Black hole perturbations in modified gravity

Hugo Roussille
Work with David Langlois and Karim Noui

December 9, 2021

2103.14744
2103.14750

GdR Gravitational Waves meeting — IPhT

Introduction

- Modified gravity theories: predictions different from GR
- Important test: quasinormal modes of black holes
- Up to now, theoretical computations are rare
- Present a systematic algorithm to extract physical information and perform numerical analysis

Quasinormal modes of a Schwarzschild black hole

Separating the degrees of freedom

1. Start with the Einstein-Hilbert action

$$S[g_{\mu\nu}] = \int \mathrm{d}^4x\, \sqrt{-g}\, R$$

2. Static spherically symmetric background

$$\bar{g}_{\mu\nu} = \mathrm{diag}(-A(r), 1/A(r), r^2, r^2 \sin^2\theta)\,, \quad A(r) = 1 - r_s/r$$

3. Perturb the metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and linearise Einstein's equations

   $\Rightarrow$ obtain 10 equations

4. Decompose the components of $h_{\mu\nu}$ over spherical harmonics
5. Separate by parity: polar (even) and axial (odd) modes
6. Gauge fixing via $h_{\mu\nu} \longrightarrow h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$:
   - Polar modes: 7 equations for K, H_0, H_1, H_2
   - Axial modes: 3 equations for h_0, h_1
7. Fourier transform: $f(t,r) = \exp(-i\omega t) f(r)$

Reducing the number of equations

Two systems with more equations than variables → overconstrained?

Axial modes

- 2 first-order equations
- 1 second-order equation

Polar modes

- 4 first-order equations
- 2 second-order equations
- 1 algebraic equation

Interestingly, each system is equivalent to a 2-dimensional system[1]:

$$\frac{\mathrm{d}X_{\text{axial}}}{\mathrm{d}r} = M_{\text{axial}}(r)X_{\text{axial}} \quad \text{and} \quad \frac{\mathrm{d}X_{\text{polar}}}{\mathrm{d}r} = M_{\text{polar}}(r)X_{\text{polar}}\,.$$

[1] *Regge, T. and Wheeler, J. A. 1957; Zerilli, F. J. 1970.*

Final system of equations

Axial modes

$$X_{\text{axial}} = {}^t\begin{pmatrix} h_0 & h_1/\omega \end{pmatrix}$$

$$M_{\text{axial}} = \begin{pmatrix} \frac{2}{r} & 2i\lambda\frac{r-r_s}{r^3} - i\omega^2 \\ -\frac{r^2}{(r-r_s)^2} & -\frac{r_s}{r(r-r_s)} \end{pmatrix}$$

$$\left(\text{set } 2(\lambda+1) = \ell(\ell+1)\right)$$

Polar modes

$$X_{\text{polar}} = {}^t\begin{pmatrix} K & H_1/\omega \end{pmatrix}$$

$$M_{\text{polar}} = \frac{1}{3r_s + 2\lambda r}\begin{pmatrix} \frac{a_{11}(r)+b_{11}(r)\omega^2}{r(r-r_s)} & \frac{a_{12}(r)+b_{12}(r)\omega^2}{r^2} \\ \frac{a_{21}(r)+b_{21}(r)\omega^2}{2(r-r_s)^2} & \frac{a_{22}(r)+b_{22}(r)\omega^2}{r(r-r_s)} \end{pmatrix}$$

$\Rightarrow$ goal to achieve: simplify these complicated differential systems

Effect of a change of variables

Changing the functions in X is not a change of basis for M!

Change of variables

$$\frac{\mathrm{d}X}{\mathrm{d}r} = M(r)X\,, \quad X = P(r)\tilde{X}$$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}r} = \tilde{M}(r)\tilde{X}\,, \quad \tilde{M} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

Main idea: find a change of variables that will put the equation into a better form

Usual change of variables: propagation equation

Canonical form for $\tilde{M}$:

$$\tilde{M} = \begin{pmatrix} 0 & 1 \\ V(r) - \frac{\omega^2}{c^2} & 0 \end{pmatrix}$$

Physical interpretation

$$\begin{cases} \tilde{X}_0' = \tilde{X}_1\,, \\ \tilde{X}_1' = (V(r) - \omega^2/c^2)\tilde{X}_0 \end{cases} \Rightarrow \quad \frac{\mathrm{d}^2\tilde{X}_0}{\mathrm{d}r_*^2} + \left(\frac{\omega^2}{c^2} - V(r)\right)\tilde{X}_0 = 0\,, \quad \frac{\mathrm{d}r}{\mathrm{d}r_*} = A(r)$$

Schrödinger equation with potential V

r_*: "tortoise coordinate", $r = r_s \longrightarrow r_* = -\infty$ and $r = +\infty \longrightarrow r_* = +\infty$

Interpretation of the equations

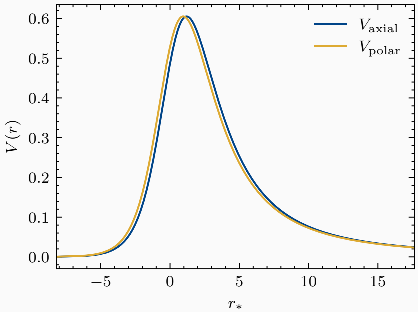
Axial case:

$$P_{\text{axial}} = \begin{pmatrix} 1 - r_s/r & r \\ ir^2/(r - r_s) & 0 \end{pmatrix}, \quad c = 1$$

At the horizon and infinity:

$$X_0(t, r) \propto e^{-i\omega(t \pm r_*)}$$

$\Rightarrow$ Propagation of waves



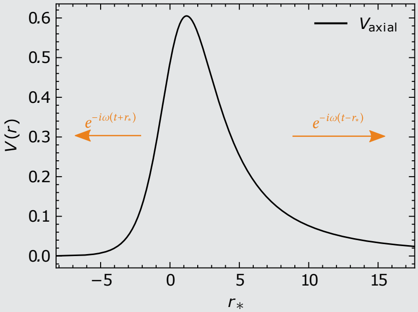
Physical interpretation

- Free propagation at $c = 1$ near the horizon and infinity
- Scattering by the potential V
- At infinity: recover gravitational waves in Minkowski

Computation of the modes

Quasinormal modes

- Waves *ingoing* at the horizon: $e^{-i\omega(t+r_*)}$
- Waves *outgoing* at infinity: $e^{-i\omega(t-r_*)}$



- 2 boundary conditions + 2$^{\mathrm{nd}}$ order system $\longrightarrow$ conditions on ω
- “Eigenvalue problem”: find values of parameter such that solutions exist
- Very different from plucked string: wave propagation at each boundary!

Quasinormal modes in modified gravity

Motivation for beyond-GR theories

Heuristic approach

- Design new tests of GR beyond a null hypothesis check
- EFT of some high energy theory

Issues of GR

- Singularities (Big Bang, black holes)
- Cosmic expansion

$\Rightarrow$ Important to look for extensions of GR

$\Rightarrow$ Quasinormal modes are a good test of both GR and the background metric

Scalar-tensor gravity

For simplicity, we consider quadratic Horndeski theory:

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x \left(F(X)R + P(X) + Q(X)\Box X + 2F'(X)\left(\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right)\right),$$

$$\phi_\mu = \nabla_\mu\phi\,, \quad X = \phi_\mu\phi^\mu$$

- New scalar degree of freedom
- Non-minimal coupling between scalar and metric
- More involved dynamics even in vaccuum

$\Rightarrow$ we are presently generalizing the results to cubic Horndeski theories

New black holes in modified gravity: BCL solution[2]

BCL solution:

$$F(X) = f_0 + f_1\sqrt{X} \quad P(X) = -p_1 X\,, \quad Q(X) = 0$$

Metric sector: RN with imaginary charge

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{r_m}{r} - \xi\frac{r_m^2}{r^2}\,, \quad \xi = \frac{f_1^2}{2f_0 p_1 r_m^2}$$

Scalar sector

$$\phi = \psi(r)\,, \quad \psi'(r) = \pm\frac{f_1}{p_1 r^2\sqrt{A(r)}}$$

$$X(r) = \frac{f_1^2}{p_1^2 r^4}$$

[2] *Babichev, E., Charmousis, C., and Lehébel, A. arXiv: 1702.01938.*

Summary: computation of QNMs in GR

1		2		3		4		5		6
Linearized Einstein's eqs	→	Gauge choice	→	Background	→	First-order system	→	Schrödinger equations	→	Numerical computation

Major difficulties:

1. Many different theories
3. Many different backgrounds
5. Highly non-trivial change of variables!

New challenges in modified gravity

New theories

Scalar-tensor: new scalar degree of freedom that couples to the polar mode

New backgrounds

BCL solution: more involved metric function

Schrödinger equation

In general, very hard to solve:

$$\begin{pmatrix} 0 & 1 \\ V(r) - \frac{\omega^2}{c^2} & 0 \end{pmatrix} = P^{-1} M P - P^{-1} \frac{\mathrm{d}P}{\mathrm{d}r}$$

$\Rightarrow$ need for a systematic approach that does not rely on specific simplifications

Example: polar BCL perturbations

$$A(r) = 1 - \frac{r_m}{r} - \xi\frac{r_m^2}{r^2}\,, \quad \xi = \frac{f_1^2}{2f_0 p_1 r_m^2}\,, \quad \phi'(r) = \pm\frac{f_1}{p_1 r^2\sqrt{A(r)}}$$

$$M(r) = \begin{pmatrix} -\frac{1}{r} + \frac{U}{2r^3A} & \frac{U}{r^4} & \frac{i(1+\lambda)}{\omega r^2} & \frac{V}{r^3} \\ \frac{\omega^2 r^2}{A^2} - \frac{\lambda}{A} - \frac{r_m}{2rA} + \frac{r_m^2 S}{4r^4A^2} & -\frac{2}{r} - \frac{UV}{2r^5A} & -\frac{i\omega r}{A} + \frac{i(1+\lambda)U}{2r^3\omega A} & -\frac{\lambda}{A} - \frac{3U}{2r^3A} - \frac{\xi^2 r_m^4}{2r^4A} \\ -\frac{i\omega V}{r^2A} & \frac{2i\omega}{r} - \frac{i\omega U}{r^3A} & -\frac{U}{r^3A} & -\frac{i\omega V}{r^2A} \\ -\frac{1}{r} + \frac{U}{2r^3A} & \frac{2}{r^2} - \frac{U^2}{2r^6A} & -\frac{i\omega}{A} + \frac{i(1+\lambda)}{\omega r^2} & \frac{1}{r} - \frac{U}{2r^3A} - \frac{UV}{2r^5A} \end{pmatrix}$$

$$U(r) = r_m(r + \xi r_m)\,, \qquad V(r) = r^2 + \xi r_m^2\,, \qquad S(r) = r^2 + 2\xi r(2r_m - r) + 2\xi^2 r_m^2\,.$$

First-order system and boundary conditions

Main idea

Skip step 5: get boundary conditions and perform numerical computations **from the first-order system**

Steps to perform

- Find asymptotic behaviour at the horizon and infinity
- Identify ingoing and outgoing modes
- Use a numerical method that does not require Schrödinger equations

Naively:

$$\frac{\mathrm{d}X}{\mathrm{d}r} = MX\,, \quad M(r) = M_p r^p + \mathcal{O}(r^{p-1}) \quad \Rightarrow \quad X \sim \exp\left(M_p \frac{r^{p+1}}{p+1}\right) X_c$$

Failure of naive approach

Axial Schwarzschild

$$M(r) = \begin{pmatrix} 0 & -i\omega^2 \\ -i & 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{r}\right)$$

$$X \sim \begin{pmatrix} e^{i\omega r} & 0 \\ 0 & e^{-i\omega r} \end{pmatrix} X_c$$

Polar Schwarzschild

$$M(r) = \begin{pmatrix} 0 & 0 \\ \frac{i\omega^2}{\lambda} & 0 \end{pmatrix} r^2 + \mathcal{O}(r)$$

$$X \sim \begin{pmatrix} 1 & 0 \\ \frac{i\omega^2}{\lambda}\frac{r^3}{3} & 1 \end{pmatrix} X_c$$

Problem

- We do not recover the $e^{\pm i\omega r_*}$ behaviour all the time!
- This is because of a *nilpotent* leading order in the polar case
- A more advanced mathematical treatment is needed

Mathematical results

Solution: behaviour studied in[3], mathematical algorithm from[4]

Mathematical algorithm

Main idea: *diagonalize M order by order* using

$$\tilde{M} = P^{-1} M P - P^{-1} \frac{\mathrm{d}P}{\mathrm{d}r}$$

⇒ important result: diagonalization is always possible!

General result:

$$M = M_p r^p + M_{p-1} r^{p-1} + \dots$$

$$\tilde{M} = D_q r^q + D_{q-1} r^{q-1} + \dots$$

$$X \sim e^{D(r)} r^{D_{-1}} F(r) X_c$$

[3] *Wasow, W. 1965.*

[4] *Balser, W. 1999.*

Example for the BCL solution: polar perturbations at infinity

$$\tilde{M} \sim \begin{pmatrix} -i\omega(1+\frac{r_m}{r}) & & & \\ & i\omega(1+\frac{r_m}{r}) & & \\ & & -\sqrt{2}\omega(1+\frac{r_m}{2r}) & \\ & & & \sqrt{2}\omega(1+\frac{r_m}{2r}) \end{pmatrix}$$

Gravitational Scalar

$$\mathfrak{g}^{\infty}_{\pm}(r) = a_{\pm} e^{\pm i\omega r} r^{\pm i\omega r_m} \,,$$
$$\mathfrak{s}^{\infty}_{\pm}(r) = b_{\pm} e^{\pm \sqrt{2}\omega r} r^{\pm \omega r_m/\sqrt{2}} \,,$$

- The modes are decoupled *locally*
- The gravitational mode propagates at $c = 1$ at infinity
- One can identify one ingoing and one outgoing gravitational mode
- The scalar mode does not propagate at infinity

Example for the BCL solution: polar perturbations at the horizon

$$\tilde{M} \sim \begin{pmatrix} -i\omega/c_0 & & & \\ & i\omega/c_0 & & \\ & & 1/2 & 1 \\ & & & 1/2 \end{pmatrix} \frac{1}{r - r_+}$$

$$\begin{aligned} \mathfrak{g}_{\pm}^{r_+}(r) &= c_{\pm}(r - r_+)^{-i\omega/c_0}\,, \\ \mathfrak{s}_1^{r_+}(r) &= (d_1 \ln(r - r_+) + d_2)\sqrt{r - r_+}\,, \\ \mathfrak{s}_2^{r_+}(r) &= d_1\sqrt{r - r_+}\,, \end{aligned}$$

- The modes are again decoupled *locally*
- The gravitational mode propagates at $c = c_0$ at the horizon
- One can identify one ingoing and one outgoing gravitational mode
- The scalar mode does not propagate at the horizon

"Recipe" for the computation of quasinormal modes

1 Linearized Einstein's eqs → 2 Gauge choice → 3 Background → 4 First-order system → 5 Asymptotical behaviour → 6 Numerical computation

- Generic algorithm that should work for any modified gravity theory
- Go around the technical difficulties of steps 1 and 3
- Caveat: we do not get the full decoupled equations for the modes ⇒ impossible to get a potential
- Asymptotical behaviour is enough to obtain boundary conditions for numerical resolution

Conclusion

- Computing quasinormal modes can be very difficult in modified theories of gravity
- We propose a new technique: use the first-order system instead of looking for Schrödinger-like equations
- A mathematical algorithm enables us to decouple the modes asymptotically, which allows us to find their physical behaviour and obtain boundary conditions
- This approach is **systematical** and theory-agnostic: it can be applied to any theory of gravity and any background

Thank you for your attention!