

GDR Ondes Gravitationnelles (CNRS)

Groupes de Travail "Formes d'Ondes" et "Théories Alternatives"

# THE QUADRUPOLE MOMENT OF COMPACT BINARIES to FOURTH POST-NEWTONIAN ORDER

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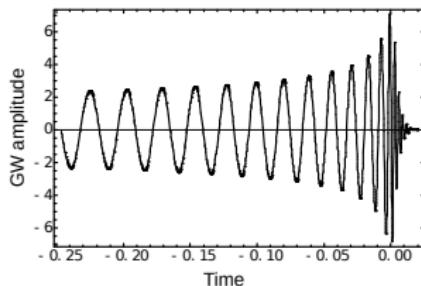
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*Based on collaborations with*

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$$4\pi R^2 \bar{G} = \frac{\kappa}{40\pi} \left[ \sum_{\mu\nu} \bar{J}_{\mu\nu} - \frac{1}{3} \left( \sum_{\mu} \bar{J}_{\mu\mu} \right)^2 \right].$$

# The gravitational chirp of compact binaries



## • **Inspiralling phase**

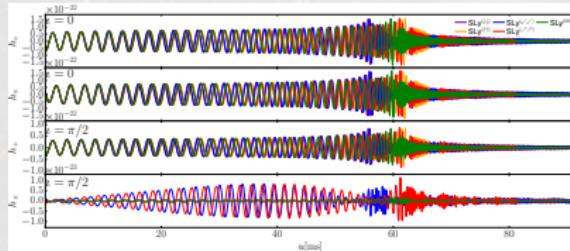
- Post-Newtonian theory
- Point-particle approximation
- Dependence on spin precession
- Universality of the signal in GR
- Effacing of the internal structure

## • **Late inspiral**

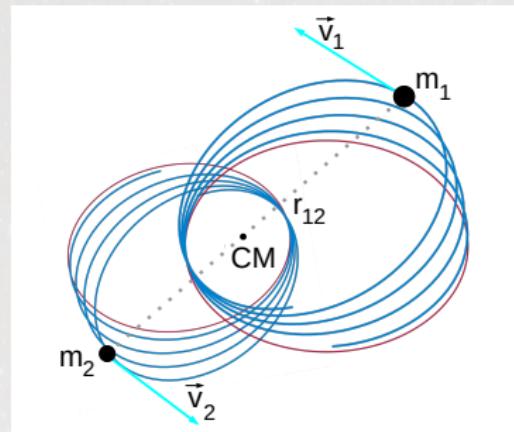
- Post-Newtonian + Effective theory
- Effects due to tidal interactions
- Dependence on the internal structure (EoS)

## • **Merger and post-merger**

- Numerical relativity
- Phenomenological models (EOB, IMR)



# Post-Newtonian equations of motion



$$\frac{d\mathbf{v}_1}{dt} = -\frac{Gm_2}{r_{12}^2} \mathbf{n}_{12} + \overbrace{\frac{1}{c^2} \left\{ \left[ \frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \dots \right] \mathbf{n}_{12} + \dots \right\}}^{1\text{PN}}$$
$$+ \underbrace{\frac{1}{c^4} [\dots]}_{2\text{PN}} + \underbrace{\frac{1}{c^5} [\dots]}_{2.5\text{PN}} + \underbrace{\frac{1}{c^6} [\dots]}_{3\text{PN}} + \underbrace{\frac{1}{c^7} [\dots]}_{3.5\text{PN}} + \underbrace{\frac{1}{c^8} [\dots]}_{4\text{PN}} + \mathcal{O}\left[\left(\frac{v}{c}\right)^9\right]$$

radiation reaction      radiation reaction      conservative & dissipative (tail)

# Methods to compute PN radiation and EoM

## ① Traditional methods in classical GR

- ADM Hamiltonian canonical formalism in GR
- Fokker EH action in harmonic coordinates
- Surface-integral approach à la EIH
- Extended fluids in the compact body limit

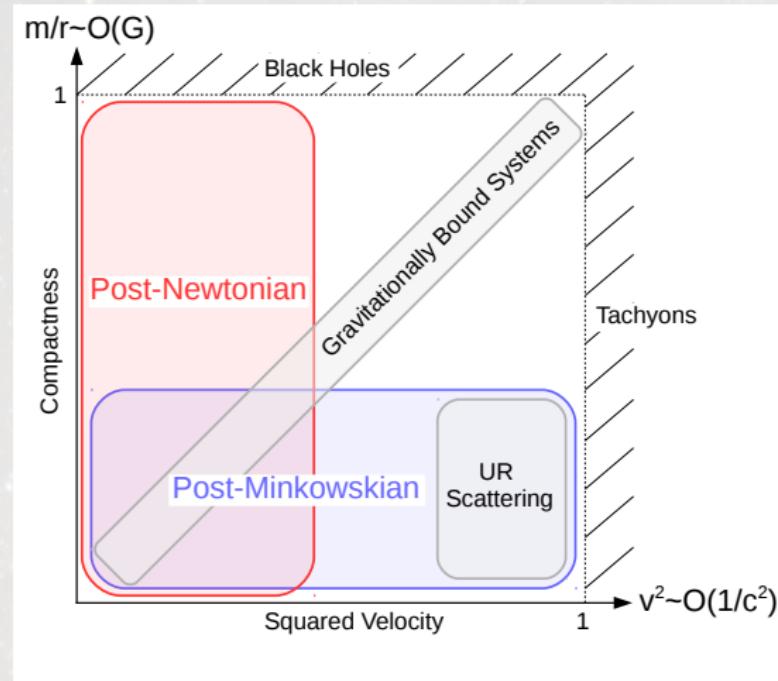
## ② QFT inspired methods

- Effective-field theory
- Scattering amplitude approach

## ③ Dimensional regularization is the common tool

- UV divergences: point particles modelling compact objects
- IR divergences: integration over all space of formal PN expansion

# Post-Newtonian versus post-Minkowskian



[see also the talk by Pierre Vanhove]

# Post-Newtonian versus post-Minkowskian

conservative sector

	special relativity	1PM	2PM	3PM	4PM	5PM	...
rest mass	$c^2$						
Newtonian	1	$G$					
1PN	$\frac{1}{c^2}$	$\frac{G}{c^2}$	$\frac{G^2}{c^2}$				
2PN	$\frac{1}{c^4}$	$\frac{G}{c^4}$	$\frac{G^2}{c^4}$	$\frac{G^3}{c^4}$			
3PN	$\frac{1}{c^6}$	$\frac{G}{c^6}$	$\frac{G^2}{c^6}$	$\frac{G^3}{c^6}$	$\frac{G^4}{c^6}$		
4PN	$\frac{1}{c^8}$	$\frac{G}{c^8}$	$\frac{G^2}{c^8}$	$\frac{G^3}{c^8}$	$\frac{G^4}{c^8}$	$\frac{G^5}{c^8}$	
5PN	$\frac{1}{c^{10}}$	$\frac{G}{c^{10}}$	$\frac{G^2}{c^{10}}$	$\frac{G^3}{c^{10}}$	$\frac{G^4}{c^{10}}$	$\frac{G^5}{c^{10}}$	...
...	...	...	...	...	...	...	...

Solving the  $n$ PM dynamics permits controlling the  $(n - 1)$ PN dynamics

# Post-Newtonian versus post-Minkowskian

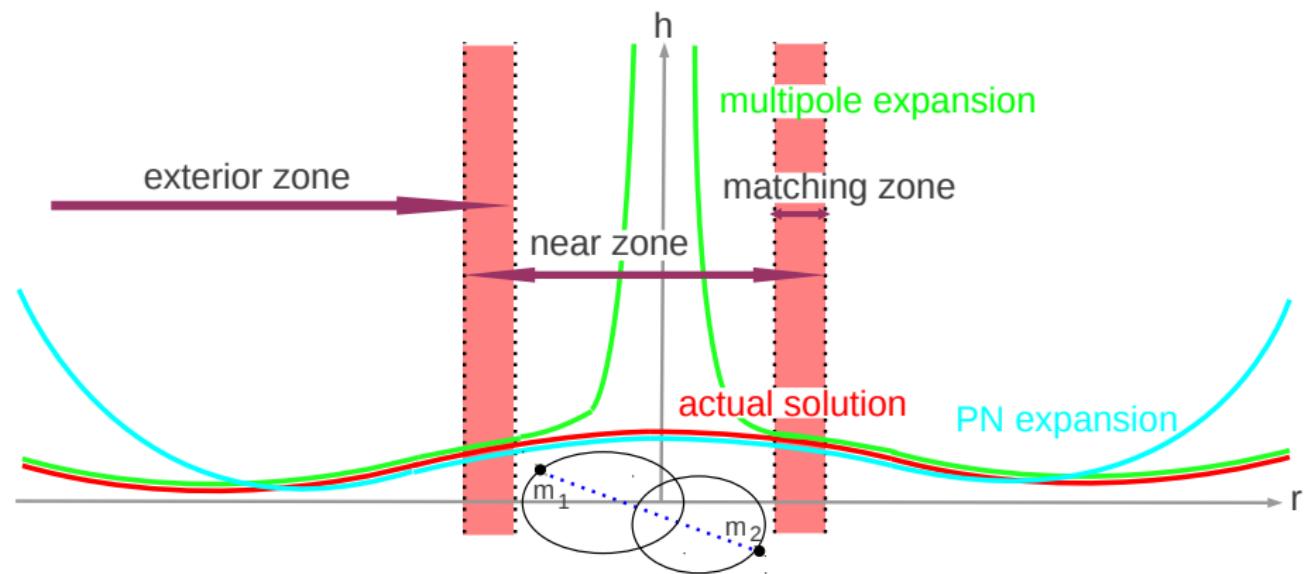
dissipative/radiation-reaction sector

	2PM	3PM	4PM	5PM	6PM	7PM	...
quadrupole formula	$\frac{G^2}{c^5}$	$\frac{G^3}{c^5}$					
1PN	$\frac{G^2}{c^7}$	$\frac{G^3}{c^7}$	$\frac{G^4}{c^7}$				
2PN	$\frac{G^2}{c^9}$	$\frac{G^3}{c^9}$	$\frac{G^4}{c^9}$	$\frac{G^5}{c^9}$			
3PN	$\frac{G^2}{c^{11}}$	$\frac{G^3}{c^{11}}$	$\frac{G^4}{c^{11}}$	$\frac{G^5}{c^{11}}$	$\frac{G^6}{c^{11}}$		
4PN	$\frac{G^2}{c^{13}}$	$\frac{G^3}{c^{13}}$	$\frac{G^4}{c^{13}}$	$\frac{G^5}{c^{13}}$	$\frac{G^6}{c^{13}}$	$\frac{G^7}{c^{13}}$	
...	...	...	...	...	...	...	...

Goal: compute the radiation at 4PN order beyond the quadrupole formula

# The Multipolar-post-Minkowskian-PN formalism

The MPM outer metric is matched to the PN inner field of the source



$$\text{matching equation} \implies \overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$$

# Field equations and Green's function in $d$ dimensions

- Einstein's field equations in harmonic (de Donder) coordinates

$$\partial_\nu h^{\mu\nu} = 0 \quad (\text{harmonic gauge condition})$$

$$\square h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu} \quad (\text{wave equation in } D = d + 1 \text{ dimensions})$$

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu} \quad (\text{matter + gravitation pseudo tensor})$$

- The Green's function is implemented in the real space-time domain

$$G_{\text{ret}}(\mathbf{x}, t) = -\frac{\tilde{k}}{4\pi} \frac{\theta(t-r)}{r^{d-1}} \gamma_{\frac{1-d}{2}} \left( \frac{t}{r} \right)$$
$$\gamma_{\frac{1-d}{2}}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)} (z^2 - 1)^{\frac{1-d}{2}}$$

# The multipole expansion outside the matter source

- The multipole expansion  $\mathcal{M}(h^{\mu\nu})$  is a retarded solution the *vacuum* field equations  $\square \mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu})$  valid formally everywhere except at  $r = 0$

$$\mathcal{M}(h^{\mu\nu}) = \underbrace{\text{FP}_{B=0} \square_{\text{ret}}^{-1} \left[ \left( \frac{r}{r_0} \right)^B \mathcal{M}(\Lambda^{\mu\nu}) \right]}_{\text{regularization when } r \rightarrow 0} - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^{\ell}}{\ell!} \hat{\partial}_L \tilde{\mathcal{F}}_L^{\mu\nu}}_{\text{retarded homogeneous solution}}$$
$$\square \tilde{\mathcal{F}}_L^{\mu\nu}(r, t) = 0 \text{ in } d \text{ dimensions}$$

- The multipole moment functions  $\mathcal{F}_L^{\mu\nu}(t)$  are symmetric-trace-free (STF) with respect to their  $\ell$  indices  $L \equiv i_1 \cdots i_\ell$

$$\tilde{\mathcal{F}}_L^{\mu\nu}(r, t) = \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} dz \gamma_{\frac{1-d}{2}}(z) \mathcal{F}_L^{\mu\nu}(t - zr)$$

# The multipole expansion matched to the PN source

- Explicit matching to a general extended PN isolated source gives

$$\mathcal{F}_L^{\mu\nu}(t) = \underbrace{\text{FP}_{B=0} \int d^d \mathbf{x} \left(\frac{r}{r_0}\right)^B}_{\text{IR regularization}} \hat{x}_L \int_{-1}^1 dz \delta_\ell^{(d)}(z) \underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x}, t + zr)}_{\text{PN expansion of the pseudo-tensor}}$$
$$\delta_\ell^{(d)}(z) \equiv \frac{\Gamma\left(\frac{d}{2} + \ell\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2} + \ell\right)} (1 - z^2)^{\frac{d-3}{2} + \ell}$$

[Blanchet 1995, 1998; Blanchet, Damour, Esposito-Farèse & Iyer 2005]

- The  $B\varepsilon$  regularization**

- first apply the limit  $B \rightarrow 0$  in generic dimensions  $d = 3 + \varepsilon$
- then the usual dimensional regularization when  $\varepsilon \rightarrow 0$

# Mass and current irreducible multipole moments

- The irreducible decomposition of  $\mathcal{F}_L^{\mu\nu}$  reads (with  $\langle \cdots \rangle$  the STF projection)

$$\begin{aligned}\mathcal{F}_L^{00} &= R_L \\ \mathcal{F}_L^{0i} &= T_{iL}^{(+)} + T_{i|\langle i_\ell L-1 \rangle}^{(0)} + \delta_{i\langle i_\ell} T_{L-1\rangle}^{(-)} \\ \mathcal{F}_L^{ij} &= U_{ijL}^{(+2)} + \underset{L}{\text{STF}} \underset{ij}{\text{STF}} \left[ U_{i|i_\ell j L-1}^{(+1)} + \delta_{ii_\ell} U_{jL-1}^{(0)} + \delta_{ii_\ell} U_{j|i_{\ell-1} L-2}^{(-1)} \right. \\ &\quad \left. + \delta_{ii_\ell} \delta_{ji_{\ell-1}} U_{L-2}^{(-2)} + W_{ij|i_\ell i_{\ell-1} L-2} \right] + \delta_{ij} V_L\end{aligned}$$

- The “mass-type” contributions  $R_L$ ,  $T_{L+1}^{(+)}$ ,  $T_{L-1}^{(-)}$ ,  $U_{L+2}^{(+2)}$ ,  $U_L^{(0)}$ ,  $U_{L-2}^{(-2)}$ ,  $V_L$  are STF in the ordinary sense
- The “current-type” contributions  $T_{i|\langle i_\ell L-1 \rangle}^{(0)}$ ,  $U_{i|i_{\ell+1} L}^{(+1)}$ ,  $U_{i|i_{\ell-1} L-2}^{(-1)}$  have more complicated symmetries

# Mass and current irreducible multipole moments

- The mass moment  $I_L$  is given by the usual STF moment, but the generalization of the current moment involves two tensors  $J_{i|L}$  and  $K_{ij|L}$  having the **symmetries of mixed Young tableaux**

$$\boxed{\begin{aligned} I_L &= \begin{array}{|c|c|c|} \hline i_\ell & \dots & i_1 \\ \hline \end{array} \\ J_{i|L} &= \begin{array}{|c|c|c|c|} \hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i & & & \\ \hline \end{array} \quad K_{ij|L} = \begin{array}{|c|c|c|c|c|} \hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & & & \\ \hline \end{array} \end{aligned}}$$

- The tensor  $K_{ij|L}$  is absent in 3 dimensions

$$\sharp(\text{components}) = \frac{(d-3)d(d-1)_{\ell-2}(2\ell+d-2)(\ell+d-1)}{2\ell(\ell+1)(\ell-2)!}$$

and plays no role with dimensional regularization

# The irreducible mass quadrupole moment

- Posing

$$\bar{\Sigma} \equiv \frac{2}{d-1} \frac{(d-2)\bar{\tau}^{00} + \bar{\tau}^{ii}}{c^2} \quad \bar{\Sigma}^i \equiv \frac{\bar{\tau}^{i0}}{c} \quad \bar{\Sigma}^{ij} \equiv \bar{\tau}^{ij}$$

$$\bar{\Sigma}_{[\ell]}(\mathbf{x}, t) = \int_{-1}^1 dz \delta_\ell^{(d)}(z) \bar{\Sigma}(\mathbf{x}, t + zr)$$

$$I_{ij} = \frac{d-1}{2(d-2)} \underset{\textcolor{red}{B=0}}{\text{FP}} \int d^d \mathbf{x} \left( \frac{r}{r_0} \right)^{\textcolor{red}{B}} \left\{ \hat{x}^{ij} \bar{\Sigma}_{[2]} - \frac{4(d+2)}{d(d+4)} \hat{x}^{ijk} \dot{\bar{\Sigma}}_{[3]}^k \right. \\ \left. + \frac{2(d+2)}{d(d+1)(d+6)} \hat{x}^{ijkl} \ddot{\bar{\Sigma}}_{[4]}^{kl} \right. \\ \left. - \frac{4(d-3)(d+2)}{d(d-1)(d+4)} \textcolor{red}{B} \hat{x}^{ijk} \frac{x^l}{r^2} \bar{\Sigma}_{[3]}^{kl} \right\}$$

- The  $B\varepsilon$  regularization is systematically applied (the limit  $B \rightarrow 0$  is finite)

# Techniques to compute the 4PN mass quadrupole

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet 2019]

- Method of super-potentials

$$\int d^3x \underbrace{r^B \hat{x}_L \overbrace{\phi}^{\text{linear potential}}}_{\text{difficult potential}} \underbrace{P}_{\text{yields a surface term}} = \int d^3x r^B \left( \Psi_L^\phi \Delta P + \underbrace{\partial_i [\partial_i \Psi_L^\phi P - \Psi_L^\phi \partial_i P]}_{\text{yields a surface term}} \right)$$

where  $\Psi_L^\phi$  is obtained from the super-potentials  $\phi_{2k}$  of  $\phi = \phi_0$  as

$$\Psi_L^\phi = \Delta^{-1}(\hat{x}_L \phi) = \sum_{k=0}^{\ell} \frac{(-2)^k \ell!}{(\ell-k)!} x_{\langle L-K} \partial_{K \rangle} \underbrace{\phi_{2k+2}}_{\Delta \phi_{2k+2} = \phi_{2k}}$$

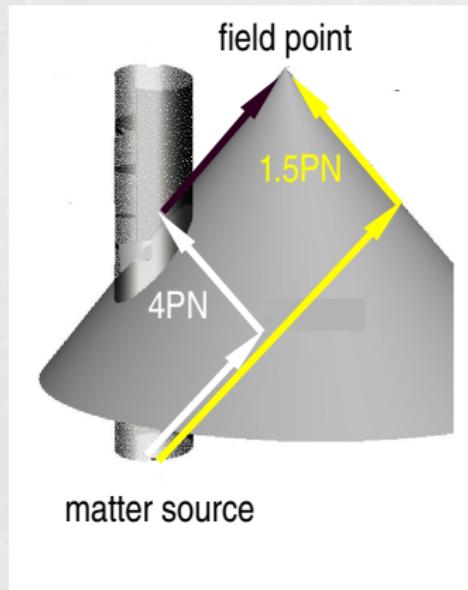
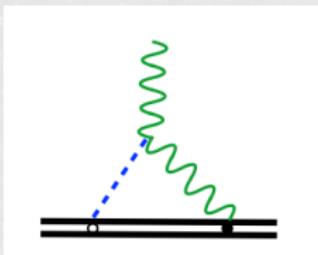
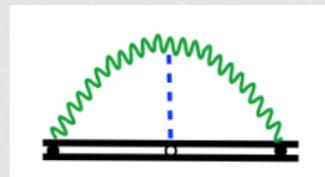
- Method of surface integrals

$$\boxed{\begin{aligned} \text{FP} \\ \text{B=0} \end{aligned} \int d^3x r^B \hat{x}_L \Delta G = -(2\ell+1) \int d\Omega \hat{n}_L X_\ell(\mathbf{n})}$$

where  $X_\ell$  is the coefficient of  $r^{-\ell-1}$  in the expansion of  $G$  when  $r \rightarrow +\infty$

- Schwartz distributional derivatives in  $d$  dimensions systematically applied

# Non-locality of the mass quadrupole at 4PN order



Effect due to propagating GW tails

- In the far zone (1.5PN order)
- In the near zone (4PN)

[Blanchet & Damour 1988, 1992; Foffa & Sturani 2012]

[Galley, Leibovich, Porto & Ross 2016]

$$I_{ij}^{\text{non-loc}} = \frac{24G^2 M}{7c^8} I_{k\langle i}(t) \int_0^{+\infty} d\tau \left[ \underbrace{-\frac{1}{2\varepsilon}}_{\text{IR pole}} + \ln \left( \frac{c\sqrt{\bar{q}}}{2\ell_0} \right) + \frac{74}{105} \right] I_{j\rangle k}^{(5)}(t - \tau)$$

# Completion of the 4PN mass quadrupole

- All UV divergences treated by dimensional regularization and all UV poles shown to be renormalized by appropriate shifts of the particles' worldlines ✓
- Presence at 4PN order of a non-local-in-time term associated with tail radiation mode and containing a crucial IR pole ✓
- IR divergences (poles  $\propto \frac{1}{d-3}$ ) appear already at 3PN order but are cancelled (as well as the finite part beyond) by poles coming from “tails-of-tails” propagating in the wave zone ✓
- At 4PN order the IR poles are cancelled by more complicated “tails-of-memory” but there remains a crucial finite contribution specifically due to dimensional regularization ✓
- Finally we have obtained the finite renormalized 4PN quadrupole moment of compact binaries ready to be used for 4PN/4.5PN templates ☺

# Towards 4.5PN templates

- 4.5PN “tail-of-tail-of-tail” completed [Marchand *et al.* 2016; Messina & Nagar 2017] ✓
- 3PN mass octupole moment [Faye *et al.* 2015] and 3PN current quadrupole moment completed [Henry *et al.* 2021] (higher-order moments known) ✓
  - UV divergences treated by dimensional regularization
  - IR divergences treated by Hadamard regularization equivalent to dimensional regularization at that order
- 2PN mass dodecapole and current octupole, as well as higher-order moments are already known ✓
- Cubic interactions at 4PN order in the radiative quadrupole moment need to be completed in (computation to be done in ordinary  $3d$ )
  - relation between radiative and canonical moments
  - relation between canonical and source/gauge moments

# Open problem: cubic interactions at 4PN order

$$\begin{aligned}
 U_{ij}(t) = & I_{ij}^{(2)}(t) + \underbrace{\frac{GM}{c^3} \int_0^{+\infty} d\tau I_{ij}^{(4)}(t-\tau) \left[ 2 \ln \left( \frac{\tau}{2\tau_0} \right) + \frac{11}{6} \right]}_{\text{1.5PN tail}} \\
 & + \frac{G}{c^5} \left\{ \underbrace{-\frac{2}{7} \int_0^{+\infty} d\tau I_{a < i}^{(3)} I_{j > a}^{(3)}(t-\tau)}_{\text{2.5PN memory}} + \text{inst. terms} \right\} \\
 & + \underbrace{\frac{G^2 M^2}{c^6} \int_0^{+\infty} d\tau I_{ij}^{(5)}(t-\tau) \left[ 2 \ln^2 \left( \frac{\tau}{2\tau_0} \right) + \frac{57}{35} \ln \left( \frac{\tau}{2\tau_0} \right) + \frac{124627}{22050} \right]}_{\text{3PN tail-of-tail}} \\
 & + \frac{G^2}{c^8} \left\{ \underbrace{\sum M \int d\tau_1 \cdots \int d\tau_2 I_{a < i}^{(n)}(t-\tau_1) I_{j > a}^{(m)}(t-\tau_2)}_{\text{4PN tail-of-memory}} + \text{inst.} \right\} \\
 & + \underbrace{\frac{G^3 M^3}{c^9} \int_0^{+\infty} d\tau I_{ij}^{(6)}(t-\tau) \left[ \frac{4}{3} \ln^3 \left( \frac{\tau}{2\tau_0} \right) + \cdots + \frac{129268}{33075} + \frac{428}{315} \pi^2 \right]}_{\text{4.5PN tail-of-tail-of-tail}} + \cdots
 \end{aligned}$$