High-order Post-Minkowskian expansion from Velocity cuts

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&

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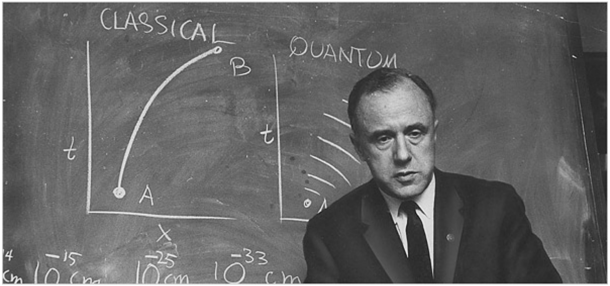
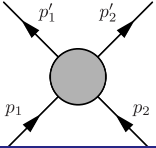
based on 2111.02976,
N.E.J. Bjerrum-Bohr, Ludovic Planté

Nove sed non nova: Classical Gravity from Amplitudes



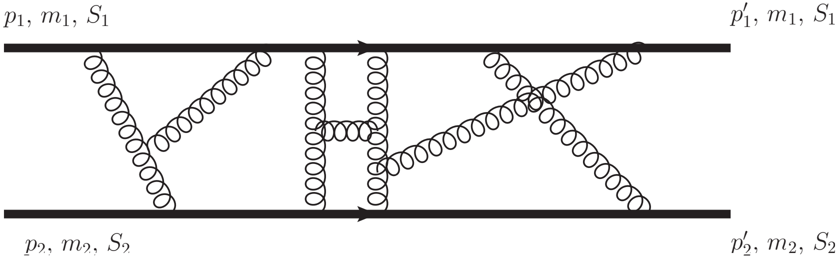
One important **new** insight is that the **classical** gravitational two-body interactions (conservative and radiation) can be extracted from **quantum scattering amplitudes**

$$\mathcal{M}(p_1 \cdot p_2, q^2) =$$



$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}^{\text{L}-\text{loop}}$$

Nove sed non nova: Classical Gravity from Amplitudes



For $\hbar, q^2 \to 0$ with $\underline{q} = q/\hbar$ fixed at each loop order the classical contribution is of order $1/\hbar$ $(\gamma = p_1 \cdot p_2/(m_1 m_2))$

$$\mathcal{M}_L(\gamma, \underline{q}, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

In this approach the **classical gravity physics contributions** are determined by the **unitarity** of the quantum scattering amplitudes

Exponentiation of the S-matrix: radial action

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

At L-loop order the singular terms for $\hbar \to 0$ contributions in the scattering matrix $\widehat{T}$ elements are *needed* so that the matrix elements of the $\widehat{N}$ operator has a smooth semi-classical limit $\hbar \to 0$

$$\hat{N}_2 = \hat{T}_2 - \frac{i}{2\hbar}(\hat{T}_0^{\text{rad}})^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_1 + \hat{T}_1\hat{T}_0) - \frac{1}{3\hbar^2}\hat{T}_0^3$$

so that the classical part leads to the radial action

$$\langle 1,2|\widehat{N}|3,4\rangle = \frac{J\pi}{2} + \int_{r_m}^{\infty}(p_r - p_\infty)dr + r_m p_\infty + O(\hbar)$$

from which gravitational observables are computed [Damour, Jaranowski, Schäfer]

The point is to use analytic continuation from the scattering regime to the bound state regime without having to use a gauge dependence Hamiltonian

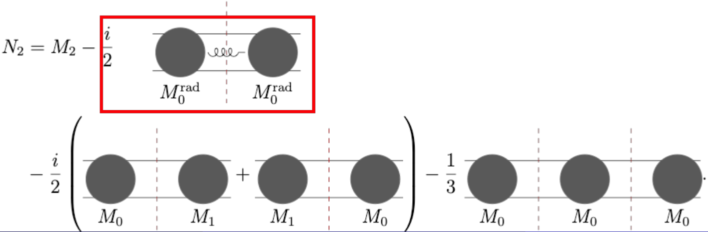
Exponentiation of the S-matrix

For unitarity of the $\widehat{S}$-matrix the completeness relation includes all the exchange of gravitons for $n \geqslant 1$

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^{(D-1)}k_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_1}} \frac{d^{(D-1)}k_2}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_2}}$$
$$\times \frac{d^{(D-1)}\ell_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{(D-1)}\ell_n}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \ldots \ell_n\rangle\langle k_1, k_2; \ell_1, \ldots \ell_n|,$$

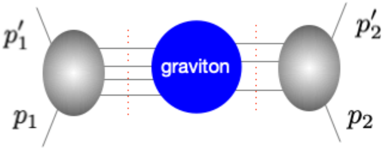
$$\widehat{N} = G_N \widehat{N}_0 + G_N^{\frac{3}{2}} \widehat{N}_0^{\text{rad}} + G_N^2 \widehat{N}_1 + G_N^{\frac{5}{2}} \widehat{N}_1^{\text{rad}} + \cdots$$

We see radiation-reaction contributions $\hat{N}^{\text{rad}}$ emerging naturally



Unitarity cuts

We can consider unitarity cuts across graviton lines



The total amplitude is the sum of cut contributions

$$\text{Cut} = \int M_r^{\text{tree}}(p_1, \ell_1, \ldots, \ell_r, p_1') M_s^{\text{tree}}(p_2, \ell_{r+1}, \ldots, \ell_{r+s}, p_2') \times M^{\text{gravity}}(\ell_1, \ldots, \ell_{r+s}) \frac{\prod_{i=1}^{r+s} d^D\ell_i}{\ell_i^2}$$

Where $M_n^{\text{tree}}(p_1, \ell_1, \ldots, \ell_n, p_1')$ and $M_n^{\text{tree}}(p_2, \ell_1, \ldots, \ell_n, p_2')$ are tree-level multigraviton emission from a massive scalar line.

Soft expansion cuts

We design an efficient organisation of the classical limit $\hbar \to 0$ in the cut

$$q^2 = (p_1 - p_1')^2 = (\hbar \underline{q})^2 \ll m_1^2, m_2^2$$

by reorganizing the multi-soft expansion of the massive scalar tree amplitude is

$$\lim_{|\vec{q}|\to 0} M_{L+1}^{\text{tree}}(p, \hbar|\underline{q}|\tilde{\ell}_2, \ldots, \hbar|\underline{q}|\tilde{\ell}_{L+2}, -p') = \frac{1}{\hbar^L |\underline{q}|^L} + O\left(\hbar|\underline{q}|\right)^{1-L}$$

so that we can *easily* isolate the classical terms in the cut

$$\mathcal{M}_L(\sigma, |q|)\Big|_{\text{classical}} = \frac{1}{\hbar} \frac{\mathcal{M}_L^{(L-2)}(\sigma, D)}{|\underline{q}|^{2-(D-3)L}},$$

From unitarity cuts…

Let's consider the 2-graviton emission

$$M_2^{\rm tree}(p,\ell_2,\ell_3,-p')$$
$$= \frac{iN_2(p,2,3,-p')^2}{(\ell_2+p)^2-m^2+i\varepsilon}+\frac{iN_2(p,3,2,-p')^2}{(\ell_3+p)^2-m^2+i\varepsilon}+\frac{i({N_2}^{[2,3]})^2}{(\ell_2+\ell_3)^2+i\varepsilon}$$

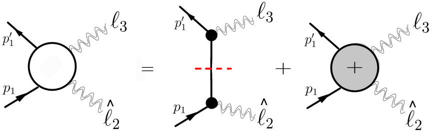
We rewrite the massive poles using

$$\frac{1}{(p+\ell)^2-m^2+i\varepsilon}=-2i\pi\delta\left((p+\ell)^2-m^2\right)+\frac{1}{(p+\ell)^2-m^2-i\varepsilon}$$

to get

$$M_2^{\rm tree}(p_1,\ell_2,\ell_3,-p_1')=M_2^{{\rm tree}(+)}(p_1,\ell_2,\ell_3,-p_1')$$
$$-2i\pi\delta((p_1+\ell_2)^2-m_1^2)M_1^{\rm tree}(p_1,\ell_2,-p_1-\ell_2)M_1^{\rm tree}(p_1+\ell_2,\ell_3,-p_1')$$

From unitarity cuts...



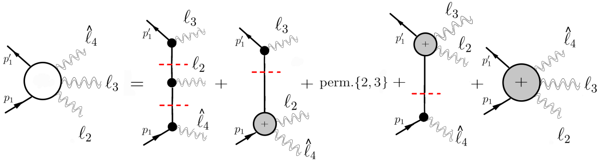
We have decomposed the scalar-graviton amplitude in unitarity cut contribution and a reminder

$$M_2^{\rm tree}(p_1,\ell_2,\ell_3,-p_1') = M_2^{{\rm tree}\,(+)}(p_1,\ell_2,\ell_3,p_1')$$
$$-2i\pi\delta((p_1+\ell_2)^2-m_1^2)M_1^{{\rm tree}\,(+)}(p_1,\ell_2,-p_1-\hat\ell_2)M_1^{{\rm tree}\,(+)}(p_1+\ell_2,\ell_3,-p_1')$$

Exhibiting an unitairty cut of the massive scalar propagator

From unitarity cuts…

In general we have



In general we have the expansion of multigraviton scalar amplitude

$$M_{L+1}^{\text{tree}} \sim (M_1^{\text{tree}(+)})^{L+1} \prod_i^L \delta_i(\dots) + (M_1^{\text{tree}(+)})^{L-1} (M_2^{\text{tree}(+)}) \prod_i^{L-1} \delta_i(\dots) + \cdots$$
$$+ M_1^{\text{tree}(+)} M_L^{\text{tree}(+)} \delta(\dots) + M_{L+1}^{\text{tree}(+)}$$

where the delta-function are unitarity cuts of the massive propagator
By factorization of the amplitude we know that the coefficients are products of on-shell amplitudes

…to velocity cuts

The classical limit is now obtained using $\ell_i = \hbar|\underline{q}|\tilde{\ell}_i$

$$\lim_{\hbar\to 0} \delta((p_1 + \hbar|\underline{q}|\tilde{\ell}_2)^2 - m_1^2) = \frac{\delta(2p_1 \cdot \tilde{\ell}_2)}{\hbar|\underline{q}|} + (\hbar|\underline{q}|)^0$$

$$\lim_{\hbar\to 0} M_{L+1}^{\text{tree}(\pm)}(p, \hbar|\underline{q}|\tilde{\ell}_2, \dots, \hbar|\underline{q}|\hat{\tilde{\ell}}_{L+2}, -p') \sim (\hbar|\underline{q}|)^0$$

In the classical limit we see the velocity cut emerging $\delta(2p_1 \cdot \tilde{\ell}_2)$ and the cut contribution is now organised in power of δ-functions

$$\mathcal{M}_L^{\text{cut}} \sim$$

$$\sim \sum_{k=0}^{2L} \hbar^{3L+1} \int \frac{(d^D\ell)^L}{\hbar^{DL}} \frac{\left(\delta((p_1 + \sum_i \ell_{\alpha_i})^2 - m_1^2)\right)^k \times (\prod M^{\text{tree}(+)}) \times (\prod M^{\text{tree}(-)\,\dagger})}{(\ell^2)^{L+1}}$$

...to velocity cuts

$$\mathcal{M}_L^{\rm cut} \sim$$

$$\sim \sum_{k=0}^{2L} \hbar^{3L+1} \int \frac{(d^D\ell)^L}{\hbar^{DL}} \frac{\left(\delta\left((p_1+\sum_i \ell_{\alpha_i})^2-m_1^2\right)\right)^k \times \left(\prod M^{\rm tree\,(+)}\right) \times \left(\prod M^{\rm tree\,(-)\,\dagger}\right)}{(\ell^2)^{L+1}}$$

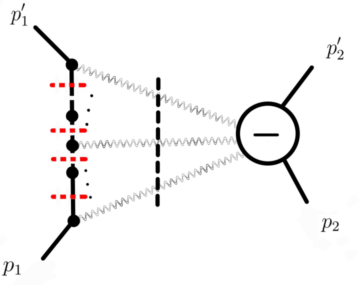
$$\sim \sum_{k=L+1}^{2L} \frac{\hbar^{L-1-k}}{|\underline{q}|^{2+k-(D-2)L}} + \frac{1}{\hbar|\underline{q}|^{2-(D-3)L}} + \sum_{k=0}^{L-1} \frac{\hbar^{L-1-k}}{|\underline{q}|^{2+k-(D-2)L}}$$

We can have three type of contributions:

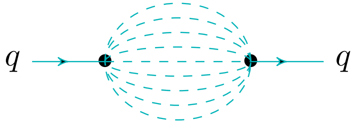
- $k=L$ delta-functions is the classical contribution. giving the radial action N_L.
- $0 \leqslant k < L$ delta-functions are quantum contributions
- $L < k \leqslant 2L$ don't contribute to the classical part but are needed for the exponentiation of the amplitude.

The leading probe contribution

The leading contribution $\nu \ll 1$ or $m_2 \ll m_1$ is the probe limit



$\propto$

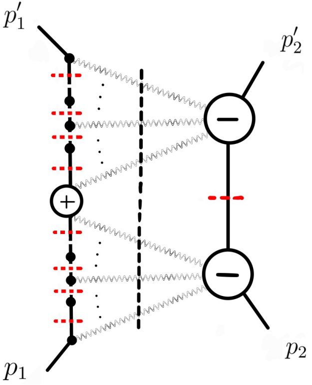


At each PM order the result is proportional to a *single* master integral: the massless sunset which is the same as for metric computation [Mougiakakos, Vanhove]

This allows a complete evaluation of the PM expansion to high-order (5PM) in the probe limit in D dimensions

Our explicit results match the schematic all-loop conjecture by [A. Brandhuber, G. Chen, G. Travaglini, C. Wen]

The full amplitude



We can organize the scattering angle in $D=4$ as [Damour]

$$\chi_{3PM}(\sigma,\nu) = \chi_{3PM}^{(0)}(\sigma,\nu) + \frac{\chi_{3PM}^{(2)}(\sigma,\nu)}{1+2\nu(\sigma-1)}$$

$$\chi_{4PM}(\sigma,\nu) = \frac{1}{\sqrt{1+2\nu(\sigma-1)}}\left(\chi_{4PM}^{(0)}(\sigma,\nu) + \frac{\chi_{4PM}^{(2)}(\sigma,\nu)}{1+2\nu(\sigma-1)}\right)$$

$$\chi_{5PM}(\sigma,\nu) = \chi_{5PM}^{(0)}(\sigma,\nu) + \frac{\chi_{5PM}^{(2)}(\sigma,\nu)}{1+2\nu(\sigma-1)} + \frac{\chi_{5PM}^{(4)}(\sigma,\nu)}{(1+2\nu(\sigma-1))^2}$$

The $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ expansion gives an easy way of reconstructing the full PM expansion by computing a few coefficients from the ν expansion. Recall $\sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$.

Conclusion

We have given a new way of organising the PM expansion for the 2-body scattering amplitude using velocity cuts

1. In practice, we need only evaluate matrix elements in the soft q^2-expansion, this means that we expand genuine unitarity cuts around the velocity cuts introduced recently [Bjerrum-Boh, Damgaard, Planté, Vanhove]
2. These velocity cuts are in correspondence with the perturbative classical wordline EFT methods
3. A new exponentiation formula and the velocity cuts males the relation between the classical part of the scattering amplitude and the effective potential simple and efficient [Bjerrum-Bohr, Damgaard, Planté, Vanhove]
4. We have derived the amplitude in D dimensions to 3PM order and to higher PM order are on the way [Bjerrum-Bohr, Planté, Vanhove, to appear]