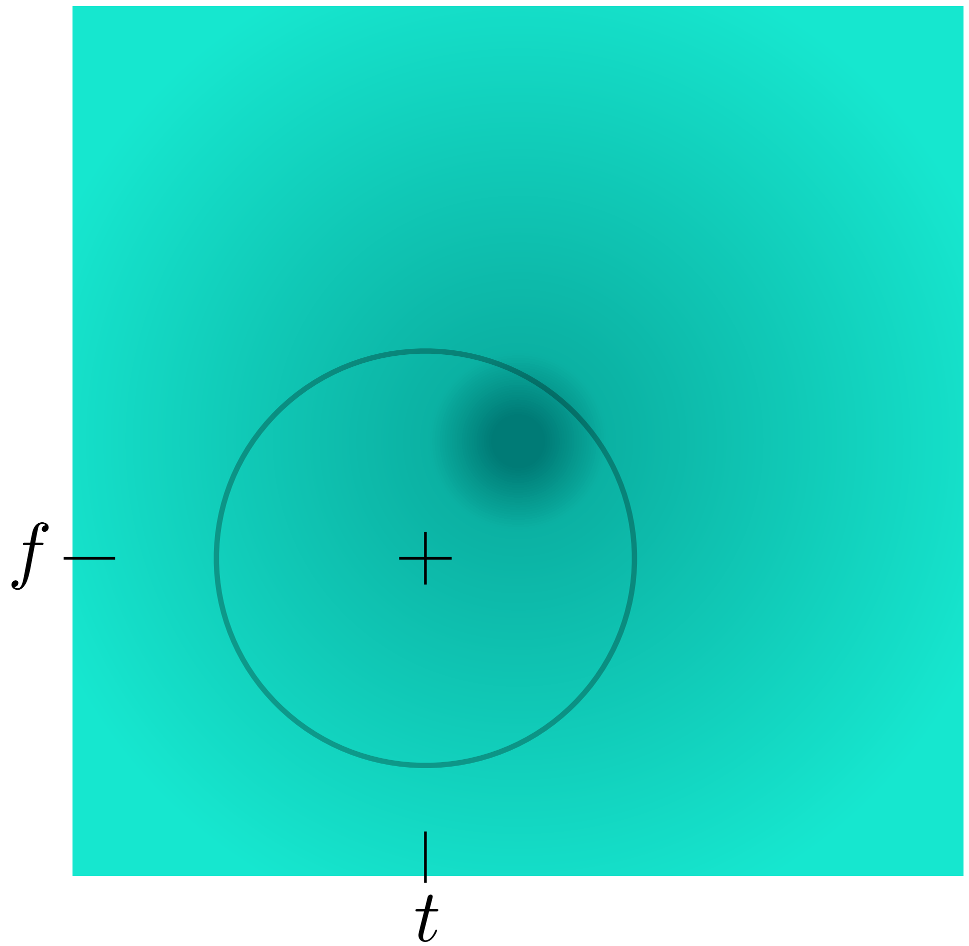
Reassignment — A mechanical analogy



Revisiting spectrograms with Wigner

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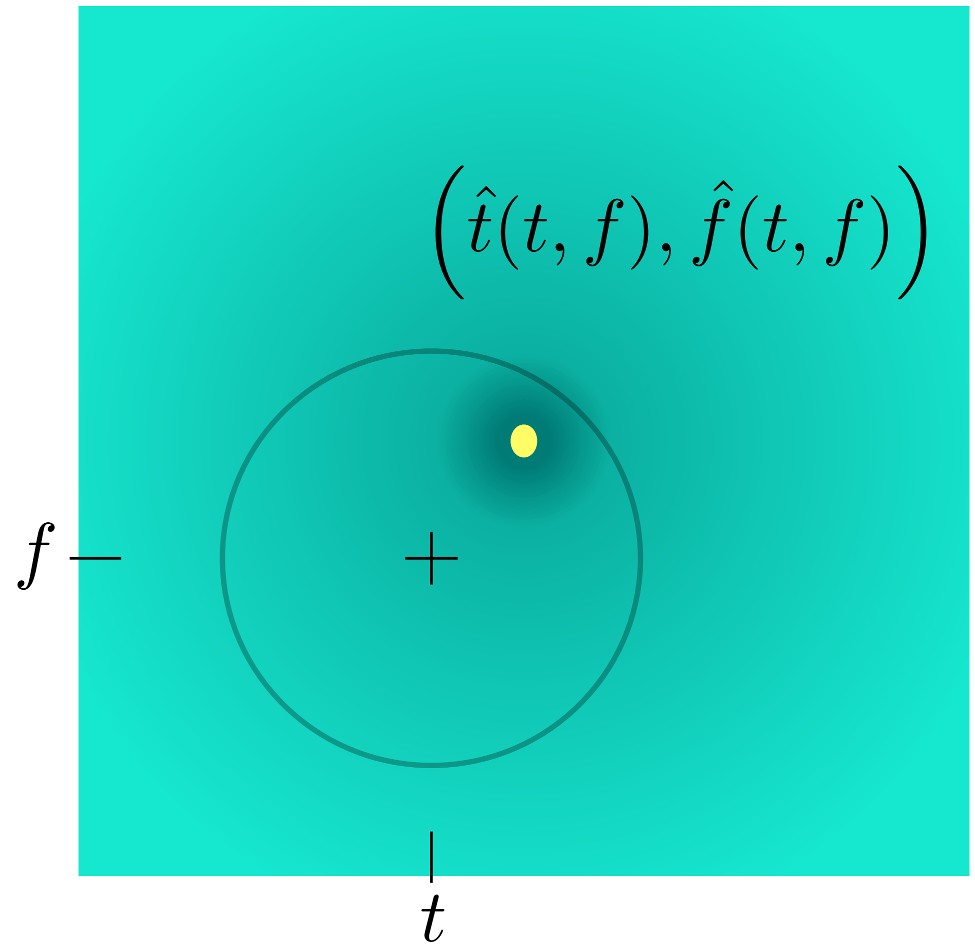
smoothing kernel

Reassignment — A mechanical analogy

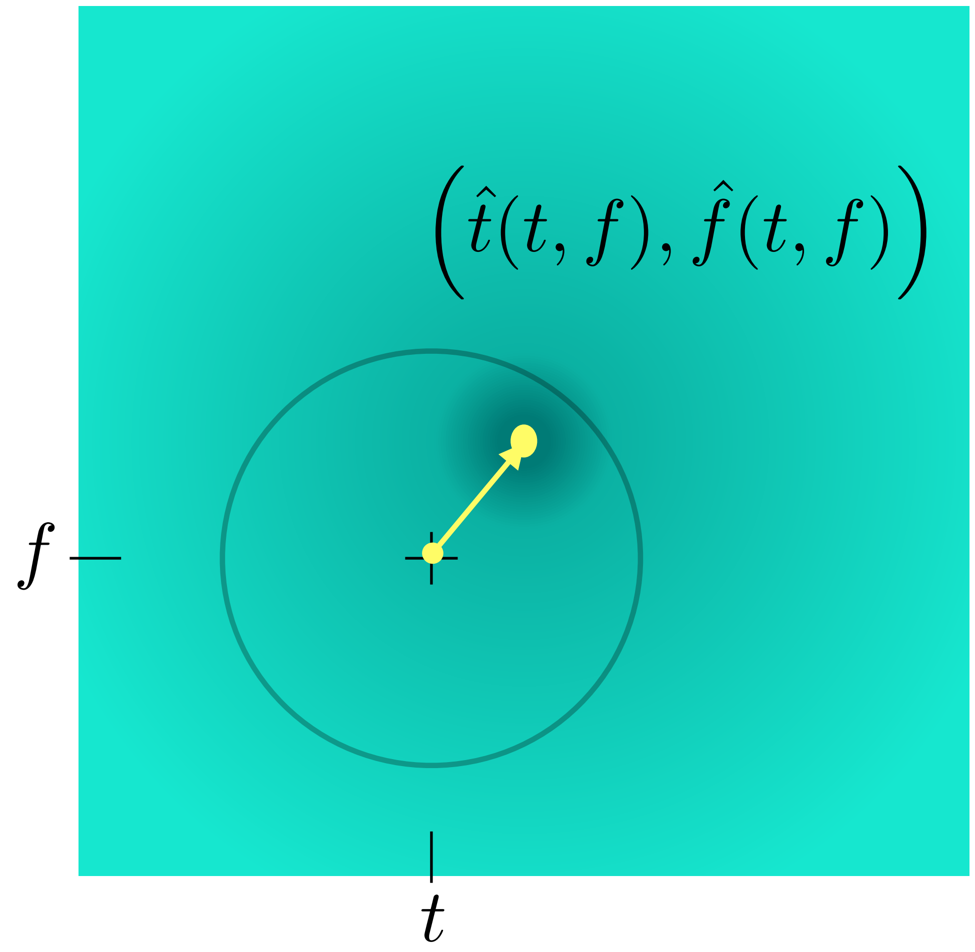


Revisiting spectrograms with Wigner

$$S_x^h(t,f) = \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi$$

smoothing kernel

Reassignment — A mechanical analogy

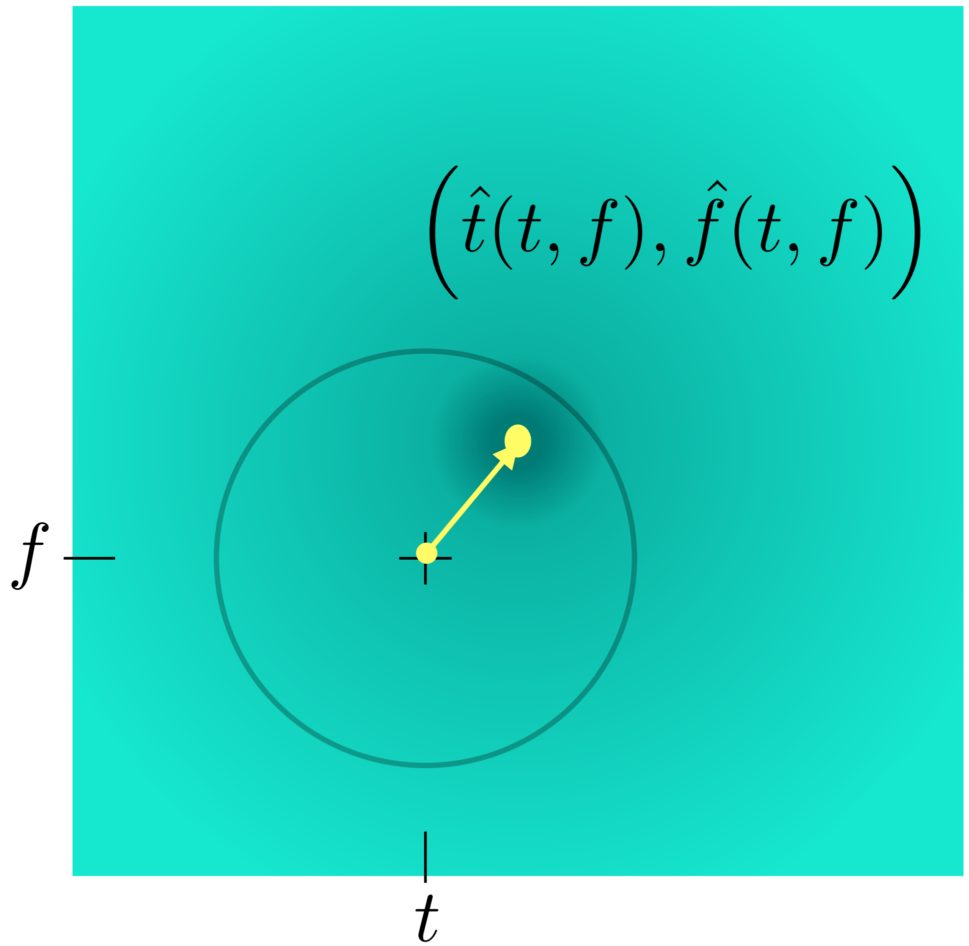


Revisiting spectrograms with Wigner

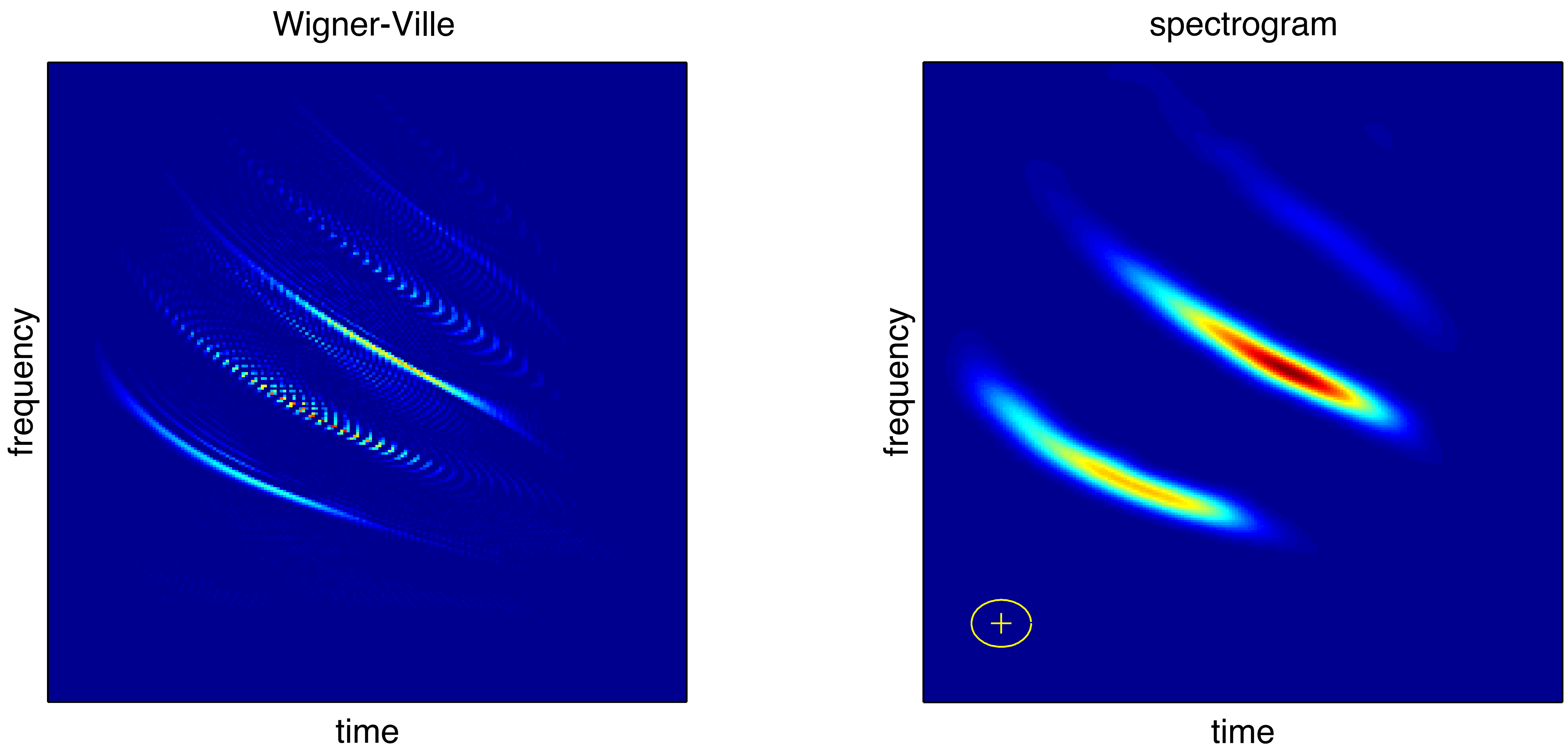
$$S_x^h(t,f) = \iint W_x(s,\xi)\, W_h(s-t,\xi-f)\, ds\, d\xi$$

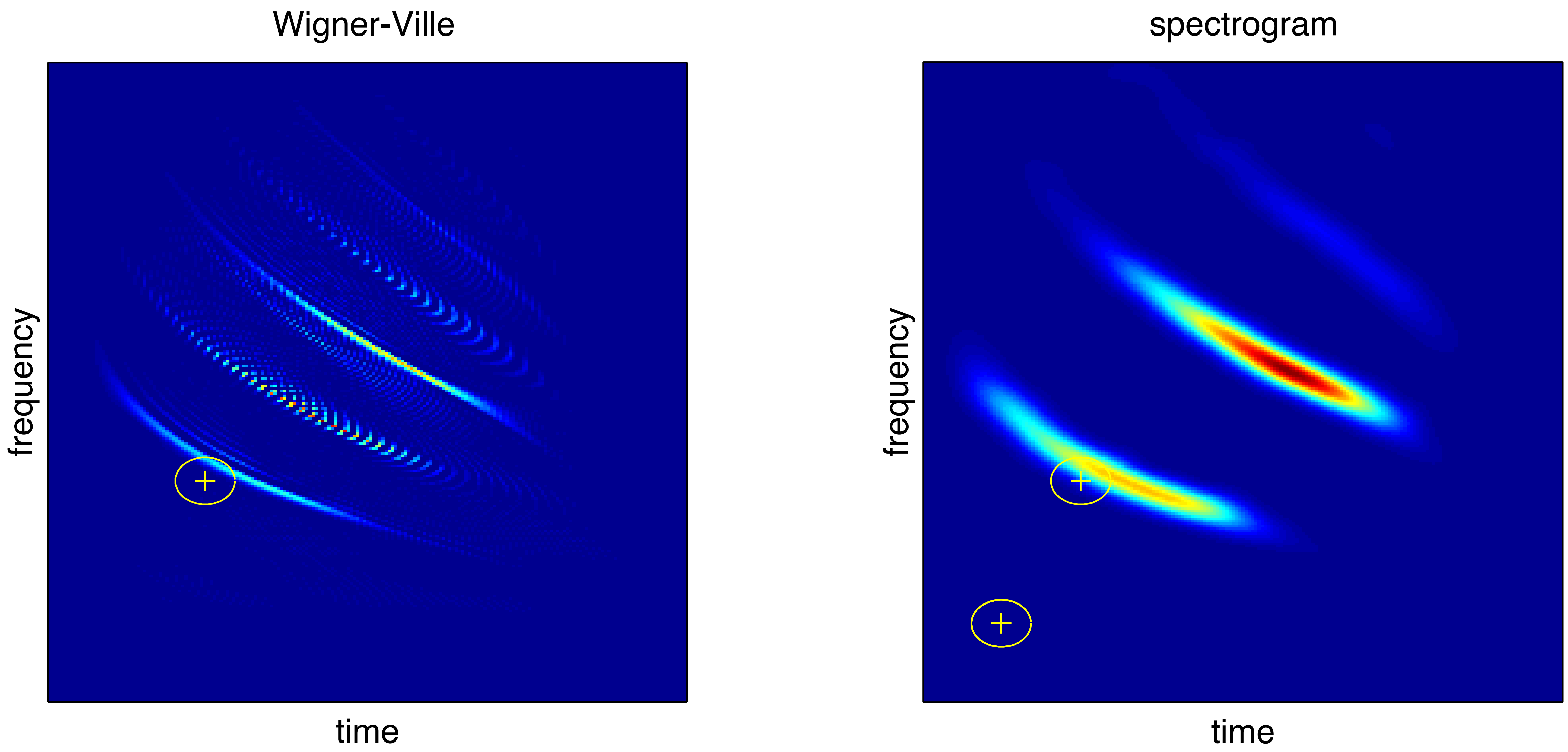
smoothing kernel

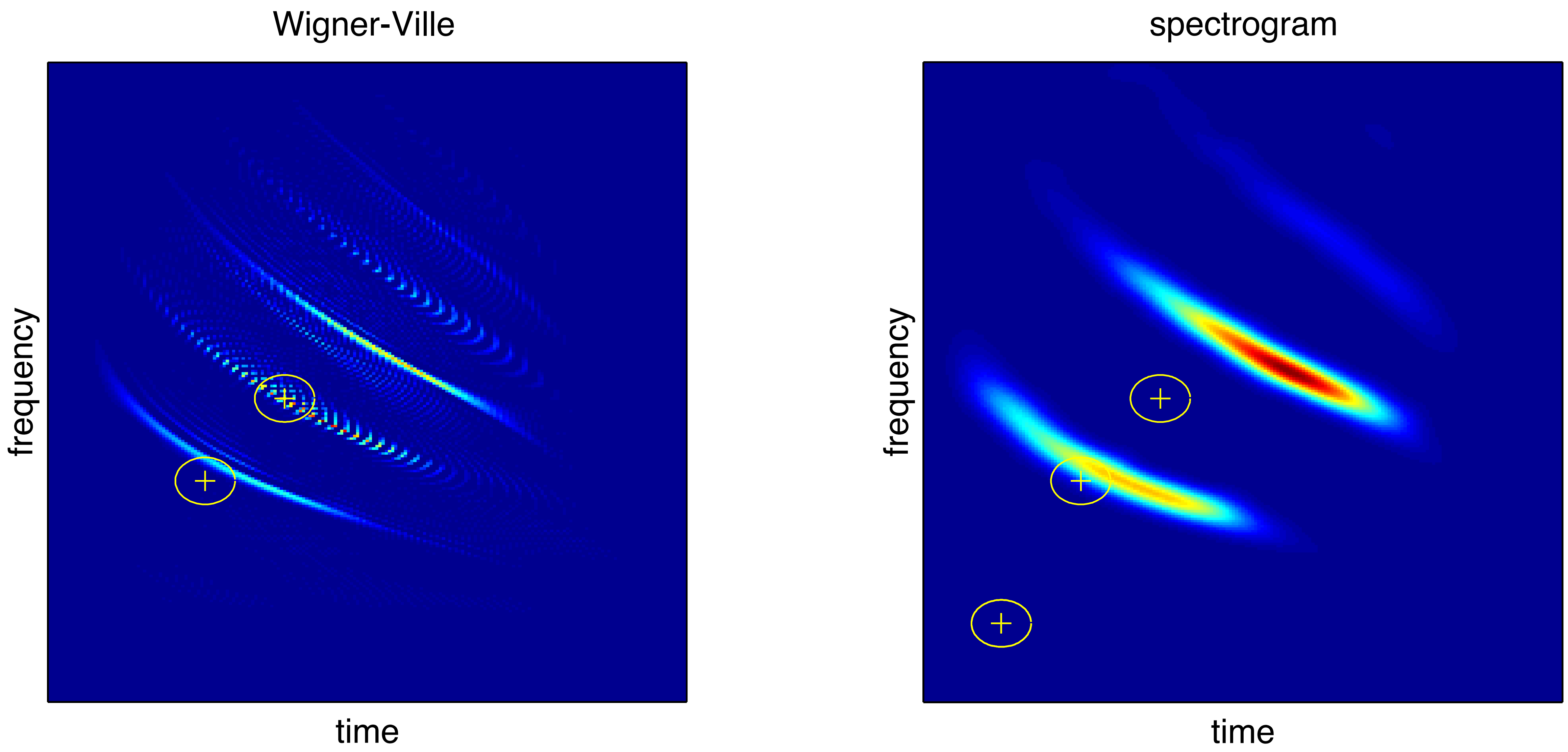
Reassignment — A mechanical analogy

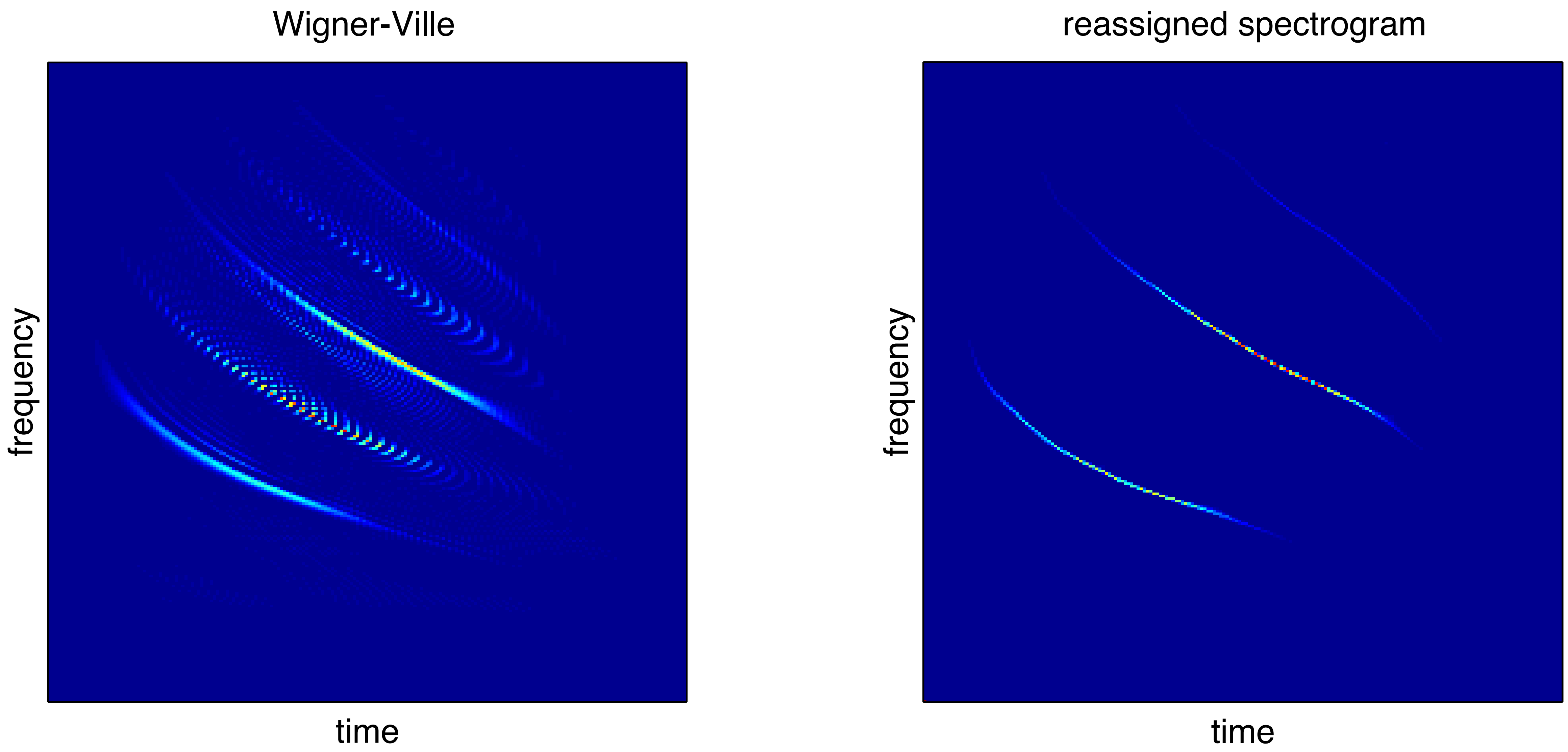


$$S_x(t,f) \to \hat{S}_x(t,f) = \iint S_x(s,\xi)\, \delta\left(t-\hat{t}(s,\xi), f-\hat{f}(s,\xi)\right)\, ds\, d\xi$$


Wigner-Ville
frequency
time
spectrogram
frequency
time


Wigner-Ville
frequency
time
spectrogram
frequency
time

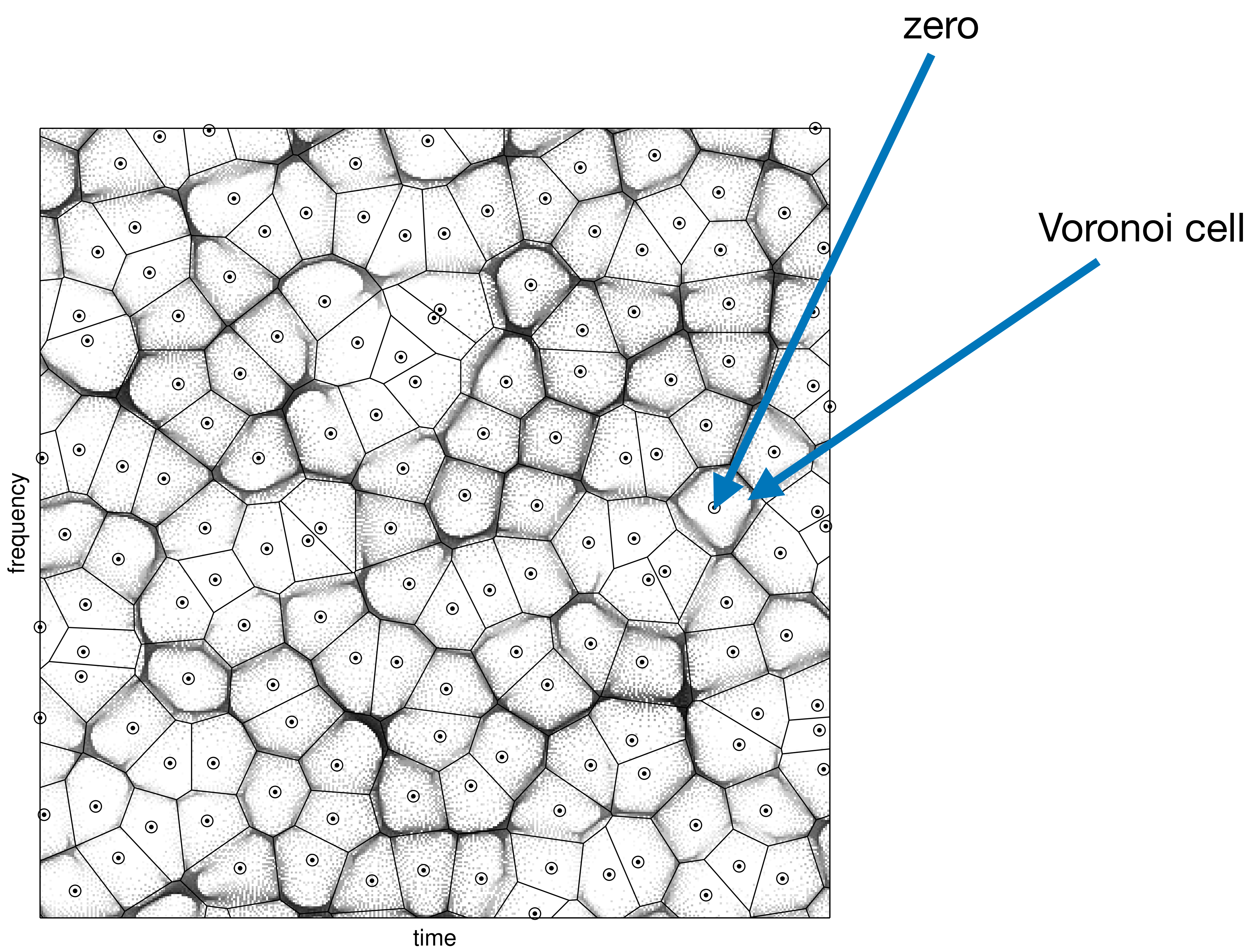

Wigner-Ville
frequency
time
spectrogram
frequency
time



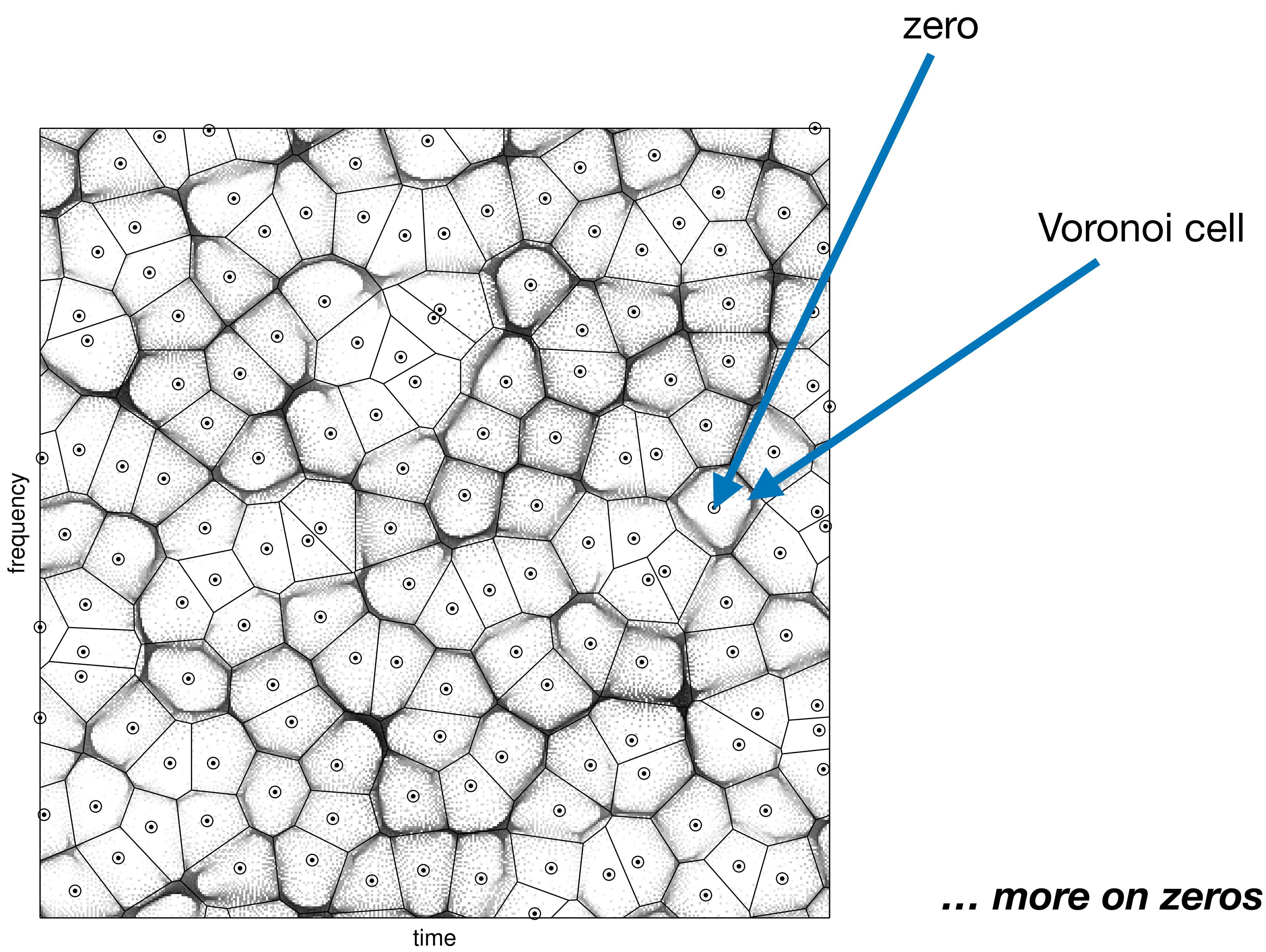
Wigner-Ville
frequency
time
reassigned spectrogram
frequency
time

A remark on reassignment, white Gaussian noise, and STFT zeros

A remark on reassignment, white Gaussian noise, and STFT zeros



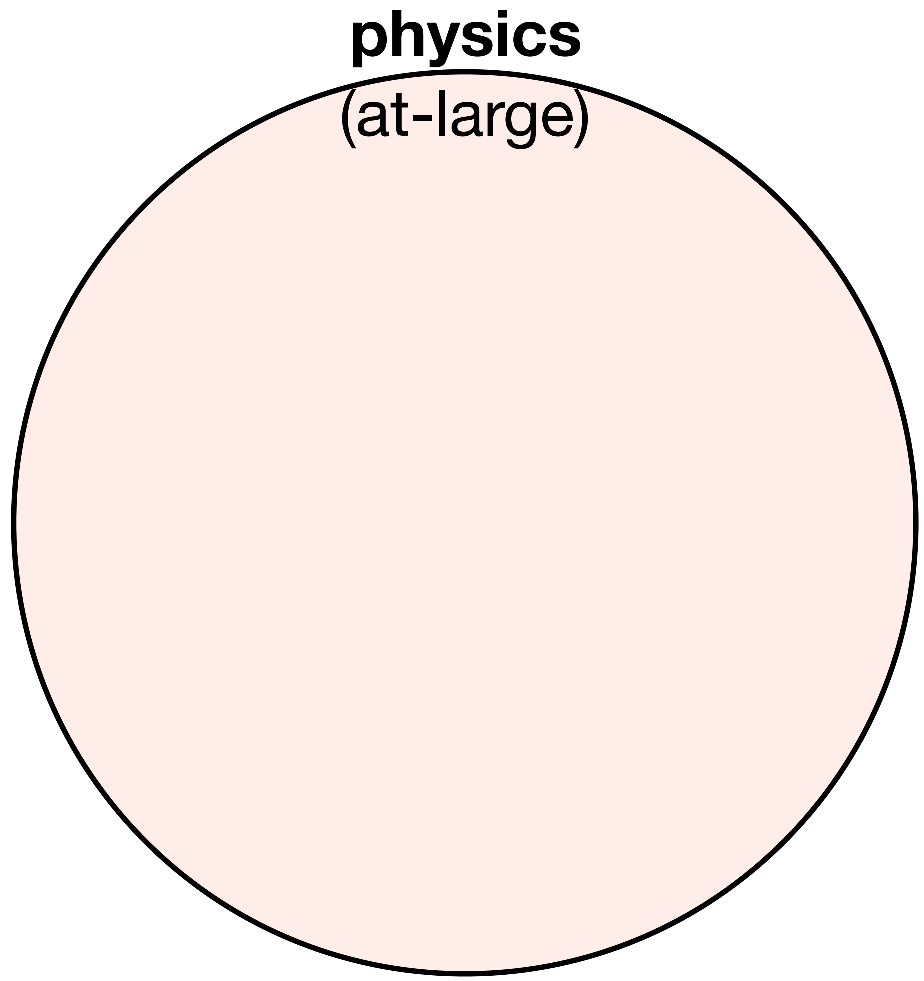
A remark on reassignment, white Gaussian noise, and STFT zeros

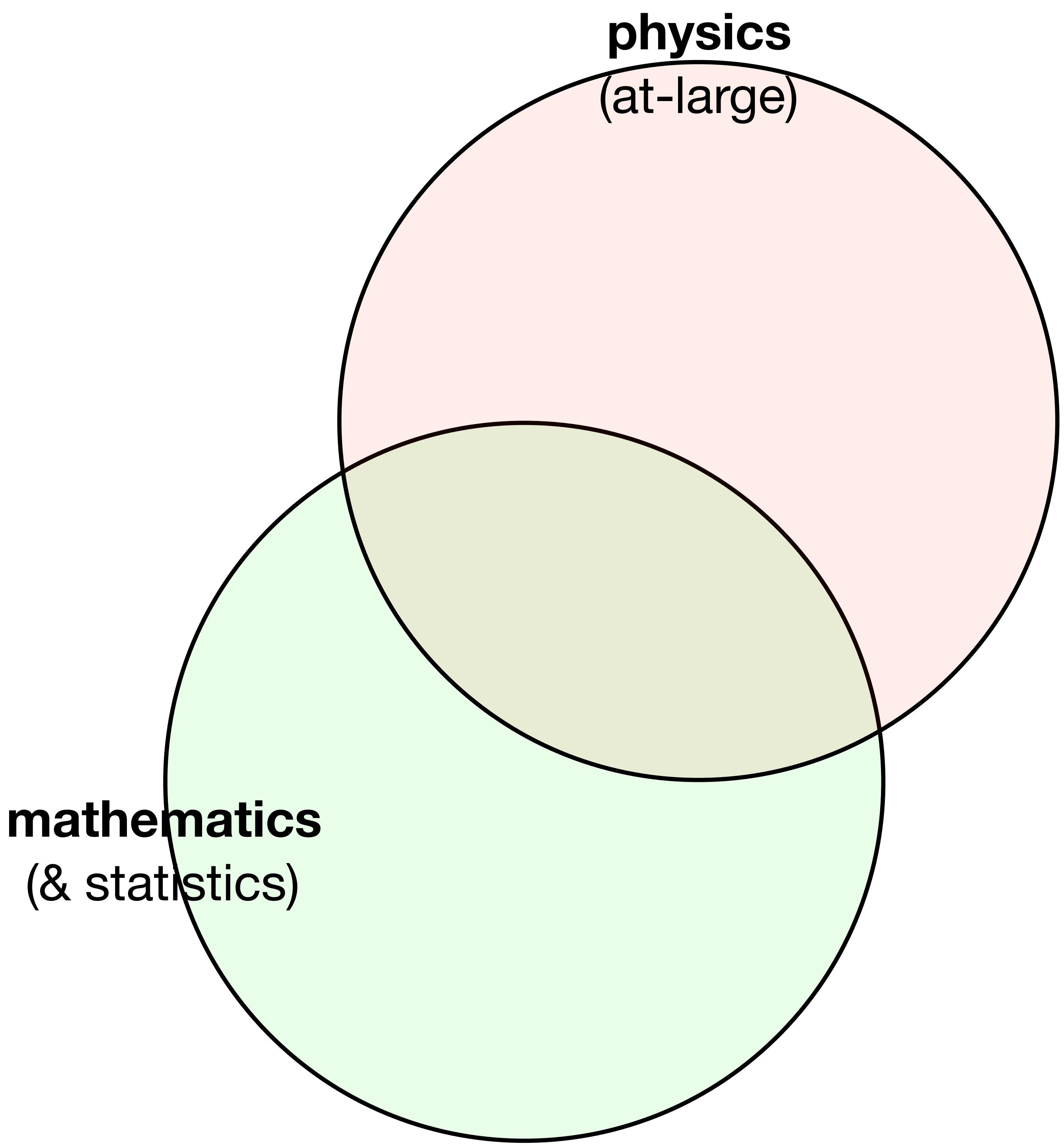


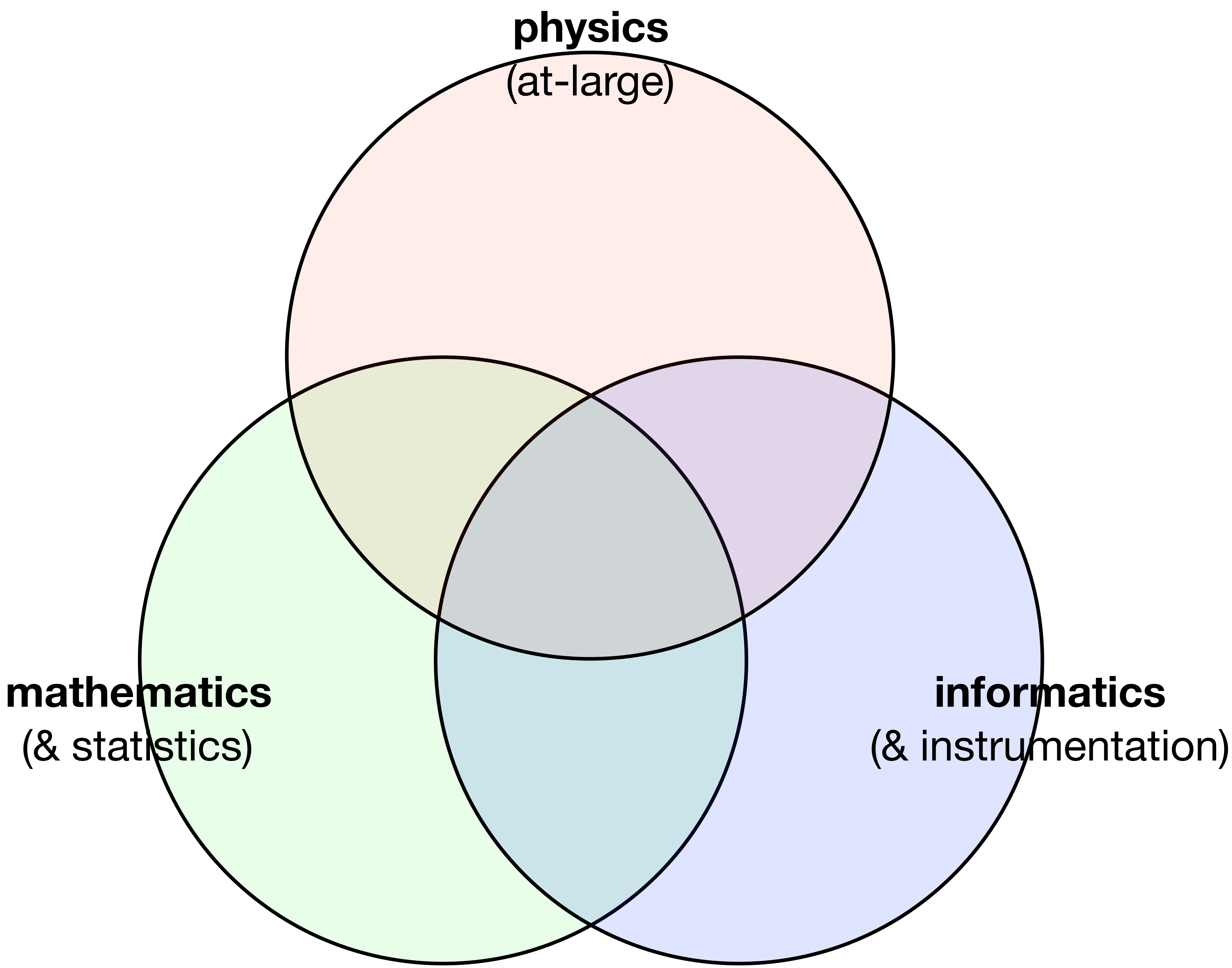
… more on zeros in other talks …

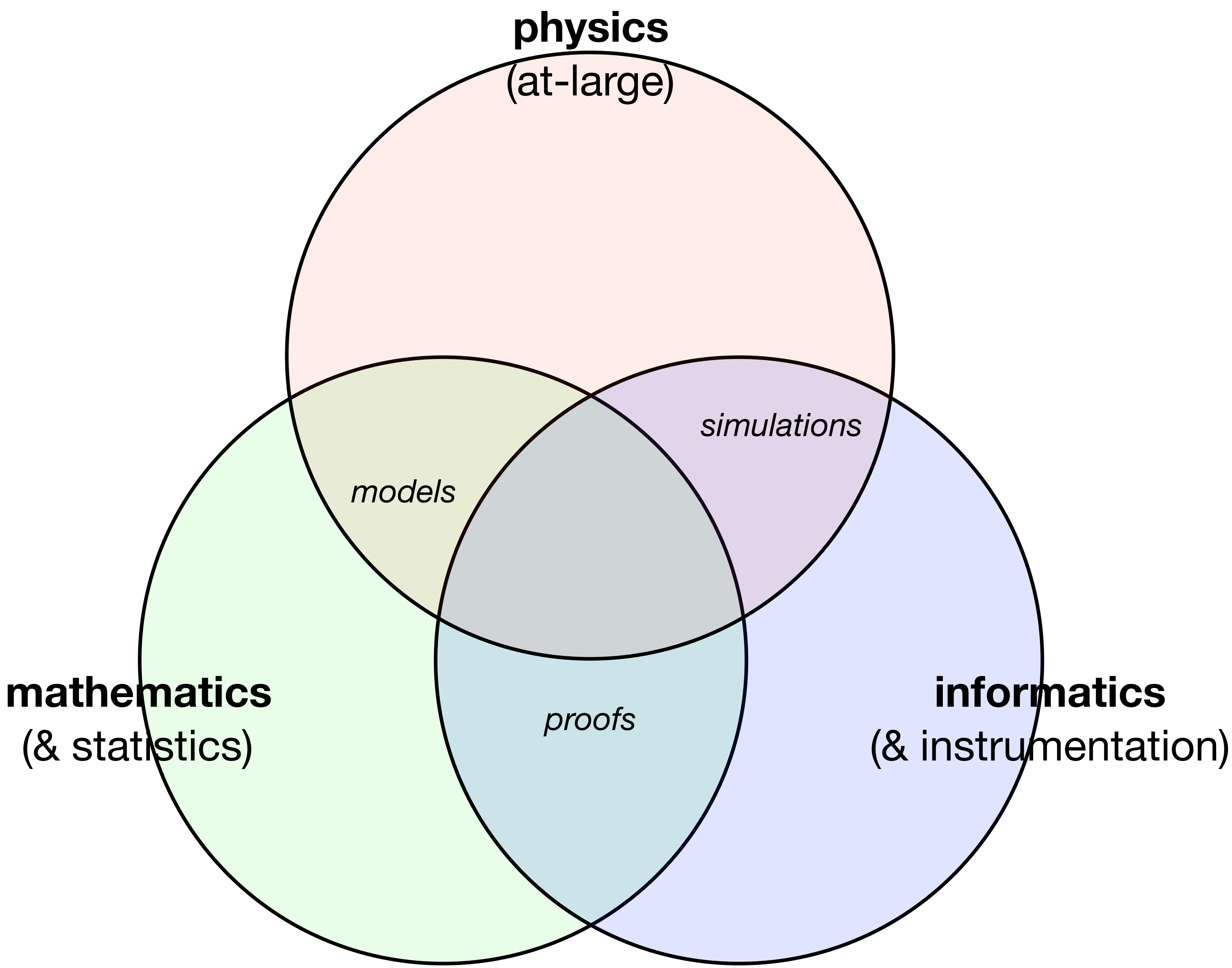
8 concluding remarks

signal as a science


physics
(at-large)


physics
(at-large)
mathematics
(& statistics)


physics
(at-large)
mathematics
(& statistics)
informatics
(& instrumentation)


physics
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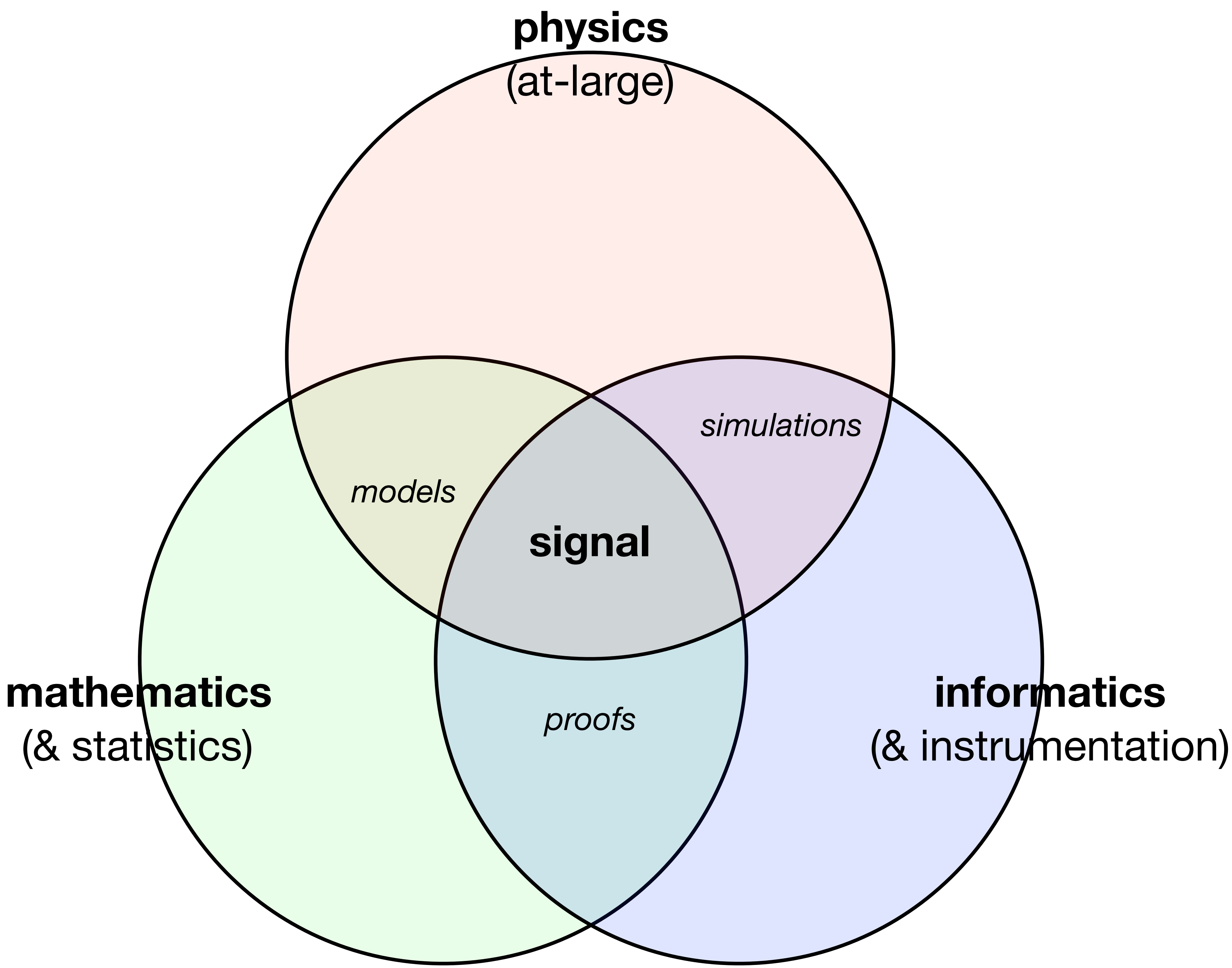
physics
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simulations
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mathematics
(& statistics)
proofs
informatics
(& instrumentation)

signal goes nonstationary

signal

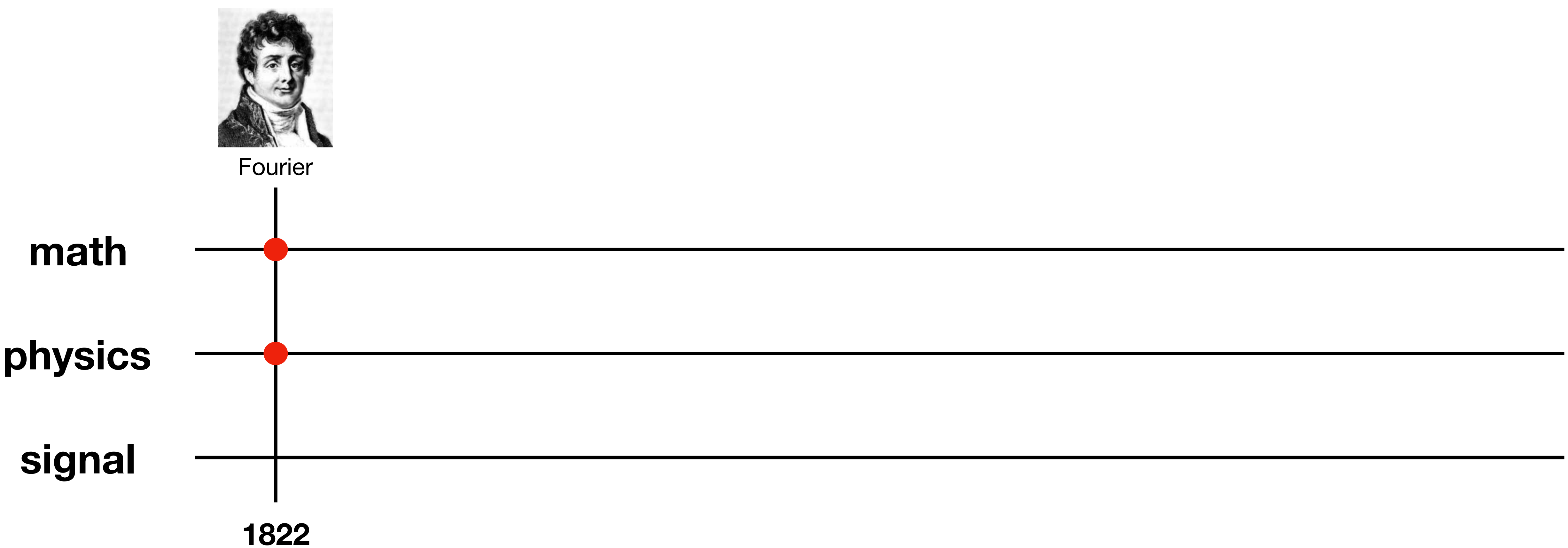
physics

signal

math

physics

signal


Fourier
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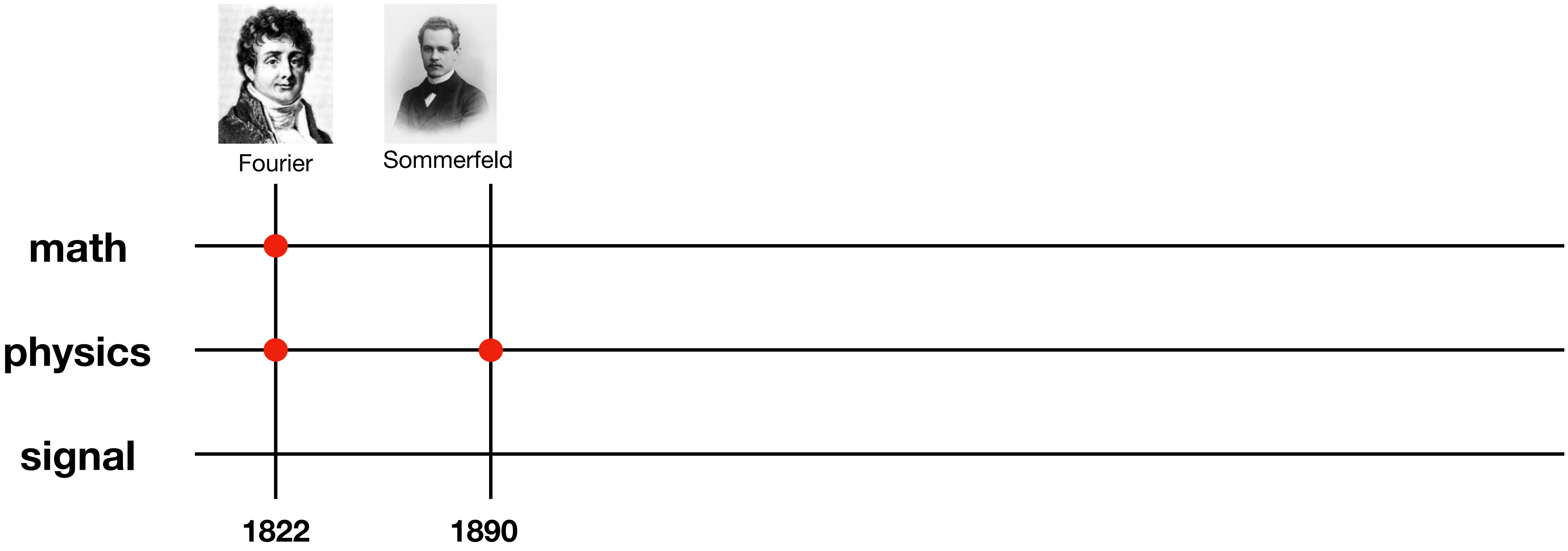
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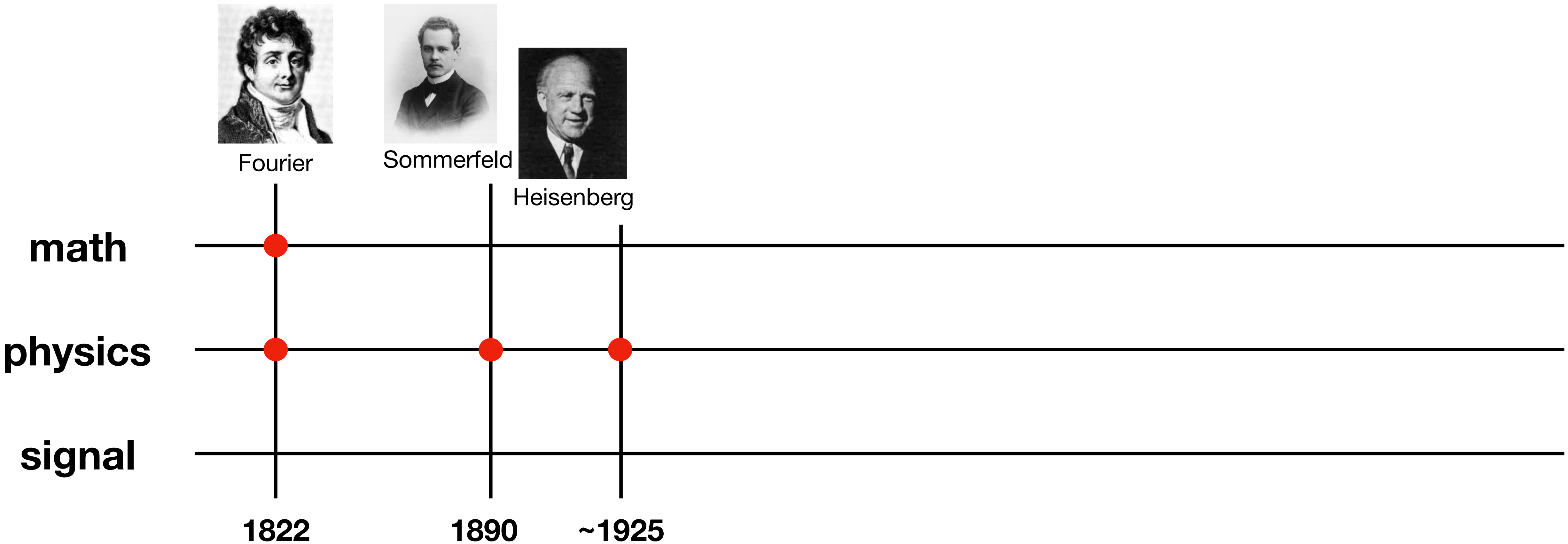
Fourier
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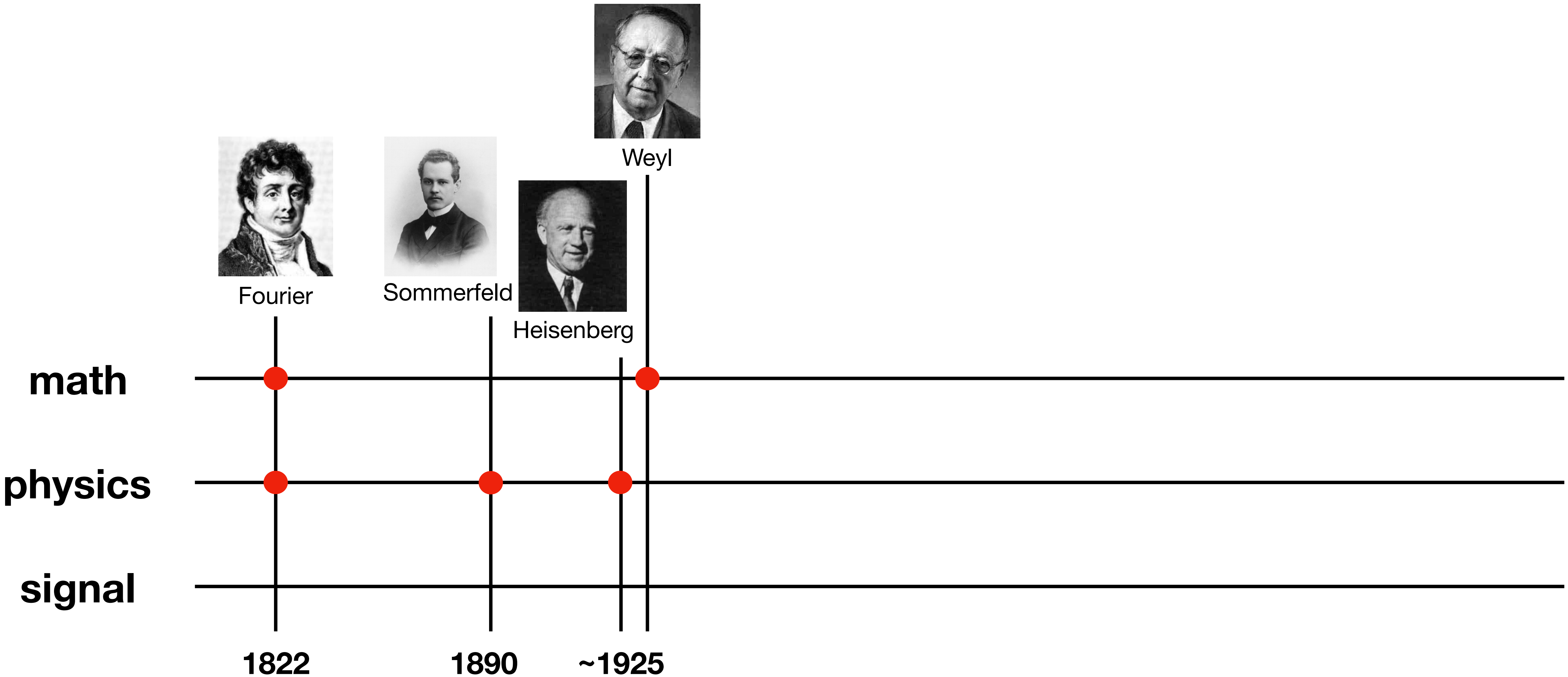
Weyl
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Heisenberg
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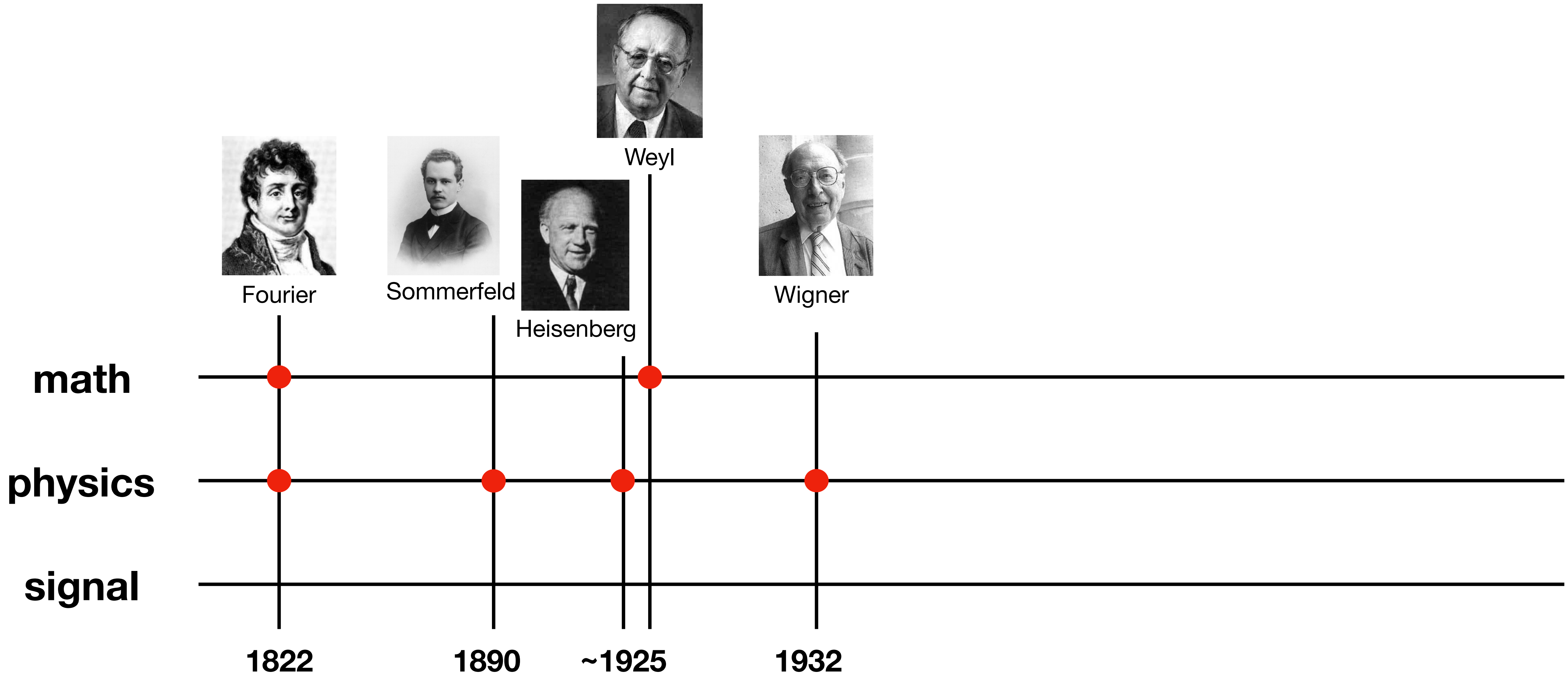
Weyl
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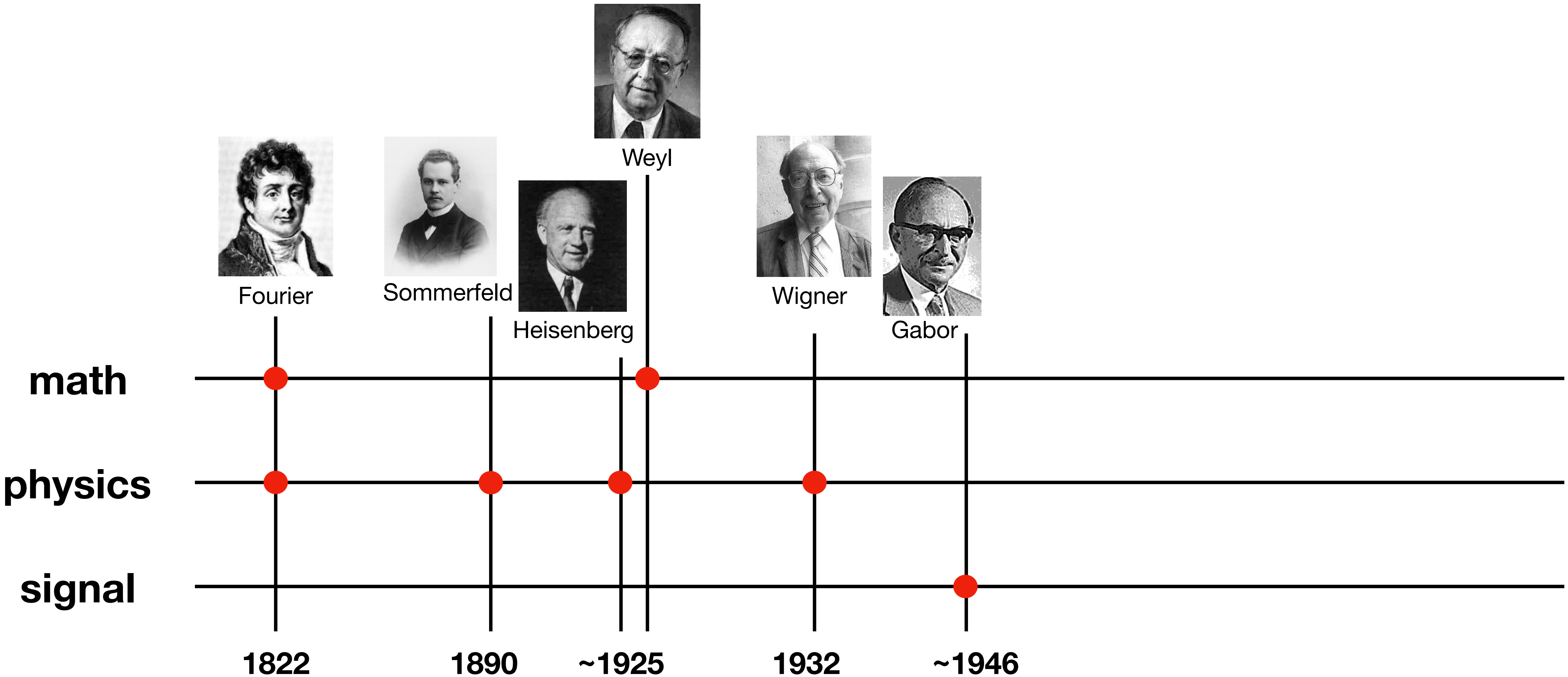

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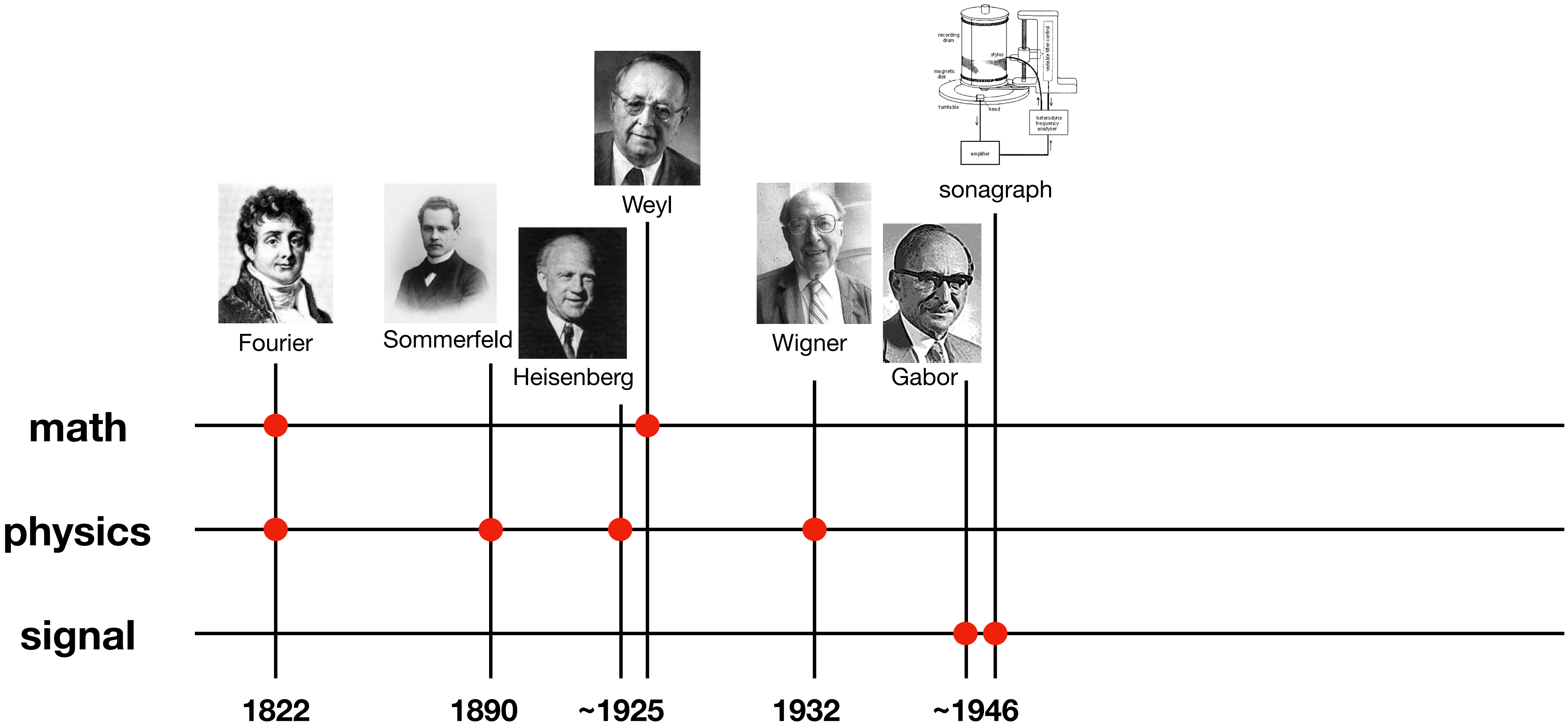
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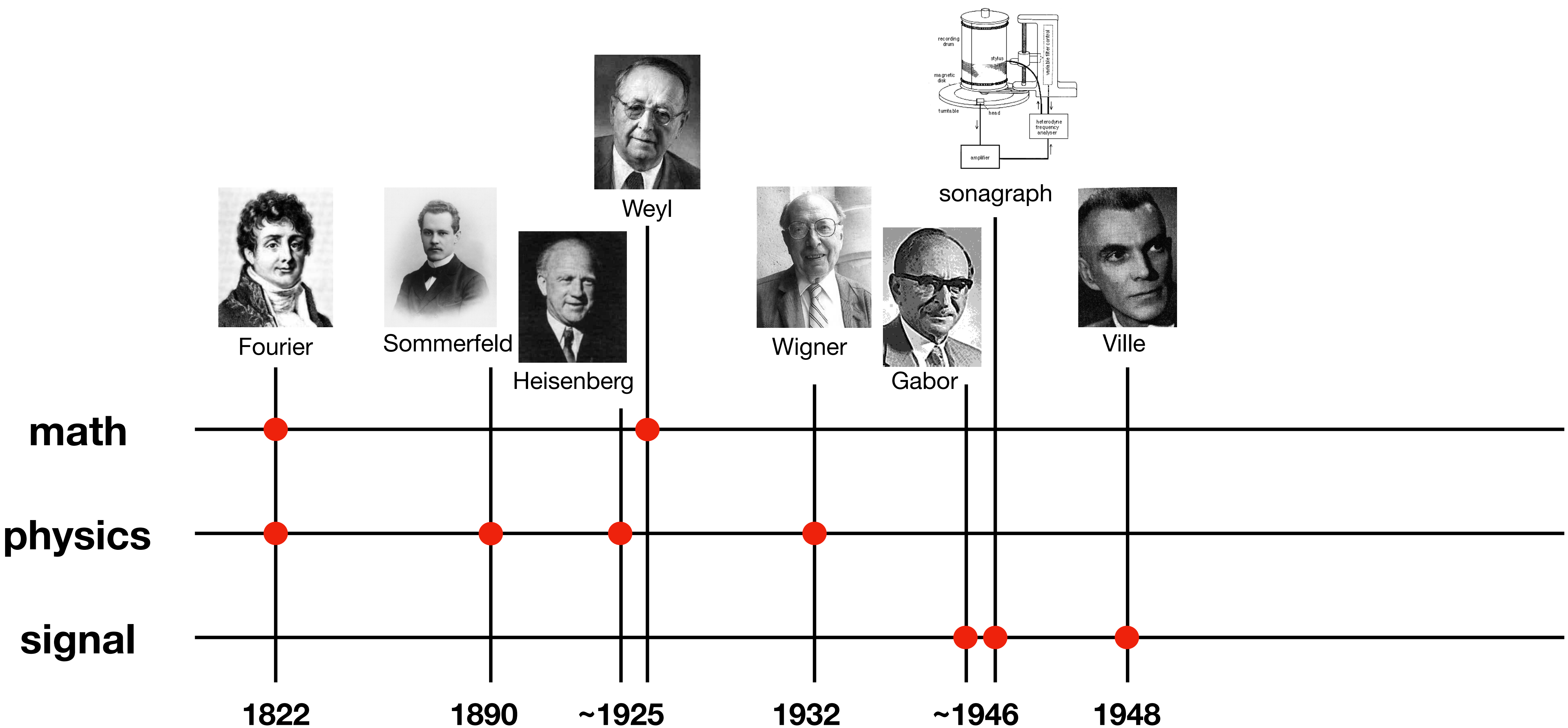
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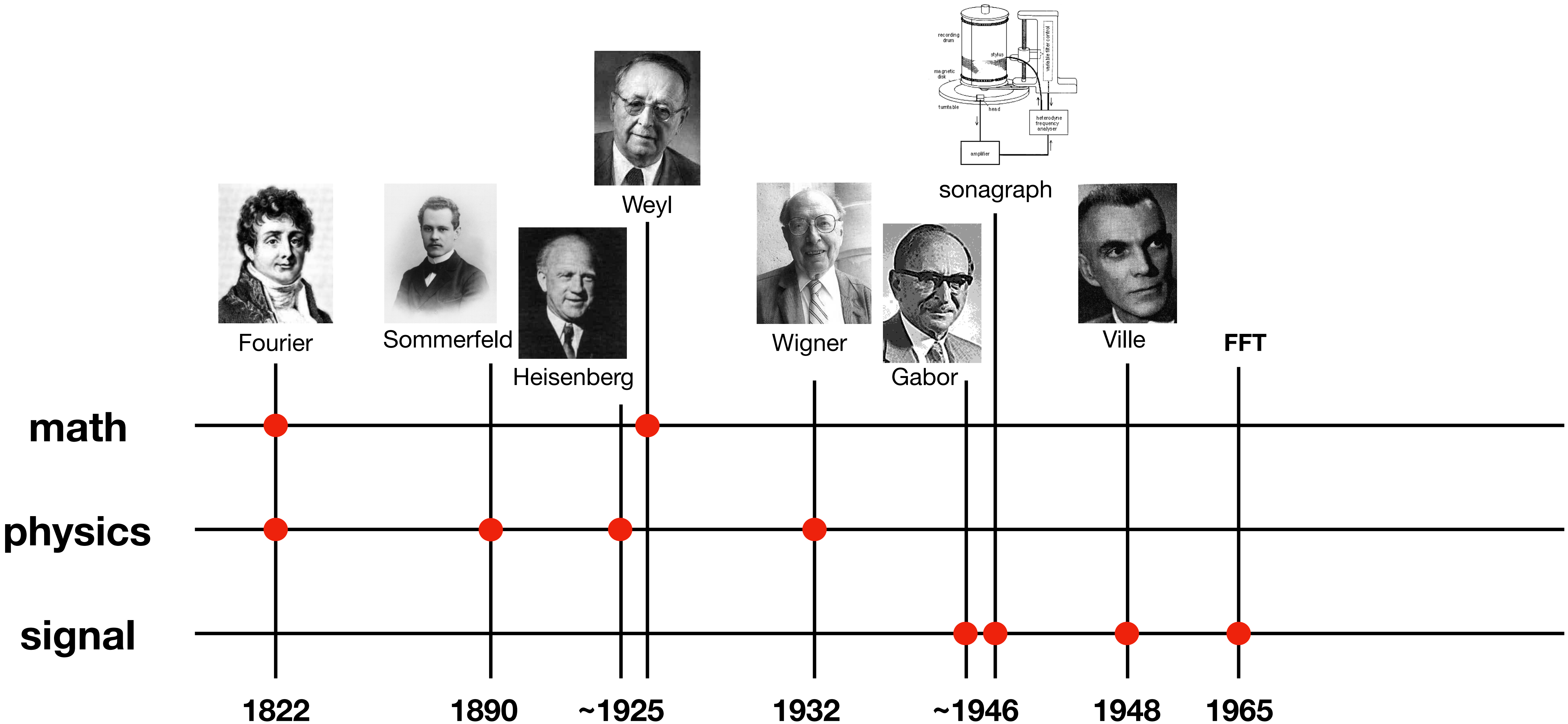

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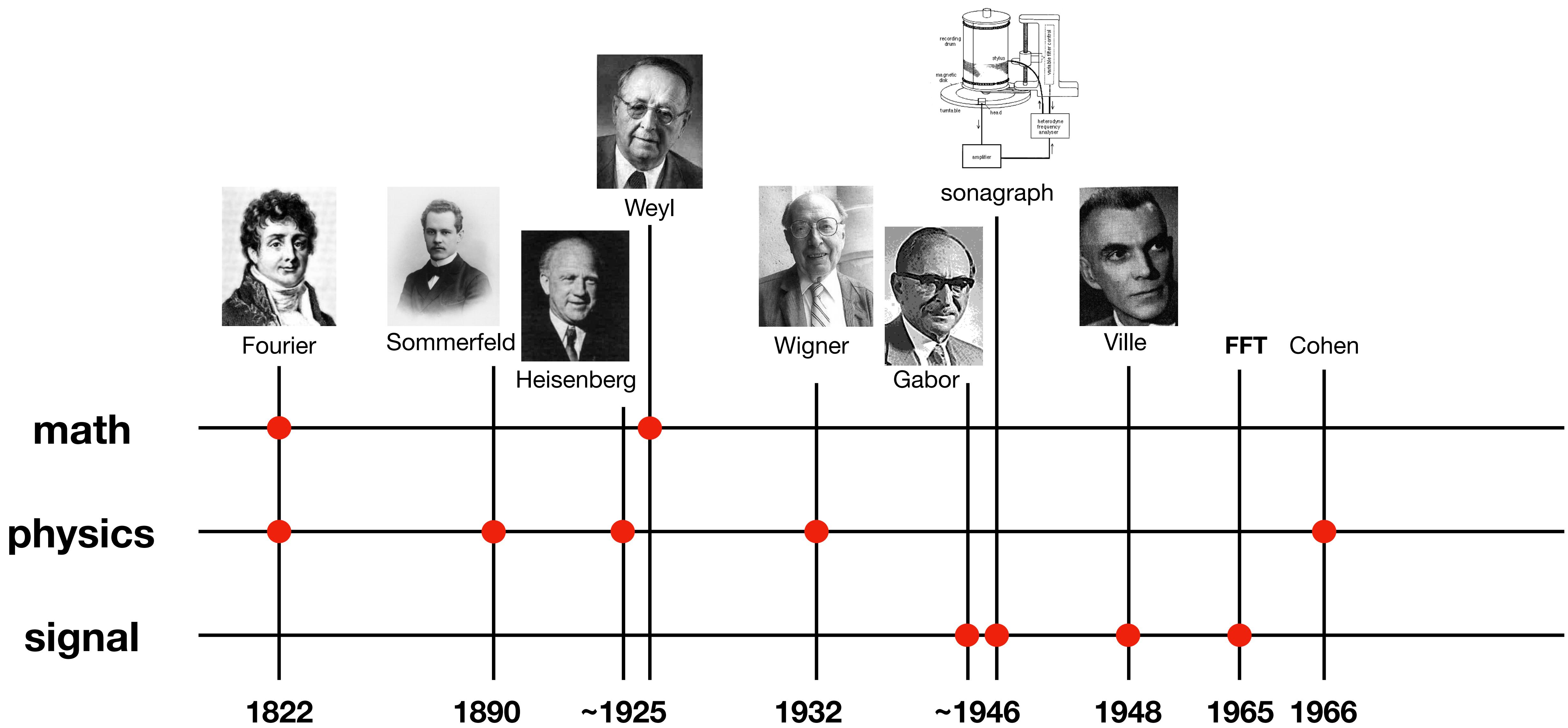
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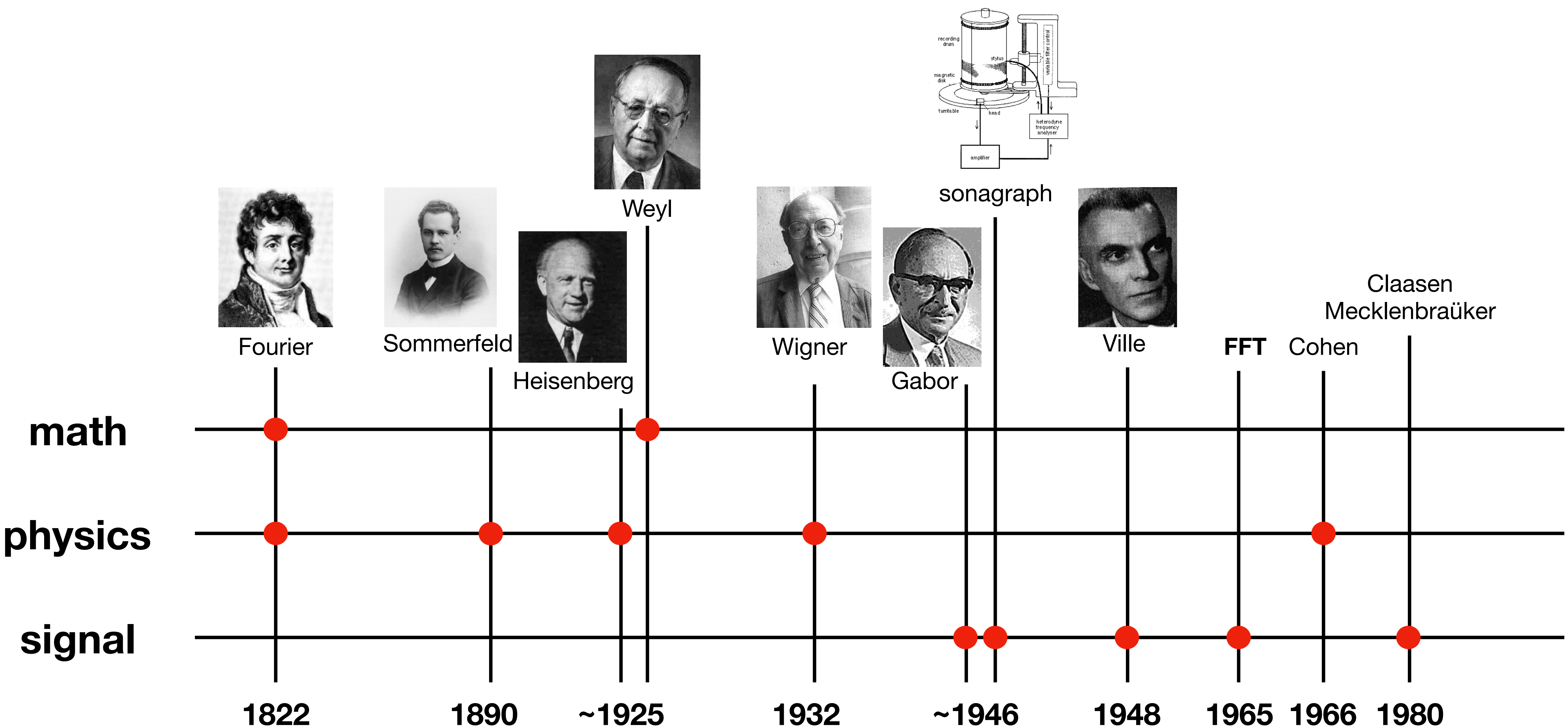

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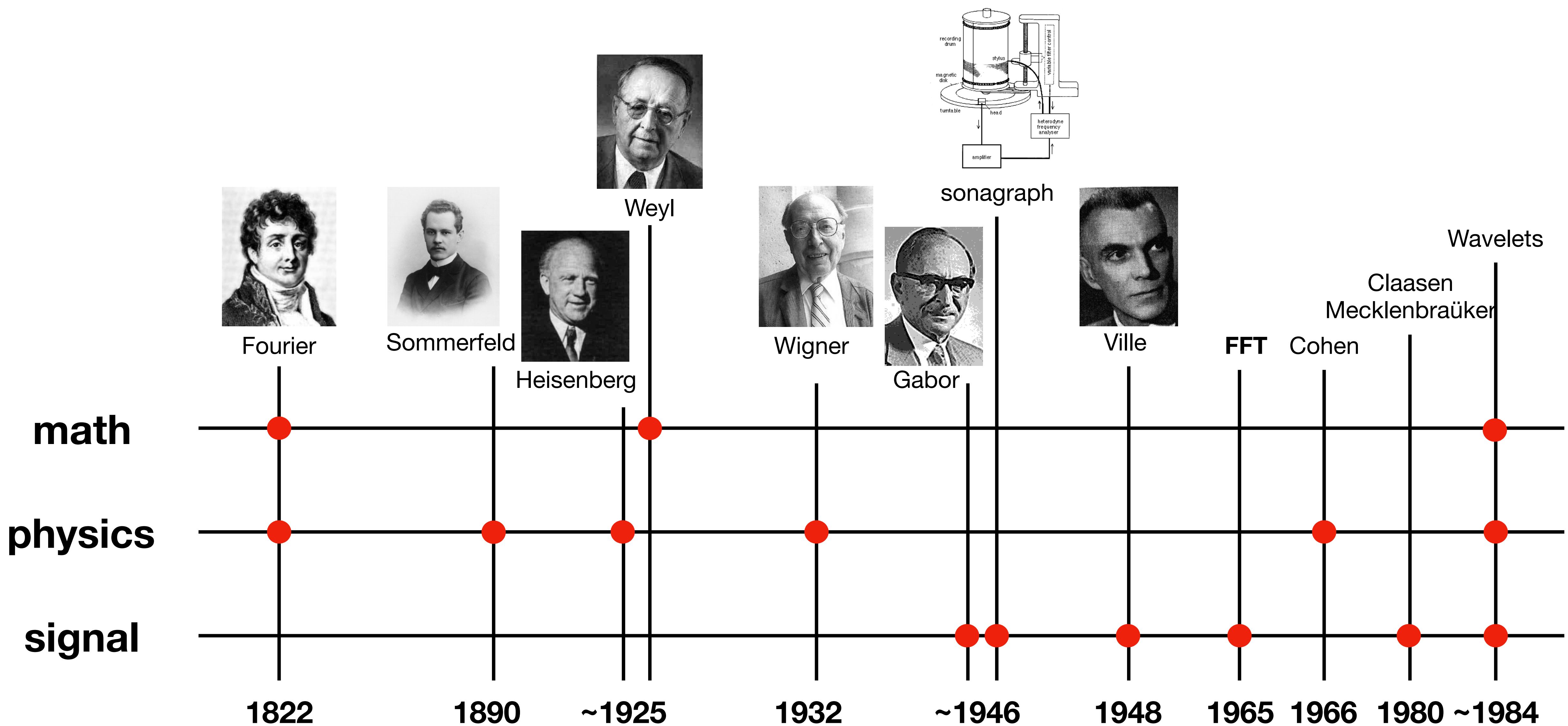

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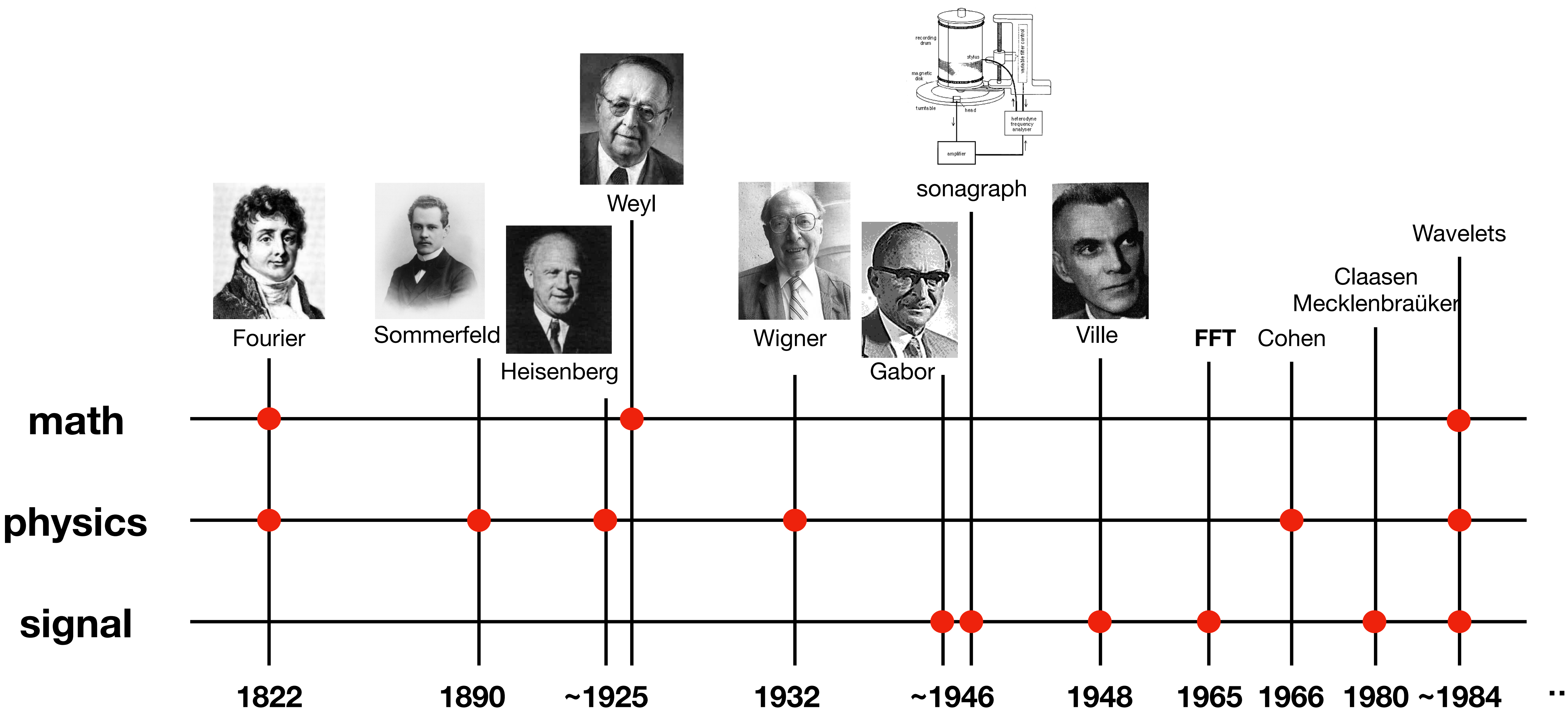

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