

# Gaussian Analytic Functions

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Let  $\mathbf{f}$  be a random analytic function on a region  $\Lambda \subset \mathbb{C}$ . If  $(\mathbf{f}(z_1), \mathbf{f}(z_2), \dots, \mathbf{f}(z_n))$  has mean 0, complex Gaussian distribution for every  $n \geq 1$  and every  $z_1, z_2, \dots, z_n \in \Lambda$ , then we say  $\mathbf{f}$  is **Gaussian analytic function (GAF)** on  $\Lambda$ .

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Examples:

1.  $\mathbf{p}(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$ , where  $(a_0, a_1, \dots, a_n)^t$  is a complex Gaussian vector.
2.  $\mathbf{f}(z) = \sum_{k \geq 0} X_k \frac{z^k}{\sqrt{k!}}$ , where  $X_k$  are i.i.d complex Gaussians



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- ▶ Random analytic functions appear in theoretical physics as Quantum Chaotic eigenstates (Bogomolny, Bohigas, Lebeouf, ...).
- ▶ GAFs appear in the study of spectrogram of signals.

# Zeros of GAF

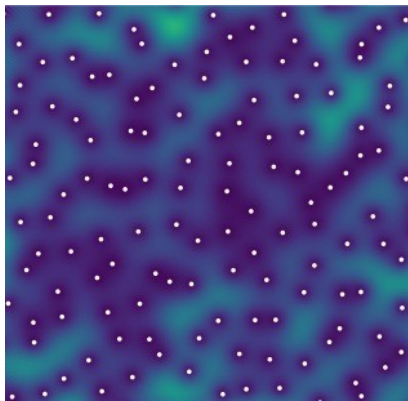


Figure: Planar GAF zeros

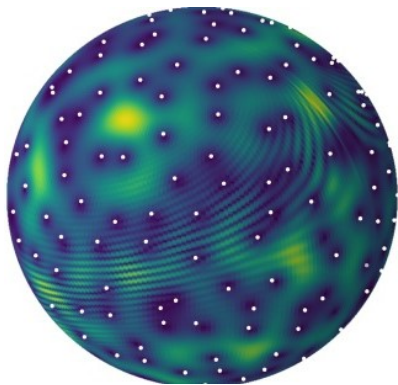


Figure: Spherical GAF zeros

## Lemma 1

Let  $p(z) = \prod_{k=1}^n (z - z_k)$  be a polynomial with coefficients of  $z^k$  as  $a_k$ ,  $0 \leq k \leq n - 1$ , then the transformation  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined by

$$T(z_1, z_2, \dots, z_n) = (a_{n-1}, \dots, a_0) \quad (1)$$

has Jacobian determinant  $\prod_{i < j} |z_i - z_j|^2$ .

# Proof of Lemma 1

- ▶ We need to compute the real Jacobian determinant which is equal to  $\left| \det \left( \frac{\partial T(z_1, z_2, \dots, z_n)}{\partial (z_1, z_2, \dots, z_n)} \right) \right|^2$  (generalizing from the case of one complex variable).

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- ▶ Write  $T_n(k) = a_{n-k} = (-1)^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} z_{i_1} z_{i_2} \dots z_{i_k}$ .

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- ▶ If  $z_i = z_j$  for some  $i \neq j$ , then  $i, j$ th columns of  $\left( \frac{\partial T(z_1, z_2, \dots, z_n)}{\partial (z_1, z_2, \dots, z_n)} \right)$  are equal and the determinant is divisible by  $(z_i - z_j)$ .

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- ▶ This implies  $\det \left( \frac{\partial T(z_1, z_2, \dots, z_n)}{\partial (z_1, z_2, \dots, z_n)} \right) = C_n \times \prod_{i < j} (z_i - z_j)$ .

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- ▶  $C_n$  can be computed to be  $(-1)^{n(n+1)/2}$ .



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Weak limits of complex Gaussian are complex Gaussian.

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## Lemma 2

*Let  $\psi_n$  be holomorphic functions on  $\Lambda$ . Assume that  $\sum_n |\psi_n(z)|^2$  converges uniformly on compact sets in  $\Lambda$ . Let  $a_n$  be i.i.d random variables with zero mean and unit variance. Then, almost surely,  $f(z) = \sum_n a_n \psi_n(z)$  converges uniformly on compact sets of  $\Lambda$  and hence defines a random analytic function. If  $a_n$  are standard complex Gaussians, then  $f(z)$  is a GAF with covariance kernel  $K(z, w) = \sum_n \psi_n(z) \bar{\psi}_n(w)$ .*

## Proof of Lemma 2

For any compact set  $K$ , regard the sequence  $X_n = \sum_{k=1}^n a_k \psi_k$  as  $L^2(K)$  valued random variable.

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$$\begin{aligned}\mathbb{E}[\|X_n\|^2] &\geq \sum_{k=1}^n \mathbb{E}[\|X_n\|^2 \mathbf{1}_{\tau=k}] \\ &= \sum_{k=1}^n \mathbb{E}[\mathbf{1}_{\tau=k} \mathbb{E}[\|X_n\|^2 | a_j, j \leq k]] \\ &\geq \sum_{k=1}^n \mathbb{E}[\mathbf{1}_{\tau=k} \|X_k\|^2] \\ &\geq \epsilon^2 \mathbb{P}(\tau \leq n).\end{aligned}$$

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Hence  $\mathbb{P}(\sup_{j \leq n} \|X_j\| \geq \epsilon) \leq \frac{1}{\epsilon^2} \sum_{j=1}^n \|\psi_j\|^2$ .

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Applying the above bound to the sequence  $\{X_{N+n} - X_N\}_n$ , we get

$$\mathbb{P}(\exists N \text{ such that } \forall n, \|X_{N+n} - X_N\| \leq \epsilon) = 1.$$

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Use Cauchy's formula ( $X_n$  is analytic) to write for any  $z \in D(z_0, R)$ ,

$$\begin{aligned} X_n(z) &= \frac{1}{2\pi i R} \int_{2R}^{3R} \int_0^{2\pi} \frac{X_n(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - z} ie^{i\theta} d\theta r dr \\ &= \frac{1}{2\pi} \int_A X_n(\zeta) \phi_z(\zeta) dm(\zeta), \end{aligned}$$

where  $A$  is the annulus and  $\phi_z$  are defined by the equality.

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As  $\{\phi_z\}_{z \in D(z_0, R)}$  are uniformly bounded in  $L^2(A)$ , uniform convergence over compact sets follows.



# Isometries of domains

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- ▶ Complex plane  $\mathbb{C}$ : The group of transformations

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where  $|\lambda| = 1$  and  $\beta \in \mathbb{C}$ . These transformations preserve Euclidean metric  $ds^2 = dx^2 + dy^2$  and the Lebesgue measure  $dm(z) = dx dy$  on the plane.

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- ▶ The sphere  $\mathbb{S}^2$ (extended complex plane): The fractional linear transformations

$$\phi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}} \quad (3)$$

with  $\alpha, \beta \in \mathbb{C}$  and  $|\alpha|^2 + |\beta|^2 = 1$ .

# Isometries of domains

$|\phi'(z)| = \frac{1}{|-\bar{\beta}z + \bar{\alpha}|^2}$  and  $\frac{|\phi'(z)|}{1+|\phi(z)|^2} = \frac{1}{1+|z|^2}$ . This shows that the spherical metric  $ds^2 = \frac{dx^2 + dy^2}{(1+|z|^2)^2}$  and spherical measure  $\frac{dm(z)}{(1+|z|^2)^2}$  are preserved by  $\phi_{\alpha,\beta}$ .

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where  $a_n$  are i.i.d standard complex Gaussians.  $f$  is random analytic in the entire plane with covariance kernel  $\exp(Lz\bar{w})$ .



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- ▶ The sphere  $\mathbb{S}^2$  : For  $L \in \mathbb{N}$ ,

$$f(z) = \sum_{n=0}^L a_n \frac{\sqrt{L(L-1)\dots(L-n+1)}}{\sqrt{n!}} z^n \quad (6)$$

where  $a_n$  are i.i.d standard complex Gaussians.  $f$  is random analytic (polynomial) in the entire plane with covariance kernel  $(1 + z\bar{w})^L$ .

- ▶ Hyperbolic plane  $\mathbb{D}$  : For  $L > 0$ ,

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{\sqrt{L(L+1)\dots(L+n-1)}}{\sqrt{n!}} z^n \quad (7)$$

where  $a_n$  are i.i.d standard complex Gaussians.  $f$  is random analytic in the unit disk  $\mathbb{D}$  with covariance kernel  $(1 - z\bar{w})^{-L}$ . For non-integer  $L$ , branch of fractional power is taken such that  $K(z, z)$  is positive.

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### Theorem 3

*The zero sets of GAF in equations (5), (6), (7) are invariant (in distribution) under transformations defined in (2), (3), (4).*

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- ▶ As  $f$  and  $h$  have the same zeros, zeros of  $f, g$  have same distribution.

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**First Intensity by Green's formula.** Let  $\phi$  be any smooth function compactly supported in  $\Lambda$ .

$f(z) = g(z) \prod_k (z - a_k)^{m_k}$ , where  $a_k$  are the zeros of  $f$  that are in the support of  $\phi$  and  $g$  is analytic with no zeros in support of  $\phi$ .

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$f(z) = g(z) \prod_k (z - a_k)^{m_k}$ , where  $a_k$  are the zeros of  $f$  that are in the support of  $\phi$  and  $g$  is analytic with no zeros in support of  $\phi$ .

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This gives

$$\int_{\Lambda} \Delta \phi(z) \frac{1}{2\pi} \log |z - a_k| dm(z) = \phi(a_k) \quad (8)$$

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$$\begin{aligned} \mathbb{E} \left[ \int_{\Lambda} \phi(z) dn_f(z) \right] &= \mathbb{E} \left[ \int_{\Lambda} \Delta\phi(z) \frac{1}{2\pi} \log |f(z)| dm(z) \right] \\ &= \int_{\Lambda} \Delta\phi(z) \frac{1}{2\pi} \mathbb{E} [|\log |f(z)||] dm(z) \\ &= \int_{\Lambda} \phi(z) \frac{1}{2\pi} \Delta \mathbb{E} [|\log |f(z)||] dm(z) \quad (10) \end{aligned}$$

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The second step is justified by Fubini's theorem.

# First Intensity Computation

Use the fact that  $\frac{f(z)}{\sqrt{K(z,z)}}$  is standard complex Gaussian to see that

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We have proved the **Edelman-Kostlan** formula.

## Theorem 4

Let  $f$  be a GAF on a domain  $\Lambda \in \mathbb{C}$ . Let  $n_f$  denote the counting measure of zero set of  $f$ ,  $\mu$  be the expectation of  $n_f$ , i.e.,  $\mu(A) = \mathbb{E}[n_f(A)]$ . Let  $\phi \in C_c^2(\Lambda)$  be a test function with compact support in  $\Lambda$ . Then for every  $\lambda > 0$ ,

$$\mathbb{P} \left[ \left| \int_{\Lambda} \phi (dn_f - d\mu) \right| \geq \lambda \right] \leq 3 \exp \left( \frac{-\pi \lambda}{\|\Delta \phi\|_{L^1}} \right)$$



# An Offord type estimate

We use the following lemma

## Lemma 5

*Let  $a$  be a complex Gaussian random variable with zero mean and variance  $\sigma^2$ . Then for any event  $E$  in the probability space, we have*

$$|\mathbb{E}[1_E \log |a|] - \mathbb{P}(E) \log(\sigma)| \leq \mathbb{P}(E) \left[ 2 \log \frac{1}{\mathbb{P}(E)} + \frac{\mathbb{P}(E)}{2} \right].$$

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**Upper bound:** w.l.o.g assume  $\sigma = 1$ . Using Jensen's inequality  $\mathbb{E}[\log |a|^2 | E] \leq \log \mathbb{E}[|a|^2 | E]$ .

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$$\frac{1}{\mathbb{P}(E)} \mathbb{E}[1_E \log |a|^2] \leq \log\left(\frac{1}{\mathbb{P}(E)}\right).$$

Thus  $\mathbb{E}[1_E \log |a|] \leq -\frac{1}{2} \mathbb{P}(E) \log \mathbb{P}(E)$ .

# An Offord type estimate

**Lower bound:** Let  $\log^- x = -\min(0, \log x)$ . Then

$$\mathbb{E}[\log |a| 1_E] \geq -\mathbb{E}[\log^- |a| 1_{E \cap \{|a| \leq \mathbb{P}(E)\}}] - \mathbb{E}[\log^- |a| 1_{E \cap \{|a| > \mathbb{P}(E)\}}].$$

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Bounding the first term by  $-2\mathbb{P}(E) \log(\frac{1}{\mathbb{P}(E)}) - \frac{1}{2}\mathbb{P}(E)^2$  and the second term by  $-\mathbb{P}(E) \log(\frac{1}{\mathbb{P}(E)})$  finishes the proof of Lemma 5.

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**Proof.**

Use (9) to get

$$\int_{\Lambda} \phi(dn_f - d\mu) = \int_{\Lambda} \Delta\phi(z) \frac{1}{2\pi} \{\log |f(z)| - \log \sqrt{K(z, z)}\} dm(z)$$

Using the above equality, Markov's inequality and applying Lemma 5 makes the proof of Theorem 4 immediate.  $\square$

## Theorem 6 (Wick / Feynman diagram formula)

Let  $(a, b) = (a_1, \dots, a_n, b_1, \dots, b_n)^t$  have  $N_{\mathbb{C}}(0, \Sigma)$  distribution, where

$$\Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{bmatrix}$$

Then,

$$E [a_1 \cdots a_n \bar{b}_1 \cdots \bar{b}_n] = \text{per}(\Sigma_{1,2})$$

In particular

$$E [ |a_1 \cdots a_n|^2 ] = \text{per}(\Sigma_{1,1})$$

For  $\{z_i\}_{i=1}^n, \{w_j\}_{j=1}^n \subset \mathbb{C}$ , we may set  $a_i = f(z_i)$  and  $b_j = f(w_j)$ .



# Kac-Rice Formula

The Kac-Rice formula can be used to compute the correlation functions of the zeros of any Gaussian random field.

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## Lemma 7 (Co-Area formula)

Let  $F : \mathbb{C}^k \mapsto \mathbb{C}^k$  be a function with continuous derivative. For any continuous function  $\varphi$  with compact support and any bounded  $B \subset \mathbb{C}^k$  we have

$$\int_B \left( \sum_{x \in F^{-1}(y)} \varphi(x) \right) dy = \int_{\mathbb{C}^k} \varphi(x) |F'(x)|^2 1_B(F(x)) dx.$$

## Proof.

Heuristically : Since  $x \in F^{-1}(y)$ , we “put  $y = F(x)$ ” on the left hand side ! □

## Theorem 8

Let  $f$  be a Gaussian analytic function on  $\Lambda$  with covariance kernel  $K$ . If  $\det K(z_i, z_j)_{i,j \leq k}$  does not vanish anywhere on  $\Lambda$ , then the  $k$ -point intensity function  $\rho_k$  exists and is given by

$$\begin{aligned} \rho_k(z_1, \dots, z_k) &= \frac{E \left[ |f'(z_1) \cdots f'(z_k)|^2 \mid f(z_1) = \dots = f(z_k) = 0 \right]}{\pi^k \det K(z_i, z_j)_{i,j \leq k}} \\ &= \frac{\text{per}(C - BA^{-1}B^*)}{\det(\pi A)}. \end{aligned}$$

where  $A, B, C$  are  $k \times k$  matrices defined by

$$\begin{aligned} A(i, j) &= E[f(z_i) \bar{f}(z_j)] \\ B(i, j) &= E[f'(z_i) \bar{f}(z_j)] \\ C(i, j) &= E[f'(z_i) \bar{f}'(z_j)]. \end{aligned}$$

## Definition 9

Fix a window function  $\phi \in L^2(\mathbb{R})$ . The short-time Fourier transform (STFT) of a function  $f \in L^2(\mathbb{R})$  with respect to  $\phi$  is defined as

$$V_\phi f(u, v) := \langle f, M_v T_u \phi \rangle, \quad u, v \in \mathbb{R}, \quad (11)$$

where  $M_v f = e^{2i\pi v \cdot} f(\cdot)$  and  $T_u f = f(\cdot - u)$ .

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Significant setting : Gaussian window  $g(t) = 2^{1/4} e^{-\pi t^2}$ ;  $\|g\|_2 = 1$ .

## Definition 10 (Bargmann transform)

The Bargmann transform of a function  $f$  on  $\mathbb{R}$  is the function  $Bf$  on  $\mathbb{C}$  defined by

$$Bf(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi t^2 - \frac{\pi}{2} z^2} dt.$$

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## Lemma 11

For  $f \in L^2(\mathbb{R})$ , it holds that

$$V_g f(u, -v) = e^{\pi i uv} e^{-\frac{\pi}{2} z^2} Bf(z),$$

where the complex number  $z$  is taken as  $z = u + iv$ .

# Time Frequency analysis and STFT

Some functions are known to have a simple closed-form Bargmann transform, such as the Hermite functions. Let  $\{H_k\}$  be Hermite polynomials



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which are orthonormal in  $L^2(\mathbb{R})$  with respect to the Gaussian window  $g$ , i.e.,

$$\int H_k(x) H_l(x) g(x) dx = \delta_{kl}, \quad k, l \geq 0,$$

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$$h_k(x) = \frac{(-1)^k}{2^{2k-1/2}\pi^k k!} e^{\pi x^2} \frac{d^k}{dx^k} \left( e^{-2\pi x^2} \right), \quad k = 0, 1, 2, \dots \quad (12)$$

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The Hermite functions  $\{h_k\}$  form an orthonormal basis of  $L^2(\mathbb{R})$ .

# Time Frequency analysis and STFT

The Bargmann transform of the Hermite function  $h_k$  is in a quite simple form as

$$Bh_k(z) = \frac{\pi^{k/2} z^k}{\sqrt{k!}}, \quad k = 0, 1, 2, \dots .$$

Based on the above formula and Lemma 11, we can see that the STFT of an Hermite function  $h_k$  with respect to the Gaussian window  $g$  could be computed via the following proposition.

## Lemma 12

*For any  $k \in \mathbb{N}$ , it holds that*

$$V_g h_k(u, v) = e^{-\pi i u v} e^{-\frac{\pi}{2} |z|^2} \frac{\pi^{k/2} \bar{z}^k}{\sqrt{k!}},$$

*where the complex number  $z$  is taken as  $z = u + iv$ .*

# Time Frequency analysis and STFT

Let  $\xi$  be Gaussian white noise on in  $L^2(\mathbb{R})$ . Then we have :

## Theorem 13

Let  $u, v \in \mathbb{R}$  and write  $z = u + iv \in \mathbb{C}$ . Then

$$V_g[\xi](z) := \psi[\xi](u, v) = \sqrt{\pi} e^{i\pi uv} e^{-\frac{\pi}{2}|z|^2} \sum_{k=0}^{\infty} \langle \xi, h_k \rangle \frac{\pi^{k/2} z^k}{\sqrt{k!}}, \quad (13)$$

where  $\{h_k\}$  are the orthonormal Hermite functions, and convergence is in  $L^2(\mu_1)$ . In addition, the random series

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- ▶  $z \in \mathbb{C}$  will be viewed as the stereographic projection of the Riemann sphere.
- ▶ Hilbert space  $\mathcal{H}$  consists of polynomials  $\psi$  of the form

$$\psi(z) = \sum_{m=-s}^s \sqrt{\binom{2s}{s+m}} a_m z^{s+m}.$$

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- ▶ Similar parametrization by the zeros can be applied to other spaces of entire functions; in particular for the case of 2D toric phase space, quantum Hall effect, and also in 2D electrons in a periodic magnetic field.
- ▶ Approach : study the evolution of the quantum system through the dynamics of a system of  $N$  particles lying in phase space.
- ▶ When the corresponding classical system is chaotic, the particles tend to fill the whole phase space in a uniform way.

# Quantum chaotic eigenstates

- ▶ “Chaotic eigenfunctions in phase space”, Nonnenmacher, S., & Voros, A. (1998). *Journal of Statistical Physics*, 92(3), 431-518.
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- ▶ “The chaotic analytic function”, Hannay, J. H., (1998) *Journal of Physics A: Mathematical and General* 31, no. 49 : L755.
- ▶ Similar to Wigner’s ansatz for studying spectra for studying spectra of heavy nuclei through random matrix eigenvalues, we study complex quantum (chaotic) systems through zeros of random polynomials.
- ▶ Random matrix : Random polynomial :: Random eigenvalues : Random zeros

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- ▶ This suggests a functional form  $K(z, z) = C \cdot \exp(z\bar{z} + \alpha z + \beta \bar{z})$ .

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# Quantum chaotic eigenstates

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- ▶ Holomorphicity requirements on  $K$  suggests a functional form  $K(z, w) = C \cdot \exp(z\bar{w})$ , which brings us back to the standard planar GAF !

- ▶ Hough, John Ben, Manjunath Krishnapur, and Yuval Peres. Zeros of Gaussian analytic functions and determinantal point processes. Vol. 51. American Mathematical Soc., 2009.