Gaussian Analytic Functions

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Definition, Examples

Let $\mathbf{f}$ be a random analytic function on a region $\Lambda \subset \mathbb{C}$. If $(\mathbf{f}(z_1), \mathbf{f}(z_2) \ldots, \mathbf{f}(z_n))$ has mean 0, complex Gaussian distribution for every $n \geq 1$ and every $z_1, z_2, \ldots, z_n \in \Lambda$, then we say $\mathbf{f}$ is **Gaussian analytic function**(GAF) on Λ.

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Examples:

1. $\mathbf{p}(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $(a_0, a_1, \ldots, a_n)^t$ is a complex Gaussian vector.
2. $\mathbf{f}(z) = \sum_{k \geq 0} X_k \frac{z^k}{\sqrt{k!}}$, where X_k are i.i.d complex Gaussians

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- The zeros of GAF with properties like ergodicity, translation invariance local repulsion makes it similar to other natural models in point processes, physics.
- Random analytic functions appear in theoretical physics as Quantum Chaotic eigenstates (Bogomolny, Bohigas, Lebeouf, ...).
- GAFs appear in the study of spectrogram of signals.

Zeros of GAF

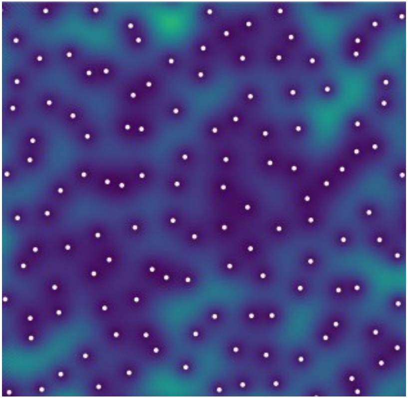

Figure: Planar GAF zeros

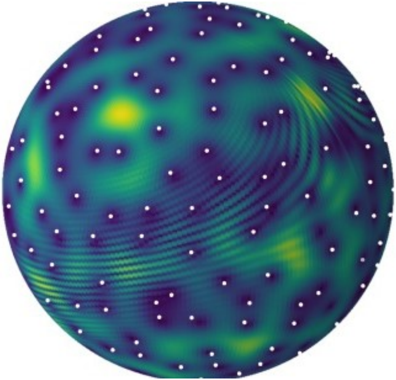

Figure: Spherical GAF zeros

Change of variables

Lemma 1

Let $p(z) = \prod_{k=1}^{n}(z - z_k)$ *be a polynomial with coefficients of* z^k *as* $a_k, 0 \leq k \leq n-1$*, then the transformation* $T : \mathbb{C}^n \to \mathbb{C}^n$ *defined by*

$$T(z_1, z_2 \ldots, z_n) = (a_{n-1}, \ldots, a_0) \tag{1}$$

has Jacobian determinant $\prod_{i<j} |z_i - z_j|^2$.

Proof of Lemma 1

- We need to compute the real Jacobian determinant which is equal to $\left|\det\left(\frac{\partial T(z_1,z_2,\ldots,z_n)}{\partial(z_1,z_2,\ldots,z_n)}\right)\right|^2$(generalizing from the case of one complex variable).

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- Write $T_n(k) = a_{n-k} = (-1)^k \sum_{1\le i_1<i_2\cdots<i_k\le n} z_{i_1} z_{i_2} \dots z_{i_k}.$

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- If $z_i = z_j$ for some $i \neq j$, then i, jth columns of $\left(\frac{\partial T(z_1,z_2,\ldots,z_n)}{\partial(z_1,z_2,\ldots,z_n)}\right)$ are equal and the determinant is divisible by $(z_i - z_j)$.

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- This implies $\det\left(\frac{\partial T(z_1,z_2,\ldots,z_n)}{\partial(z_1,z_2,\ldots,z_n)}\right) = C_n \times \prod_{i<j}(z_i - z_j)$.

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- C_n can be computed to be $(-1)^{n(n+1)/2}$.

Complex Gaussian Distribution

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Weak limits of complex Gaussian are complex Gaussian.

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Lemma 2

Let ψ_n be holomorphic functions on Λ. Assume that $\sum_n |\psi_n(z)|^2$ converges uniformly on compact sets in Λ. Let a_n be i.i.d random variables with zero mean and unit variance. Then, almost surely, $f(z) = \sum_n a_n \psi_n(z)$ converges uniformly on compact sets of Λ and hence defines a random analytic function. If a_n are standard complex Gaussians, then f(z) *is a GAF with covariance kernel $K(z, w) = \sum_n \psi_n(z) \bar{\psi}_n(w)$.*

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$$\begin{aligned}
\mathbb{E}[\|X_n\|^2] &\geq \sum_{k=1}^{n} \mathbb{E}[\|X_n\|^2 1_{\tau=k}] \\
&= \sum_{k=1}^{n} \mathbb{E}\left[1_{\tau=k}\mathbb{E}[\|X_n\|^2 | a_j, j \leq k]\right] \\
&\geq \sum_{k=1}^{n} \mathbb{E}[1_{\tau=k}\|X_k\|^2] \\
&\geq \epsilon^2 \mathbb{P}(\tau \leq n).
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Hence $\mathbb{P}(\sup_{j\leq n}\|X_j\| \geq \epsilon) \leq \frac{1}{\epsilon^2}\sum_{j=1}^{n}\|\psi_j\|^2$.

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Applying the above bound to the sequence $\{X_{N+n} - X_N\}_n$, we get

$$\mathbb{P}(\exists N \text{ such that } \forall n, \|X_{N+n} - X_N\| \leq \epsilon) = 1.$$

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Thus almost surely X_n converges.
Use Cauchy's formula(X_n is analytic) to write for any $z \in D(z_0, R)$,

$$\begin{aligned} X_n(z) &= \frac{1}{2\pi i R} \int_{2R}^{3R} \int_{0}^{2\pi} \frac{X_n(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - z} ie^{i\theta} d\theta r dr \\ &= \frac{1}{2\pi} \int_{A} X_n(\zeta) \phi_z(\zeta) dm(\zeta), \end{aligned}$$

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where A is the annulus and ϕ_z are defined by the equality.
As $\{\phi_z\}_{z \in D(z_0,R)}$ are uniformly bounded in $L^2(A)$, uniform convergence over compact sets follows.

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- Complex plane $\mathbb{C}$: The group of transformations

$$\phi_{\lambda,\beta}(z) = \lambda z + \beta \tag{2}$$

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- The sphere $\mathbb{S}^2$(extended complex plane): The fractional linear transformations

$$\phi_{\alpha,\beta}(z) = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}} \tag{3}$$

  with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

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- The sphere $\mathbb{S}^2$: For $L \in \mathbb{N}$,

$$f(z) = \sum_{n=0}^{L} a_n \frac{\sqrt{L(L-1)\dots(L-n+1)}}{\sqrt{n!}} z^n \tag{6}$$

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Isometry-invariant zero sets

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Theorem 3

The zero sets of GAF in equations (5), (6), (7) *are invariant (in distribution) under transformations defined in* (2), (3), (4)*.*

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- $$\begin{aligned} K_g(z,w) &= K_f(\lambda z + \beta, \lambda w + \beta) \\ &= \exp(Lz\bar{w} + Lz\lambda\bar{\beta} + L\bar{w}\bar{\lambda}\beta + L|\beta|^2) \end{aligned}$$

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- As f and h have the same zeros, zeros of f, g have same distribution.

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This gives

$$\int_\Lambda \Delta\phi(z) \frac{1}{2\pi} \log|z - a_k| dm(z) = \phi(a_k) \tag{8}$$

First Intensity Computation

This implies

$$\int_{\Lambda} \Delta\phi(z) \frac{1}{2\pi} \log|f(z)| dm(z) = \int_{\Lambda} \phi(z) dn_f(z) \tag{9}$$

First Intensity Computation

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Now for random analytic function **f**, we get

$$\begin{aligned}\mathbb{E}\left[\int_{\Lambda} \phi(z)dn_f(z)\right] &= \mathbb{E}\left[\int_{\Lambda} \Delta\phi(z)\frac{1}{2\pi}\log|f(z)|dm(z)\right] \\ &= \int_{\Lambda} \Delta\phi(z)\frac{1}{2\pi}\mathbb{E}\left[\log|f(z)|\right]dm(z) \\ &= \int_{\Lambda} \phi(z)\frac{1}{2\pi}\Delta\mathbb{E}\left[\log|f(z)|\right]dm(z) \qquad (10)\end{aligned}$$

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\end{aligned}$$

The second step is justified by Fubini's theorem.

First Intensity Computation

Use the fact that $\frac{f(z)}{\sqrt{K(z,z)}}$ is standard complex Gaussian to see that

$$\mathbb{E}\left[\log|f(z)|\right] = \mathbb{E}\left[\log|a|\right] + \frac{1}{2}\log K(z,z)$$

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We have proved the **Edelman-Kostlan** formula.

An Offord type estimate

Theorem 4

Let f *be a GAF on a domain* $\Lambda \in \mathbb{C}$. *Let* n_f *denote the counting measure of zero set of* f, μ *be the expectation of* n_f, *i.e,* $\mu(A) = \mathbb{E}[n_f(A)]$. *Let* $\phi \in C_c^2(\Lambda)$ *be a test function with compact support in* Λ. *Then for every* $\Lambda > 0$,

$$\mathbb{P}\left[\left|\int_\Lambda \phi(dn_f - d\mu)\right| \geq \lambda\right] \leq 3\exp\left(\frac{-\pi\lambda}{\|\Delta\phi\|_{L^1}}\right)$$

An Offord type estimate

We use the following lemma

Lemma 5

Let a be a complex Gaussian random variable with zero mean and variance σ^2. Then for any event E in the probability space, we have

$$\left|\mathbb{E}[1_E \log |a|] - \mathbb{P}(E)\log(\sigma)\right| \le \mathbb{P}(E)\left[2\log\frac{1}{\mathbb{P}(E)} + \frac{\mathbb{P}(E)}{2}\right].$$

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Upper bound: w.l.o.g assume $\sigma = 1$. Using Jensen's inequality $\mathbb{E}[\log |a|^2 | E] \leq \log \mathbb{E}[|a|^2 | E]$.

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Thus $\mathbb{E}[1_E \log |a|] \leq -\frac{1}{2}\mathbb{P}(E)\log \mathbb{P}(E)$.

An Offord type estimate

Lower bound: Let $\log^- x = -\min(0, \log x)$. Then

$$\mathbb{E}[\log|a|1_E] \geq -\mathbb{E}[\log^-|a|1_{E\cap\{|a|\leq\mathbb{P}(E)\}}] - \mathbb{E}[\log^-|a|1_{E\cap\{|a|>\mathbb{P}(E)\}}].$$

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Bounding the first term by $-2\mathbb{P}(E)\log(\frac{1}{\mathbb{P}(E)}) - \frac{1}{2}\mathbb{P}(E)^2$ and the second term by $-\mathbb{P}(E)\log(\frac{1}{\mathbb{P}(E)})$ finishes the proof of Lemma 5.

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Proof.

Use (9) to get

$$\int_{\Lambda} \phi(dn_f - d\mu) = \int_{\Lambda} \Delta\phi(z)\frac{1}{2\pi}\{\log|f(z)| - \log\sqrt{K(z,z)}\}dm(z)$$

Using the above equality, Markov's inequality and applying Lemma 5 makes the proof of Theorem 4 immediate. □