Critical phenomena in gravitational collapse beyond spherical symmetry:

Comparison of linear Brill and Teukolsky waves

Isabel Suárez Fernández, PhD student @ CENTRA, Lisbon

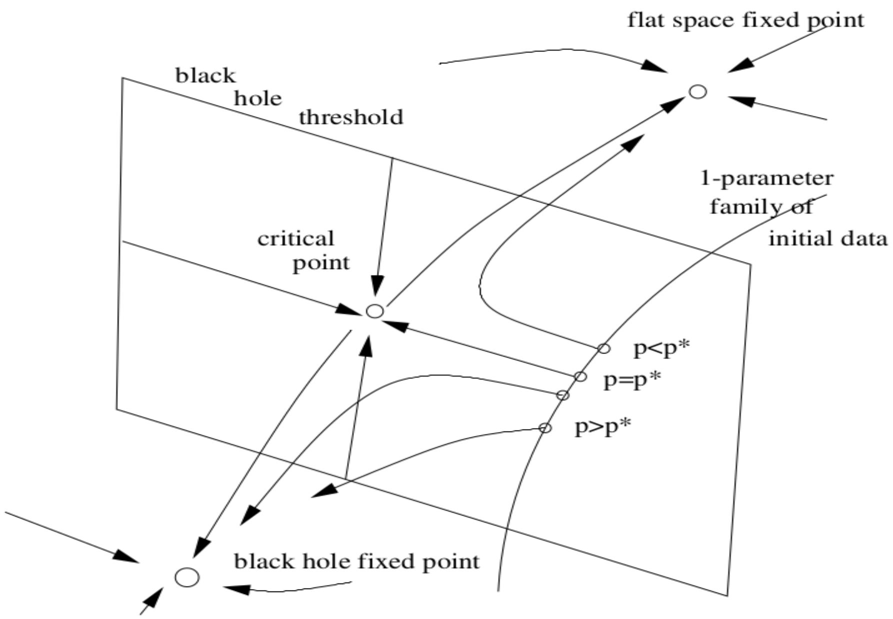
Supervision: David Hilditch, Miguel Zilhao and Vítor Cardoso

9th December, 2021
IPhT CEA-Saclay

Roadmap of the talk

- Critical phenomena in gravitational collapse
  - Critical phenomena with electromagnetic waves as Initial Data (ID)
- Linear Brill and Teukolsky waves
  - The special case: Holz data
- Comparison
  - Gauge change
  - Moncrief formalism
  - Kretschmann scalar
- Conclusions

What are Critical Phenomena?



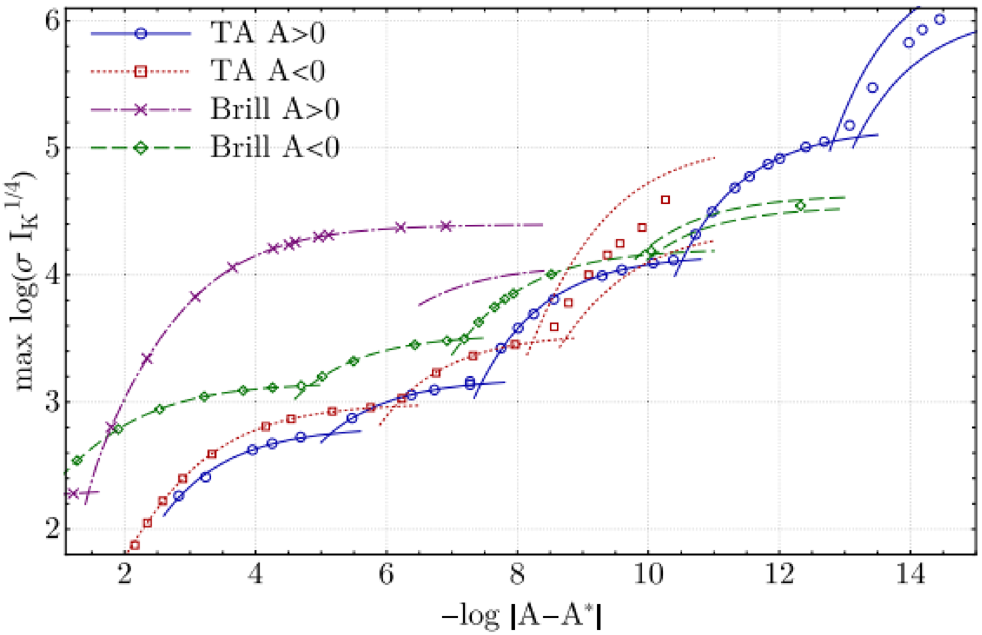
- Universality
- Self-similar behavior
- Power law behavior near the threshold

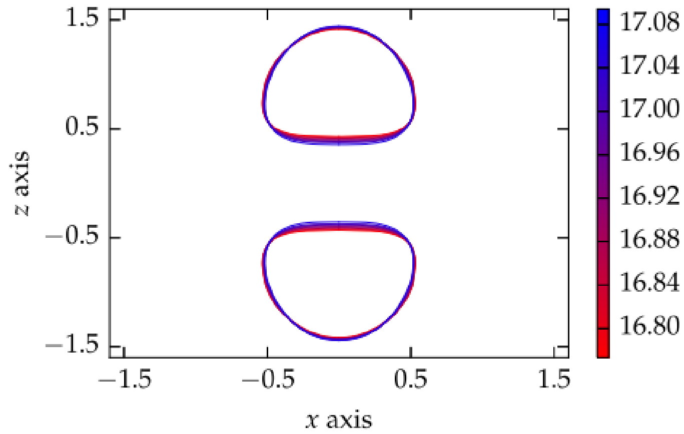
$$M \simeq (p - p_*)^\gamma$$

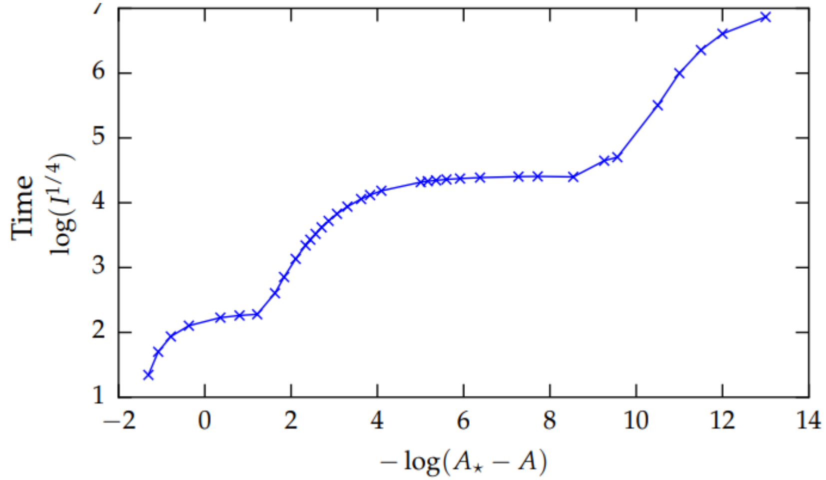
[1] Gundlach, C., Martín-García, J.M. Critical Phenomena in Gravitational Collapse. Living Rev. Relativ. 10, 5 (2007)
[2] M. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field Phys. Rev. Lett. 70, 9. (1993)

Collapse of Gravitational Waves in vacuum

- Evidence of different threshold solutions



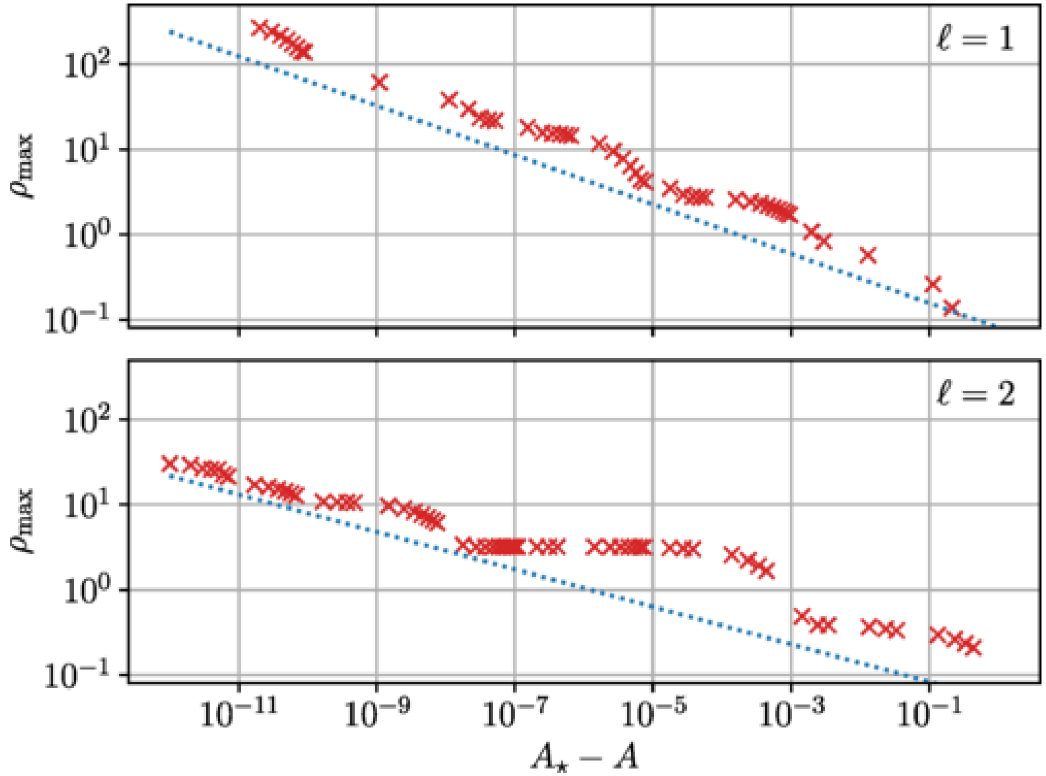




[3] T. Ledvinka and A. Khirnov. Universality of Curvature Invariants in Critical Vacuum Gravitational Collapse Phys. Rev. Lett. 127, 011104
[4] D. Hilditch, A. Weyhausen, and B. Brügmann. Evolutions of centered Brill waves with a pseudospectral method.Phys. Rev. D 96, 104051

Why different critical solutions?

- In [4] they study the collapse of electromagnetic waves
  - Dipoles (l=1)
  - Quadrupoles (l=2)
- Absence of spherical solution, as GW
- Different critical solutions
- Speculation about the different multipoles of the initial data



[5] M. F. Perez Mendoza and T. W. Baumgarte. Critical phenomena in the gravitational collapse of electromagnetic dipole and quadrupole waves. Phys. Rev. D 103, 124048

Comparison of linear Brill and Teukolsky waves

Brill Wave

- Multipolar
- More numerically unstable
- Easy to obtain as ID, (just solving a elliptic equation)
- q(t,r)
- Fully nonlinear
- $h_{ij}^{\mathrm{B}} = \gamma_{ij}^{\mathrm{B}} - \eta_{ij}$

Teukolsky Wave

- Quadrupolar
- More numerically stable
- More difficult to obtain as ID (solving Hamiltonian constraint)
- F(t,r)
- Linear before solving H constraint
- $h_{ij}^{\mathrm{T}} = \gamma_{ij}^{\mathrm{T}} - \eta_{ij}$

[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Teukolsky waves

Metric ansatz

$$ds^2 = -dt^2 + dr^2\,\{1 + Af_{rr}\} + r\,drd\theta\,\{2Bf_{r\theta}\} + r\sin(\theta)\,drd\phi\,\{2Bf_{r\phi}\} + r^2\,d\theta^2\,\{1 + Cf^{(1)}_{\theta\theta} + Af^{(2)}_{\theta\theta}\} + r^2\sin(\theta)\,d\theta d\phi\,\{2(A-2C)f_{\theta\phi}\} + r^2\sin^2(\theta)\,d\phi^2\,\{1 + Cf^{(1)}_{\phi\phi} + Af^{(2)}_{\phi\phi}\},$$

$$\Downarrow$$

$$h_{rr} = Af_{rr},$$
$$h_{r\theta} = rBf_{r\theta},$$
$$h_{\theta\theta} = r^2(Cf^{(1)}_{\theta\theta} + Af^{(2)}_{\theta\theta}),$$
$$h_{\phi\phi} = r^2\sin^2\theta(Cf^{(1)}_{\phi\phi} + Af^{(2)}_{\phi\phi})$$

- f_{ij} → Angular functions
- A, B, C → Coefficients depending on F(t,r) and derivatives
- F(t,r) → Seed function

Chosen seed function

$$F(t,r) = A_T\lambda^4\Big((t-r)e^{-((r-t)/\lambda)^2} - (r+t)e^{-((r+t)/\lambda)^2}\Big)$$

Coefficients

$$A^{\mathrm{T}} = -24\mathcal{A}_{\mathrm{T}}e^{-(r/\lambda)^2},$$
$$B^{\mathrm{T}} = \frac{8\mathcal{A}_{\mathrm{T}}}{\lambda^2}e^{-(r/\lambda)^2}(2r^2 - 3\lambda^2),$$
$$C^{\mathrm{T}} = \frac{8\mathcal{A}_{\mathrm{T}}}{\lambda^4}e^{-(r/\lambda)^2}(r^4 - 4r^2\lambda^2 + 3\lambda^4).$$

[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Brill waves

Metric ansatz

$$\gamma_{ij} dx^i dx^j = \psi^4 \left(e^{2q} (dr^2 + r^2 d\theta^2) + r^2 \sin^2(\theta) d\varphi^2 \right)$$

- q(t,r) → Seed function
- Ψ → Conformal factor

Linear Hamiltonian constraint

$$\nabla^2 \psi = -\frac{\psi}{4} \tau \qquad \tau \equiv \frac{\partial^2 q}{\partial \rho^2} + \frac{\partial^2 q}{\partial z^2}$$

$$\psi = 1 + u \Rightarrow \boxed{\nabla^2 u = -\frac{1}{4} \tau}$$

[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

The special case of Holz data

Holz seed function

$$q(r,\theta) = \mathcal{A}_\mathrm{B} r^2 \sin^2(\theta)\sigma^{-2} e^{-(r/\sigma)^2}$$
$$= \mathcal{A}_\mathrm{B}\rho^2\sigma^{-2}e^{-(\rho^2+z^2)/\sigma^2},$$

$$q(r,\theta) = q_{00}(r)Y_{00}(\theta) + q_{20}(r)Y_{20}(\theta)$$

$$q_{00}(r) = \sqrt{\pi}\,\frac{4\mathcal{A}_\mathrm{B}}{3}\left(\frac{r}{\sigma}\right)^2 e^{-(r/\sigma)^2}$$
$$q_{20}(r) = -\sqrt{\frac{\pi}{5}}\,\frac{4\mathcal{A}_\mathrm{B}}{3}\left(\frac{r}{\sigma}\right)^2 e^{-(r/\sigma)^2}$$

- Centered gaussian
- The most common choice
- Purely quadrupolar!

$$u_{00}(r) = -\frac{\sqrt{\pi}}{6\sigma^2}\mathcal{A}_\mathrm{B} e^{-(r/\sigma)^2}\left(2r^2+\sigma^2\right)$$
$$u_{20}(r) = -\sqrt{\frac{\pi}{5}}\,\frac{\mathcal{A}_\mathrm{B}}{24r^3\sigma^2}\Big(3\sqrt{\pi}\sigma^5\mathrm{erf}\left(\frac{r}{\sigma}\right) - 2re^{-(r/\sigma)^2}\left(2r^2\sigma^2+4r^4+3\sigma^4\right)\Big)$$

[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Comparison → Gauge transformation

We write Brill Waves in the Transverse Traceless (TT) gauge

Metric perturbation

$$
\begin{aligned}
h_{rr} &= A f_{rr}, \\
h_{r\theta} &= r B f_{r\theta}, \\
h_{\theta\theta} &= r^2 (C f_{\theta\theta}^{(1)} + A f_{\theta\theta}^{(2)}), \\
h_{\phi\phi} &= r^2 \sin^2\theta (C f_{\phi\phi}^{(1)} + A f_{\phi\phi}^{(2)})
\end{aligned}
$$

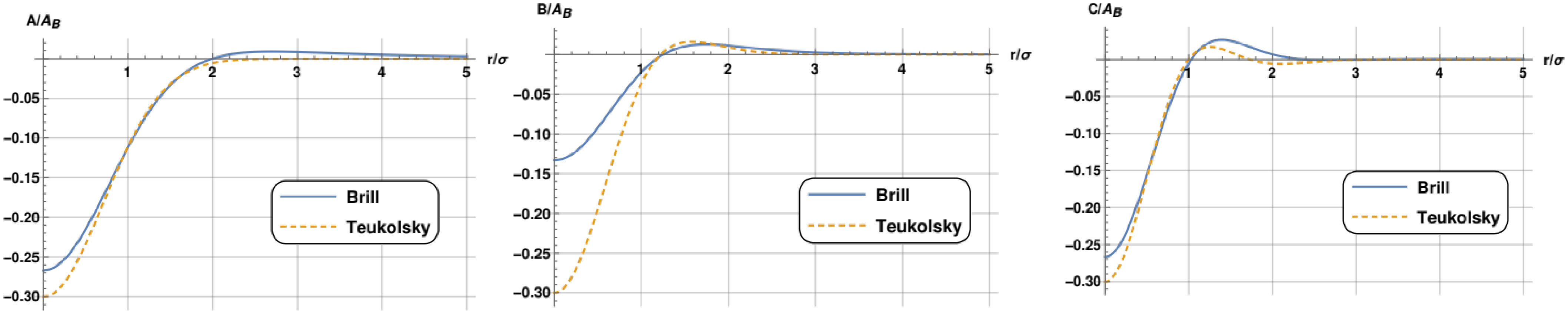
Coefficients

$$
\begin{aligned}
A^{\mathrm{B}} = \frac{\mathcal{A}_{\mathrm{B}}\sigma^2}{8r^5}\Big[&2re^{-(r/\sigma)^2}\left(4r^2+9\sigma^2\right)+ \\
&\sqrt{\pi}\sigma\left(2r^2-9\sigma^2\right)\operatorname{erf}\left(\frac{r}{\sigma}\right)\Big], \\
B^{\mathrm{B}} = -\frac{\mathcal{A}_{\mathrm{B}}}{12r^5}\Big[&2re^{-(r/\sigma)^2}\left(4r^4+6r^2\sigma^2+9\sigma^4\right)- \\
&9\sqrt{\pi}\sigma^5\operatorname{erf}\left(\frac{r}{\sigma}\right)\Big], \\
C^{\mathrm{B}} = \frac{\mathcal{A}_{\mathrm{B}}}{96r^5\sigma^2}\Big[&2re^{-\frac{r^2}{\sigma^2}}\left(16r^6+36r^2\sigma^4+63\sigma^6\right)+ \\
&3\sqrt{\pi}\sigma^5\left(2r^2-21\sigma^2\right)\operatorname{erf}\left(\frac{r}{\sigma}\right)\Big]
\end{aligned}
$$

[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Comparison → Gauge transformation

$$A_T = A_B/80 \quad \lambda = \sigma$$



[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Comparison → Gauge invariant Moncrief formalism

Moncrief function

$$R_{lm} = \frac{r[l(l+1)k_{1lm} + 4(1-2M/r)^2 k_{2lm}]}{l(l+1)[(l-1)(l+2)+6M/r]}$$

$$\begin{aligned}
k_{1lm} &= K_{lm} + l(l+1)G_{lm} + 2\left(1-\frac{2M}{r}\right)\left(r\partial_r G_{lm} - \frac{h_{1lm}}{r}\right), \\
k_{2lm} &= \frac{H_{2lm}}{2(1-2M/r)} - \frac{1}{2\sqrt{1-2M/r}}\partial_r\left(\frac{r[K_{lm}+l(l+1)G_{lm}]}{\sqrt{1-2M/r}}\right)
\end{aligned}$$

$$\begin{aligned}
H_{2lm} &= \int\left(1-\frac{2M}{r}\right)\gamma_{rr}Y^*_{lm}d\Omega \\
h_{1lm} &= \frac{1}{l(l+1)}\int\sigma^{ab}(E^{lm}_b)^*\gamma_{ra}d\Omega \\
&= \frac{1}{l(l+1)}\int\left((\partial_\theta Y_{lm})^*\gamma_{r\theta} + (\partial_\phi Y_{lm})^*\frac{\gamma_{r\phi}}{\sin^2\theta}\right)d\Omega \\
K_{lm} &= \frac{1}{2r^2}\int\gamma_{cd}\sigma^{cd}Y^*_{lm}d\Omega = \frac{1}{2r^2}\int\gamma_+ Y^*_{lm}d\Omega \\
G_{lm} &= \frac{1}{(l-1)l(l+1)(l+2)r^2}\int\gamma_{cd}(Z^{cd}_{lm})^*d\Omega
\end{aligned}$$

[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

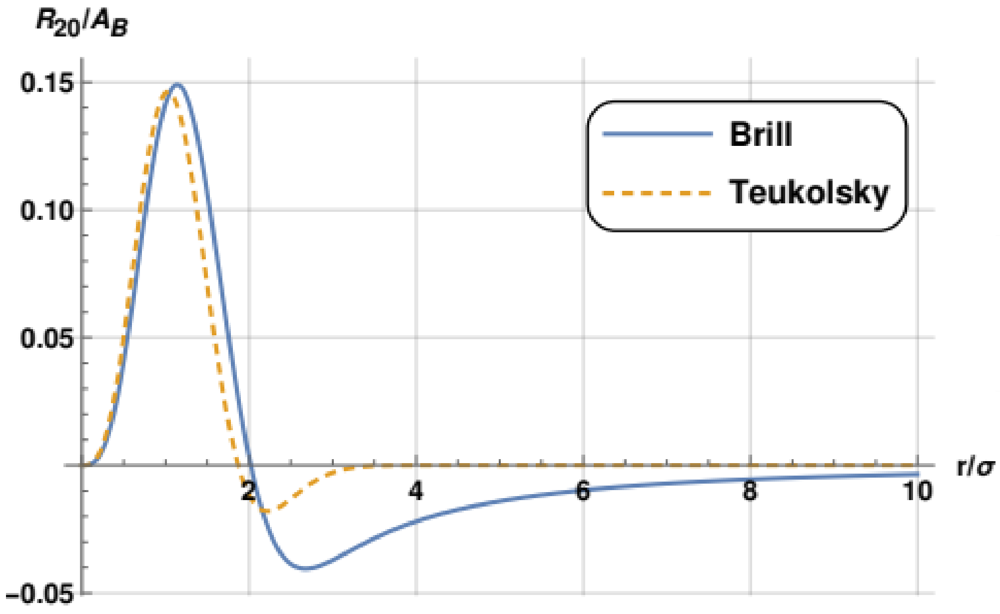
Comparison → Gauge invariant Moncrief formalism

$$R^{\mathrm{T}}_{20} = -\sqrt{\frac{\pi}{5}}\frac{8\mathcal{A}_{\mathrm{T}}}{\lambda^4} r^3 e^{-(r/\lambda)^2}\left(2r^2 - 7\lambda^2\right)$$

$$R^{\mathrm{B}}_{20} = \sqrt{\frac{\pi}{5}}\mathcal{A}_{\mathrm{B}}\left[\frac{1}{6r\sigma^2}e^{-(r/\sigma)^2}\left(4r^4 + 2r^2\sigma^2 + 3\sigma^4\right) - \frac{\sqrt{\pi}}{4}\frac{\sigma^3}{r^2}\mathrm{erf}\left(\frac{r}{\sigma}\right)\right] \tag{40}$$

$$\boxed{R_{20} = \frac{r}{6}\sqrt{\frac{\pi}{5}}\left(r\partial_r A - 6A - 6B + 12C\right)}$$

$A_T = A_B/80 \quad \lambda = \sigma$

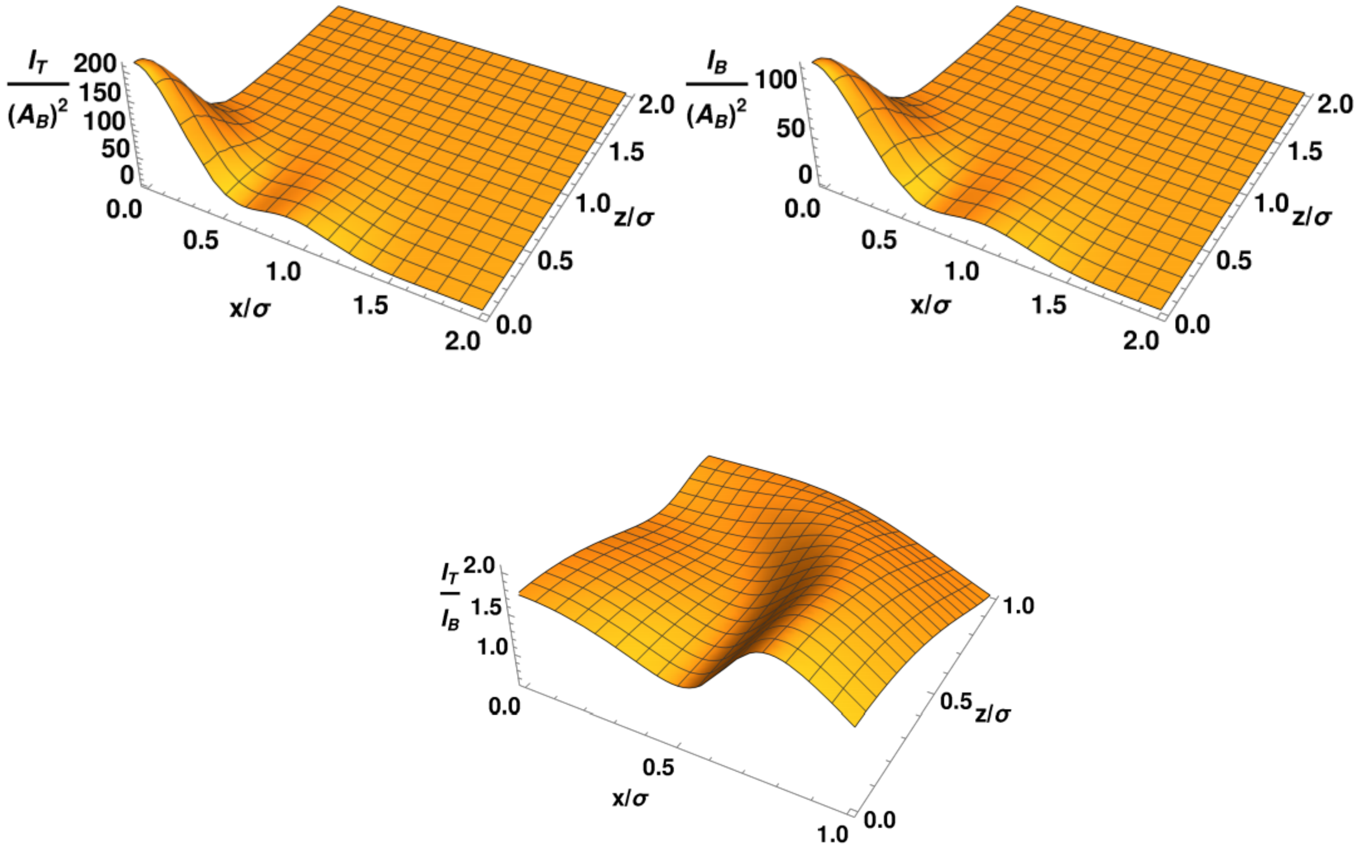


[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Comparison → Kretschmann scalar

$$I = 8\,\gamma^{ij}\gamma^{kl}R_{ik}R_{jl}$$

$A_T = A_B/80 \quad \lambda = \sigma$



[6] I. Suárez Fernández, T. W. Baumgarte and D. Hilditch. Comparison of linear Brill and Teukoslky waves. arXiv:2111.04752

Conclusions

- The Holz data, the most common choice, is purely dipolar, as Teukolsky.
- Compare in 3 different ways Brill and Teukolsky waves with coherent results
- Brill Waves in TT can be expressed as Teukolsky waves, but we are missing F(t,r)
- The Moncrief function and the Kretschmann scalar are very similar qualitatively
- The different multipoles of Brill Waves coupling to different parts of the Einstein equations does not seem the reason of the differences between Teukolsky and Brill waves behavior.

Conclusions

- The Holz data, the most common choice, is purely dipolar
- Compare in 3 different ways Brill and Teukolsky waves with coherent results
- Brill Waves in TT can be expressed as Teukolsky waves, but we are missing F(t,r)
- The Moncrief function and the Kretschmann scalar are very similar qualitatively
- The different multipoles of Brill Waves coupling to different parts of the Einstein equations does not seem the reason of the differences between Teukolsky and Brill waves behavior.

Thank you