Theoretical analysis of material constants in thermopiezoelasticity

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Abstract

Materials with thermopiezoelastic properties have both theoretical and practical significance in solid-state physics and materials science. Thermopiezoelastic media exhibit coupling among the thermal, electric and elastic fields. Strain, electric displacement and entropy can be written in terms of stress, electric field and temperature. Investigation is concerned with the analysis of nonlinear thermopiezoelasticity using thermodynamic principles. Strain, electric displacement and entropy are expanded into Taylor series. Zeroth to eighth rank tensors are derived for describing material constants. Relationship of material constants of strain, electric displacement and entropy are obtained in thermopiezoelasticity. Due to intrinsic coupling behavior, thermopiezoelastic materials are widely used as sensors and actuators in sensing, actuation, and control of smart structures. Mathematical expressions of material constants may be useful for future investigation of the mechanics and physics of nonlinear thermopiezoelasticity.

Introduction

Mechanical, electrical, and thermal fields are coupled in thermopiezoelasticity. Nonlinear analysis of thermopiezoelasticity has been made for determining their material constants. In the previous studies, Bao and co-workers (1998) introduced static, dynamic, and control characteristics of a nonlinear piezoelectric laminated beam subjected to mechanical, temperature, and electric excitations. Wang et al. (1999) studied the nonlinear electromechanical behavior piezoelectric ceramic in a wide electric field and frequency range. Zhou and Tzou (2000) developed nonlinear electromechanics and active control of piezoelectric laminated circular spherical shallow shells. Hall (2001) gave an overview of experimental evidence and understanding of nonlinear dielectric, elastic and piezoelectric relationships in piezoelectric ceramics.

Furthermore, Altay and Dokmeci (2002) described a nonlinear rod theory for high-frequency vibrations of thermopiezoelectric materials. Warkusz and Linek (2003) analyzed material constants from zeroth to sixth rank tensors in nonlinear mechanical, electrical and thermal phenomena in piezoelectric crystals.Wagner (2004) presented nonlinear longitudinal vibrations of non-slender piezoceramic rods. Mukherjee and Chaudhuri (2005) deduced a generalized formulation for nonlinear dynamic analysis of piezoelectric structures. Blackburn and Cain (2006) examined nonlinear piezoelectric resonance.

Among some recent works, Balakina (2007) deduced analytical expressions for the pyro- and piezoelectric coefficients of nonlinear optical polymer electrets. Grigoriev et al. (2008) discussed nonlinear piezoelectricity in epitaxial ferroelectrics at high electric fields. Xia and Shen (2009) proposed an analysis with the nonlinear vibration and dynamic response of a shear deformable functionally graded material (FGM) plate with surface-bonded piezoelectric fiber reinforced composite actuators (PFRC) in thermal environments. Yiqi and Yiming (2010) considered nonlinear dynamic response and active vibration control of the piezoelectric functionally graded plate.

Present analysis deals with thermodynamic Gibbs function. Strain, electric displacement and entropy are produced simultaneously by stress, electric field and temperature change. In order to derive material constants from zeroth to eighth rank tensors, thermodynamic Gibbs function is expressed in Taylor series up to fourth order differentiation.

Thermopiezoelasticity and thermodynamics

In thermopiezoelasticity, thermodynamics Gibbs function G (or the Gibbs potential) can be written as

$$G = U - \varepsilon_{ij}\sigma_{ij} - E_m D_m - TS \tag{1}$$

In equation (1), U, T and S are internal energy, temperature change and entropy, respectively. Subscripts indices i, j, m are ranging 1 to 3. It is noted that all other subscripts indices denoting in other equations range from 1 to 3 unless otherwise specified. Strain, stress, electric field and electric displacement are denoted by ε_{ij}, σ_{ij}, E_m and D_m. Following Einstein's summation convention, the summation sign $\sum$ is omitted. For example, its detail can be found in Weber, Balashova and Kizhaev (2000).

Differential form of Gibbs function is expressed as

$$\mathrm{d}G = \mathrm{d}U - \varepsilon_{ij}\mathrm{d}\sigma_{ij} - \sigma_{ij}\mathrm{d}\varepsilon_{ij} - E_m\mathrm{d}D_m - D_m\mathrm{d}E_m - S\,\mathrm{d}T - T\,\mathrm{d}S \tag{2}$$

As far as the first and second law of thermodynamics are concerned, differential form of internal energy is represented by

$$\mathrm{d}U = \sigma_{ij}\mathrm{d}\varepsilon_{ij} + E_m\mathrm{d}D_m + T\,\mathrm{d}S \tag{3}$$

It follows from equations (2) and (3) that

$$\mathrm{d}G = -\varepsilon_{ij}\mathrm{d}\sigma_{ij} - D_m\mathrm{d}E_m - S\,\mathrm{d}T\,. \tag{4}$$

Strain, electric displacement and entropy can be written in the following forms

$$\varepsilon_{ij} = f(\sigma_{kl}, E_n, T) \tag{5}$$

$$D_m = f(\sigma_{kl}, E_n, T) \tag{6}$$

$$S = f(\sigma_{kl}, E_n, T) \tag{7}$$

Further, thermodynamics Gibbs function can be written as

$$\mathrm{d}G = \left.\frac{\partial G}{\partial \sigma_{ij}}\right|_{E,T}\mathrm{d}\sigma_{ij} + \left.\frac{\partial G}{\partial E_m}\right|_{\sigma,T}\mathrm{d}E_m + \left.\frac{\partial G}{\partial T}\right|_{\sigma,E}\mathrm{dT} \tag{8}$$

In equation (8), indices in the lower part of vertical line on the partial derivatives indicate the variable that must be held constant during differentiation.

Also, equations (4) and (8) yield

$$\varepsilon_{ij} = -\left.\frac{\partial G}{\partial \sigma_{ij}}\right|_{E,T} \tag{9}$$

$$D_m = -\left.\frac{\partial G}{\partial E_m}\right|_{\sigma,T} \tag{10}$$

$$S = -\left.\frac{\partial G}{\partial T}\right|_{\sigma,E} \tag{11}$$

Material constants are obtained by the thermodynamic Gibbs function. Strain, electric displacement and entropy are expanded into Taylor series.

Nonlinear equations in thermopiezoelasticity

Up to third order differentiation, Strain, electric displacement and entropy can be expressed in Taylor Series as follows:

$$\varepsilon_{ij} = (\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}}\sigma_{pq} + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial E_n}E_n + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial T}T$$

$$+\frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\sigma_{pq}E_n + \frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\sigma_{pq}T)\sigma_{kl}$$
$$+(\frac{\partial \varepsilon_{ij}}{\partial E_n} + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial E_n}\sigma_{kl} + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial E_n\partial E_t}E_t + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial E_n\partial T}T$$
$$+\frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{pq}\partial E_t\partial E_n}\sigma_{pq}E_t + \frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{pq}\partial E_n\partial T}\sigma_{pq}T)E_n$$
$$+(\frac{\partial \varepsilon_{ij}}{\partial T} + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial T}\sigma_{kl} + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial E_n\partial T}E_n + \frac{1}{2!}\frac{\partial^2 \varepsilon_{ij}}{\partial T^2}T$$
$$+\frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{pq}\partial E_n\partial T}\sigma_{pq}E_n + \frac{1}{3!}\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial T^2}\sigma_{kl}T)T \tag{12}$$

$$D_m = (\frac{\partial D_m}{\partial \sigma_{kl}} + \frac{1}{2!}\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}}\sigma_{pq} + \frac{1}{2!}\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial E_n}E_n + \frac{1}{2!}\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial T}T$$
$$+\frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\sigma_{pq}E_n + \frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\sigma_{pq}T)\sigma_{kl}$$
$$+(\frac{\partial D_m}{\partial E_n} + \frac{1}{2!}\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial E_n}\sigma_{kl} + \frac{1}{2!}\frac{\partial^2 D_m}{\partial E_n\partial E_t}E_t + \frac{1}{2!}\frac{\partial^2 D_m}{\partial E_n\partial T}T$$
$$+\frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{pq}\partial E_t\partial E_n}\sigma_{pq}E_t + \frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{pq}\partial E_n\partial T}\sigma_{pq}T)E_n$$
$$+(\frac{\partial D_m}{\partial T} + \frac{1}{2!}\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial T}\sigma_{kl} + \frac{1}{2!}\frac{\partial^2 D_m}{\partial E_n\partial T}E_n + \frac{1}{2!}\frac{\partial^2 D_m}{\partial T^2}T$$
$$+\frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{pq}\partial E_n\partial T}\sigma_{pq}E_n + \frac{1}{3!}\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial T^2}\sigma_{kl}T)T \tag{13}$$

$$S = (\frac{\partial S}{\partial \sigma_{kl}} + \frac{1}{2!}\frac{\partial^2 S}{\partial \sigma_{kl}\partial \sigma_{pq}}\sigma_{pq} + \frac{1}{2!}\frac{\partial^2 S}{\partial \sigma_{kl}\partial E_n}E_n + \frac{1}{2!}\frac{\partial^2 S}{\partial \sigma_{kl}\partial T}T$$
$$+\frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\sigma_{pq}E_n + \frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\sigma_{pq}T)\sigma_{kl}$$
$$+(\frac{\partial S}{\partial E_n} + \frac{1}{2!}\frac{\partial^2 S}{\partial \sigma_{kl}\partial E_n}\sigma_{kl} + \frac{1}{2!}\frac{\partial^2 S}{\partial E_n\partial E_t}E_t + \frac{1}{2!}\frac{\partial^2 S}{\partial E_n\partial T}T$$
$$+\frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{pq}\partial E_t\partial E_n}\sigma_{pq}E_t + \frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{pq}\partial E_n\partial T}\sigma_{pq}T)E_n$$
$$+(\frac{\partial S}{\partial T} + \frac{1}{2!}\frac{\partial^2 S}{\partial \sigma_{kl}\partial T}\sigma_{kl} + \frac{1}{2!}\frac{\partial^2 S}{\partial E_n\partial T}E_n + \frac{1}{2!}\frac{\partial^2 S}{\partial T^2}T$$

$$+\frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{pq}\partial E_n\partial T}\sigma_{pq}E_n + \frac{1}{3!}\frac{\partial^3 S}{\partial \sigma_{kl}\partial T^2}\sigma_{kl}T)T \tag{14}$$

Material constants obtained from strain ε_{ij}

Taking equation (9), expressions of material constants can be obtained as

Second order elasticity constant (fourth rank tensor)

$$\left.\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}}\right|_{E,T} = \frac{\partial}{\partial \sigma_{kl}}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^2 G}{\partial \sigma_{kl}\partial \sigma_{ij}}\right|_{E,T} = s^{E,T}_{ijkl} \tag{15a}$$

In equation (15a) and all other equations of material constants, superscript indices are constant. It should be noted that superscripts indices are not taken as powers.

Second order piezoelectric constant (third rank tensor)

$$\left.\frac{\partial \varepsilon_{ij}}{\partial E_n}\right|_{\sigma,T} = \frac{\partial}{\partial E_n}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^2 G}{\partial E_n\partial \sigma_{ij}}\right|_{T} = d^{T}_{ijn} \tag{15b}$$

Second order piezo-calorific constant (second rank tensor)

$$\left.\frac{\partial \varepsilon_{ij}}{\partial T}\right|_{\sigma,E} = \frac{\partial}{\partial T}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^2 G}{\partial T\partial \sigma_{ij}}\right|_{E} = \alpha^{E}_{ij} \tag{15c}$$

Third order elasticity constant (sixth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}}\right|_{E,T} = \frac{\partial^2}{\partial \sigma_{kl}\partial \sigma_{pq}}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{ij}}\right|_{E,T} = s^{E,T}_{ijklpq} \tag{15d}$$

Third order piezoelectric constant (fifth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial E_n}\right|_{T} = \frac{\partial^2}{\partial \sigma_{kl}\partial E_n}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial E_n\partial \sigma_{ij}}\right|_{T} = d^{T}_{ijk\ln} \tag{15e}$$

Third order piezo-calorific constant (fourth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{ij}}{\partial \sigma_{kl}\partial T}\right|_{E} = \frac{\partial^2}{\partial \sigma_{kl}\partial T}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial T\partial \sigma_{ij}}\right|_{E} = \alpha^{E}_{ijkl} \tag{15f}$$

Fourth order elasticity constant (eighth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\right|_{E,T} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}\partial \sigma_{ij}}\right|_{E,T} = s^{E,T}_{ijklpqrs} \tag{15g}$$

Fourth order piezoelectric constant (seventh rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\right|_{T} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n\partial \sigma_{ij}}\right|_{T} = d^{T}_{ijklpqn} \tag{15h}$$

Fourth order piezo-calorific constant (sixth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T\partial \sigma_{ij}}\right|_{E} = \alpha^{E}_{ijklpq} \tag{15i}$$

Third order electrostriction constant (fourth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{ij}}{\partial E_n\partial E_t}\right|_{\sigma,T} = \frac{\partial^2}{\partial E_n\partial E_t}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^3 G}{\partial E_n\partial E_t\partial \sigma_{ij}}\right|_{T} = o^{T}_{ijnt} \tag{15j}$$

Third order electro-thermo-elastic constant (third rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{ij}}{\partial E_n\partial T}\right|_{\sigma} = \frac{\partial^2}{\partial E_n\partial T}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^3 G}{\partial E_n\partial T\partial \sigma_{ij}}\right| = k_{ijn} \tag{15k}$$

Fourth order electrostriction constant (sixth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{pq}\partial E_t\partial E_n}\right|_{T} = \frac{\partial^3}{\partial \sigma_{pq}\partial E_t\partial E_n}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_t\partial E_n\partial \sigma_{ij}}\right|_{T} = o^{T}_{ijpqtn} \tag{15l}$$

Fourth order electro-thermo-elastic constant (fifth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{pq}\partial E_n\partial T}\right| = \frac{\partial^3}{\partial \sigma_{pq}\partial E_n\partial T}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_n\partial T\partial \sigma_{ij}}\right| = k_{ijpqn} \tag{15m}$$

Third order thermal expansion constant (second rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{ij}}{\partial T^2}\right|_{\sigma,E} = \frac{\partial^2}{\partial T^2}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^3 G}{\partial T^2\partial \sigma_{ij}}\right|_{E} = r^{E}_{ij} \tag{15n}$$

Fourth order thermal expansion constant (fourth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{ij}}{\partial \sigma_{kl}\partial T^2}\right|_{E} = \frac{\partial^3}{\partial \sigma_{kl}\partial T^2}\left(-\frac{\partial G}{\partial \sigma_{ij}}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial T^2\partial \sigma_{ij}}\right|_{E} = r^{E}_{ijkl} \tag{15o}$$

Material constants obtained from electric displacement D_m

Considering equation (10), the material constants can be derived as

Second order piezoelectric constant (third rank tensor)

$$\left.\frac{\partial D_m}{\partial \sigma_{kl}}\right|_{E,T} = \frac{\partial}{\partial \sigma_{kl}}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^2 G}{\partial \sigma_{kl}\partial E_m}\right|_{T} = d^{T}_{klm} \tag{16a}$$

Second order permittivity constant (second rank tensor)

$$\left.\frac{\partial D_m}{\partial E_n}\right|_{\sigma,T} = \frac{\partial}{\partial E_n}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^2 G}{\partial E_n\partial E_m}\right|_{\sigma,T} = \eta^{\sigma,T}_{mn} \tag{16b}$$

Second order electric heat constant (first rank tensor)

$$\left.\frac{\partial D_m}{\partial T}\right|_{\sigma,E} = \frac{\partial}{\partial T}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^2 G}{\partial T\partial E_m}\right|_{\sigma} = p^{\sigma}_{m} \tag{16c}$$

Third order piezoelectric constant (fifth rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}}\right|_{E,T} = \frac{\partial^2}{\partial \sigma_{kl}\partial \sigma_{pq}}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_m}\right|_{T} = d^{T}_{klpqm} \tag{16d}$$

Third order electrostriction constant (fourth rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial E_n}\right|_{T} = \frac{\partial^2}{\partial \sigma_{kl}\partial E_n}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial E_n\partial E_m}\right|_{T} = o^{T}_{klmn} \tag{16e}$$

Third order electro-thermo-elastic constant (third rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial T}\right|_{E} = \frac{\partial^2}{\partial \sigma_{kl}\partial T}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial T\partial E_m}\right| = k_{klm} \tag{16f}$$

Fourth order piezoelectric constant (seventh rank tensor)

$$\left.\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\right|_{E,T} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}\partial E_m}\right|_{T} = d^{T}_{klpqrsm} \tag{16g}$$

Fourth order electrostriction constant (sixth rank tensor)

$$\left.\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\right|_{T} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n\partial E_m}\right|_{T} = o^{T}_{klpqnm} \tag{16h}$$

Fourth order electro-thermo-elastic constant (fifth rank tensor)

$$\left.\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T\partial E_m}\right| = k_{klpqm} \tag{16i}$$

Third order permittivity constant (third rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial E_n\partial E_t}\right|_{\sigma,T} = \frac{\partial^2}{\partial E_n\partial E_t}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^3 G}{\partial E_n\partial E_m\partial E_t}\right|_{\sigma,T} = \eta^{\sigma,T}_{mnt} \tag{16j}$$

Third order electric heat constant (second rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial E_n\partial T}\right|_{\sigma} = \frac{\partial^2}{\partial E_n\partial T}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^3 G}{\partial E_n\partial T\partial E_m}\right|_{\sigma} = p^{\sigma}_{mn} \tag{16k}$$

Fourth order electro-elastic constant (fifth rank tensor)

$$\left.\frac{\partial^3 D_m}{\partial \sigma_{pq}\partial E_t\partial E_n}\right|_{T} = \frac{\partial^3}{\partial \sigma_{pq}\partial E_t\partial E_n}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_t\partial E_n\partial E_m}\right|_{T} = \Gamma^{T}_{pqtnm} \tag{16l}$$

Fourth order electro-thermo-electro-elastic constant (fourth rank tensor)

$$\left.\frac{\partial^3 E_m}{\partial \sigma_{pq}\partial E_n\partial T}\right| = \frac{\partial^3}{\partial \sigma_{pq}\partial E_n\partial T}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_n\partial T\partial E_m}\right| = \beta_{pqnm} \tag{16m}$$

Third order pyroelectric constant (first rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial T^2}\right|_{\sigma,E} = \frac{\partial^2}{\partial T^2}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^3 G}{\partial T^2\partial E_m}\right|_{\sigma} = u^{\sigma}_{m} \tag{16n}$$

Fourth order thermo-electro-elastic constant (third rank tensor)

$$\left.\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial T^2}\right|_{E} = \frac{\partial^3}{\partial \sigma_{kl}\partial T^2}\left(-\frac{\partial G}{\partial E_m}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial T^2\partial E_m}\right| = \omega_{klm} \tag{16o}$$

Material constants obtained from entropy S

Using equation (11), the material constants can be expressed as

Second order piezo-calorific constant (second rank tensor)

$$\left.\frac{\partial S}{\partial \sigma_{kl}}\right|_{E,T} = \frac{\partial}{\partial \sigma_{kl}}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^2 G}{\partial \sigma_{kl}\partial T}\right|_{E} = \alpha^{E}_{kl} \tag{17a}$$

Second order electric heat constant (first rank tensor)

$$\left.\frac{\partial S}{\partial E_n}\right|_{\sigma,T} = \frac{\partial}{\partial E_n}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^2 G}{\partial E_n\partial T}\right|_{\sigma} = p^{\sigma}_{n} \tag{17b}$$

Second order thermal capacity constant (zeroth rank tensor)

$$\left.\frac{\partial S}{\partial T}\right|_{\sigma,E} = \frac{\partial}{\partial T}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^2 G}{\partial T^2}\right|_{\sigma,E} = \xi^{\sigma,E} \tag{17c}$$

Third order piezo-calorific constant (fourth rank tensor)

$$\left.\frac{\partial^2 S}{\partial \sigma_{kl}\partial \sigma_{pq}}\right|_{E,T} = \frac{\partial^2}{\partial \sigma_{kl}\partial \sigma_{pq}}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = \alpha^{E}_{klpq} \tag{17d}$$

Third order electro-thermo-elastic constant (third rank tensor)

$$\left.\frac{\partial^2 S}{\partial \sigma_{kl}\partial E_n}\right|_{T} = \frac{\partial^2}{\partial \sigma_{kl}\partial E_n}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial E_n\partial T}\right| = k_{k\ln} \tag{17e}$$

Third order thermal expansion constant (second rank tensor)

$$\left.\frac{\partial^2 S}{\partial \sigma_{kl}\partial T}\right|_{E} = \frac{\partial^2}{\partial \sigma_{kl}\partial T}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial T^2}\right|_{E} = r^{E}_{kl} \tag{17f}$$

Fourth order piezo-calorific constant (sixth rank tensor)

$$\left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\right|_{E,T} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}\partial T}\right|_{E} = \alpha^{E}_{klpqrs} \tag{17g}$$

Fourth order electro-thermo-elastic constant (fifth rank tensor)

$$\left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\right|_{T} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n\partial T}\right| = k_{klpqn} \tag{17h}$$

Fourth order thermal expansion constant (fourth rank tensor)

$$\left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = \frac{\partial^3}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T^2}\right|_{E} = r^{E}_{klpq} \tag{17i}$$

Third order electric heat constant (second rank tensor)

$$\left.\frac{\partial^2 S}{\partial E_n\partial E_t}\right|_{\sigma,T} = \frac{\partial^2}{\partial E_n\partial E_t}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial E_n\partial E_t\partial T}\right|_{\sigma} = p^{\sigma}_{nt} \tag{17j}$$

Third order pyroelectric constant (first rank tensor)

$$\left.\frac{\partial^2 S}{\partial E_n\partial T}\right|_{\sigma} = \frac{\partial^2}{\partial E_n\partial T}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial E_n\partial T^2}\right|_{\sigma} = u^{\sigma}_{n} \tag{17k}$$

Fourth order electro-thermo-electro-elastic constant (fourth rank tensor)

$$\left.\frac{\partial^3 S}{\partial \sigma_{pq}\partial E_t\partial E_n}\right|_{T} = \frac{\partial^3}{\partial \sigma_{pq}\partial E_t\partial E_n}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_n\partial E_t\partial T}\right| = \beta_{pqnt} \tag{17l}$$

Fourth order thermo-electro-elastic constant (third rank tensor)

$$\left.\frac{\partial^3 S}{\partial \sigma_{pq}\partial E_n\partial T}\right| = \frac{\partial^3}{\partial \sigma_{pq}\partial E_n\partial T}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_n\partial T^2}\right| = \omega_{pqn} \tag{17m}$$

Third order thermal capacity constant (zeroth rank tensor)

$$\left.\frac{\partial^2 S}{\partial T^2}\right|_{\sigma,E} = \frac{\partial^2}{\partial T^2}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial T^3}\right|_{\sigma,E} = P^{\sigma,E} \tag{17n}$$

Fourth order thermo-elastic (second rank tensor)

$$\left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial T^2}\right|_{E} = \frac{\partial^3}{\partial \sigma_{kl}\partial T^2}\left(-\frac{\partial G}{\partial T}\right) = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial T^3}\right|_{E} = \mu^{E}_{kl} \tag{17o}$$

Relationship of thermopiezoelastic material constants

Considering equations (15a) to (17o), the material constants can be further written as

Second order piezoelectric constant (third rank tensor)

$$\left.\frac{\partial \varepsilon_{kl}}{\partial E_n}\right|_{\sigma,T} = \left.\frac{\partial D_n}{\partial \sigma_{kl}}\right|_{E,T} = -\left.\frac{\partial^2 G}{\partial \sigma_{kl}\partial E_n}\right|_{T} = d^{T}_{klm} \tag{18a}$$

Second order piezo-calorific constant (second rank tensor)

$$\left.\frac{\partial \varepsilon_{kl}}{\partial T}\right|_{\sigma,E} = \left.\frac{\partial S}{\partial \sigma_{kl}}\right|_{E,T} = -\left.\frac{\partial^2 G}{\partial \sigma_{kl}\partial T}\right|_{E} = \alpha^{E}_{kl} \tag{18b}$$

Third order piezoelectric constant (fifth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{kl}}{\partial \sigma_{kl}\partial E_n}\right|_{\sigma,T} = \left.\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}}\right|_{E,T} = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_m}\right|_{T} = d^{T}_{klpqm} \tag{18c}$$

Third order piezo-calorific constant (fourth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{kl}}{\partial \sigma_{pq}\partial T}\right|_{E} = \left.\frac{\partial^2 S}{\partial \sigma_{kl}\partial \sigma_{pq}}\right|_{E,T} = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = \alpha^{E}_{klpq} \tag{18d}$$

Fourth order piezoelectric constant (seventh rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{kl}}{\partial \sigma_{pql}\partial \sigma_{rs}\partial E_m}\right|_{T} = \left.\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\right|_{E,T} = -\left.\frac{\partial^r G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}\partial E_m}\right|_{T} = d^{T}_{klpqrsm} \tag{18e}$$

Fourth order piezo-calorific constant (sixth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{kl}}{\partial \sigma_{pq}\partial \sigma_{rs}\partial T}\right|_{E} = \left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial \sigma_{rs}}\right|_{E,T} = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T\partial \sigma_{rs}}\right|_{E} = \alpha^{E}_{klpqrs} \tag{18f}$$

Third order electrostriction constant (fourth rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{kl}}{\partial E_n\partial E_t}\right|_{\sigma,T} = \left.\frac{\partial^2 D_m}{\partial \sigma_{kl}\partial E_t}\right|_{T} = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial E_n\partial E_t}\right|_{T} = o^{T}_{k\ln t} \tag{18g}$$

Third order electro-thermo-elastic constant (third rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{kl}}{\partial E_n\partial T}\right|_{\sigma} = \left.\frac{\partial^2 D_n}{\partial \sigma_{kl}\partial T}\right|_{E} = \left.\frac{\partial^2 S}{\partial \sigma_{kl}\partial E_n}\right|_{T} = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial E_n\partial T}\right| = k_{k\ln} \tag{18h}$$

Fourth order electrostriction constant (sixth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{kl}}{\partial \sigma_{pq}\partial E_t\partial E_n}\right|_{T} = \left.\frac{\partial^3 D_n}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_t}\right|_{T} = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n\partial E_t}\right|_{T} = o^{T}_{klpqnt} \tag{18i}$$

Fourth order electro-thermo-elastic constant (fifth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{kl}}{\partial \sigma_{pq}\partial E_n\partial T}\right| = \left.\frac{\partial^3 D_n}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = \left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial E_n}\right|_{T} = -\left.\frac{\partial^4 G}{\sigma_{kl}\partial \sigma_{pq}\partial E_n\partial T}\right| = k_{klpqn} \tag{18j}$$

Third order thermal expansion constant (second rank tensor)

$$\left.\frac{\partial^2 \varepsilon_{kl}}{\partial T^2}\right|_{\sigma,E} = \left.\frac{\partial^2 S}{\partial \sigma_{kl}\partial T}\right|_{E} = -\left.\frac{\partial^3 G}{\partial \sigma_{kl}\partial T^2}\right|_{E} = r^{E}_{kl} \tag{18k}$$

Fourth order thermal expansion constant (fourth rank tensor)

$$\left.\frac{\partial^3 \varepsilon_{kl}}{\partial \sigma_{pq}\partial T^2}\right|_{E} = \left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T}\right|_{E} = -\left.\frac{\partial^4 G}{\partial \sigma_{kl}\partial \sigma_{pq}\partial T^2}\right|_{E} = r^{E}_{klpq} \tag{18l}$$

Second order electric heat constant (first rank tensor)

$$\left.\frac{\partial D_n}{\partial T}\right|_{\sigma,E} = \left.\frac{\partial S}{\partial E_n}\right|_{\sigma,T} = -\left.\frac{\partial^2 G}{\partial E_n\partial T}\right|_{\sigma} = p^{\sigma}_{n} \tag{18m}$$

Third order electric heat constant (second rank tensor)

$$\left.\frac{\partial^2 D_m}{\partial E_n\partial T}\right|_{\sigma} = \left.\frac{\partial^2 S}{\partial E_n\partial E_m}\right|_{\sigma,T} = -\left.\frac{\partial^3 G}{\partial E_n\partial E_m\partial T}\right|_{\sigma} = p^{\sigma}_{nm} \tag{18n}$$

Fourth order electro-thermo-electro-elastic constant (fourth rank tensor)

$$\left.\frac{\partial^3 E_m}{\partial \sigma_{pq}\partial E_n\partial T}\right| = \left.\frac{\partial^3 S}{\partial \sigma_{pq}\partial E_m\partial E_n}\right|_{T} = -\left.\frac{\partial^4 G}{\partial \sigma_{pq}\partial E_n\partial E_m\partial T}\right| = \beta_{pqmn} \tag{18o}$$

Third order pyroelectric constant (first rank tensor)

$$\left.\frac{\partial^2 D_n}{\partial T^2}\right|_{\sigma,E} = \left.\frac{\partial^2 S}{\partial E_n\partial T}\right|_{\sigma} = -\left.\frac{\partial^3 G}{\partial T^2\partial E_n}\right|_{\sigma} = u^{\sigma}_{n} \tag{18p}$$

Fourth order thermo-electro-elastic constant (third rank tensor)

$$\left.\frac{\partial^3 D_m}{\partial \sigma_{kl}\partial T^2}\right|_{E} = \left.\frac{\partial^3 S}{\partial \sigma_{kl}\partial E_m\partial T}\right| = -\left.\frac{\partial^4 G}{\sigma_{kl}\partial E_m\partial T^2}\right| = \omega_{klm} \tag{18q}$$

By using equations (15a) to (17o) into equations (12) to (14), this results nonline thermopiezoelastic equations

$$\varepsilon_{ij} = (s^{E,T}_{ijkl} + \frac{1}{2!}s^{E,T}_{ijklpq}\sigma_{pq} + \frac{1}{2!}d^{T}_{ijk\ln}E_n + \frac{1}{2!}\alpha^{E}_{ijkl}T$$
$$+\frac{1}{3!}s^{E,T}_{ijklpqrs}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}d^{T}_{ijklpqn}\sigma_{pq}E_n + \frac{1}{3!}\alpha^{E}_{ijklpq}\sigma_{pq}T)\sigma_{kl}$$
$$+(d^{T}_{ijn} + \frac{1}{2!}d^{T}_{ijk\ln}\sigma_{kl} + \frac{1}{2!}o^{T}_{ijnt}E_t + \frac{1}{2!}k_{ijn}T$$
$$+\frac{1}{3!}d^{T}_{ijklpqn}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}o^{T}_{ijpqtn}\sigma_{pq}E_t + \frac{1}{3!}k_{ijpqn}\sigma_{pq}T)E_n$$
$$+(\alpha^{E}_{ij} + \frac{1}{2!}\alpha^{E}_{ijkl}\sigma_{kl} + \frac{1}{2!}k_{ijn}E_n + \frac{1}{2!}r^{E}_{ij}T$$
$$+\frac{1}{3!}\alpha^{E}_{ijklpq}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}k_{ijpqn}\sigma_{pq}E_n + \frac{1}{3!}r^{E}_{ijkl}\sigma_{kl}T)T \tag{19}$$

$$D_m = (d^{T}_{klm} + \frac{1}{2!}d^{T}_{klpqm}\sigma_{pq} + \frac{1}{2!}o^{T}_{klmn}E_n + \frac{1}{2!}k_{klm}T$$
$$+\frac{1}{3!}d^{T}_{klpqrsm}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}o^{T}_{klpqnm}\sigma_{pq}E_n + \frac{1}{3!}k_{klpqm}\sigma_{pq}T)\sigma_{kl}$$
$$+(\eta^{\sigma,T}_{mn} + \frac{1}{2!}o^{T}_{klmn}\sigma_{kl} + \frac{1}{2!}\eta^{\sigma,T}_{mnt}E_t + \frac{1}{2!}p^{\sigma}_{mn}T$$

$$+\frac{1}{3!}o^{T}_{klpqnm}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\Gamma^{T}_{pqtnm}\sigma_{pq}E_t + \frac{1}{3!}\beta_{pqnm}\sigma_{pq}T)E_n$$
$$+(p^{\sigma}_{m} + \frac{1}{2!}k_{klm}\sigma_{kl} + \frac{1}{2!}p^{\sigma}_{mn}E_n + \frac{1}{2!}u^{\sigma}_{m}T$$
$$+\frac{1}{3!}k_{klpqm}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\beta_{pqnm}\sigma_{pq}E_n + \frac{1}{3!}\omega_{klm}\sigma_{kl}T)T \tag{20}$$

$$S = (\alpha^{E}_{kl} + \frac{1}{2!}\alpha^{E}_{klpq}\sigma_{pq} + \frac{1}{2!}k_{k\ln}E_n + \frac{1}{2!}r^{E}_{kl}T$$
$$+\frac{1}{3!}\alpha^{E}_{klpqrs}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}k_{klpqn}\sigma_{pq}E_n + \frac{1}{3!}r^{E}_{klpq}\sigma_{pq}T)\sigma_{kl}$$
$$+(p^{\sigma}_{n} + \frac{1}{2!}k_{k\ln}\sigma_{kl} + \frac{1}{2!}p^{\sigma}_{nt}E_t + \frac{1}{2!}u^{\sigma}_{n}T$$
$$+\frac{1}{3!}k_{klpqn}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\beta_{pqnt}\sigma_{pq}E_t + \frac{1}{3!}\omega_{pqn}\sigma_{pq}T)E_n$$
$$+(\xi^{\sigma,E} + \frac{1}{2!}r^{E}_{kl}\sigma_{kl} + \frac{1}{2!}u^{\sigma}_{n}E_n + \frac{1}{2!}P^{\sigma,E}T$$
$$+\frac{1}{3!}r^{E}_{klpq}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\omega_{pqn}\sigma_{pq}E_n + \frac{1}{3!}\mu^{E}_{kl}\sigma_{kl}T)T \tag{21}$$

Nonlinearity and linearity of thermopiezoelasticity

It is convenient to write nonlinear constitutive equations (19) to (21) in matrix form as

$$\begin{bmatrix}\varepsilon_{ij}\\ D_m\\ S\end{bmatrix} = \begin{bmatrix}A & D & E\\ D & B & F\\ E & F & C\end{bmatrix}\begin{bmatrix}\sigma_{kl}\\ E_n\\ T\end{bmatrix} \tag{22}$$

In equation (22), A, B, C are basic effects, and D, E, F are conjugate effe respectively. The basic and conjugate effects are given by

$$A = s^{E,T}_{ijkl} + \frac{1}{2!}s^{E,T}_{ijklpq}\sigma_{pq} + \frac{1}{2!}d^{T}_{ijk\ln}E_n + \frac{1}{2!}\alpha^{E}_{ijkl}T$$
$$+\frac{1}{3!}s^{E,T}_{ijklpqrs}\sigma_{pq}\sigma_{rs} + \frac{1}{3!}d^{T}_{ijklpqn}\sigma_{pq}E_n + \frac{1}{3!}\alpha^{E}_{ijklpq}\sigma_{pq}T \tag{23a}$$

$$B = \eta^{\sigma,T}_{mn} + \frac{1}{2!}o^{T}_{klmn}\sigma_{kl} + \frac{1}{2!}\eta^{\sigma,T}_{mnt}E_t + \frac{1}{2!}p^{\sigma}_{mn}T$$
$$+\frac{1}{3!}o^{T}_{klpqnm}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\Gamma^{T}_{pqtnm}\sigma_{pq}E_t + \frac{1}{3!}\beta_{pqnm}\sigma_{pq}T \tag{23b}$$

$$C = \xi^{\sigma,E} + \frac{1}{2!}r^{E}_{kl}\sigma_{kl} + \frac{1}{2!}u^{\sigma}_{n}E_n + \frac{1}{2!}P^{\sigma,E}T$$
$$+\frac{1}{3!}r^{E}_{klpq}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\omega_{pqn}\sigma_{pq}E_n + \frac{1}{3!}\mu^{E}_{kl}\sigma_{kl}T \tag{23c}$$

$$D = d^{T}_{ijn} + \frac{1}{2!}d^{T}_{ijk\ln}\sigma_{kl} + \frac{1}{2!}o^{T}_{ijnt}E_t + \frac{1}{2!}k_{ijn}T$$
$$+\frac{1}{3!}d^{T}_{ijklpqn}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}o^{T}_{ijpqtn}\sigma_{pq}E_t + \frac{1}{3!}k_{ijpqn}\sigma_{pq}T \tag{23d}$$

$$E = \alpha^{E}_{ij} + \frac{1}{2!}\alpha^{E}_{ijkl}\sigma_{kl} + \frac{1}{2!}k_{ijn}E_n + \frac{1}{2!}r^{E}_{ij}T$$
$$+\frac{1}{3!}\alpha^{E}_{ijklpq}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}k_{ijpqn}\sigma_{pq}E_n + \frac{1}{3!}r^{E}_{ijkl}\sigma_{kl}T \tag{23e}$$

$$F = p^{\sigma}_{m} + \frac{1}{2!}k_{klm}\sigma_{kl} + \frac{1}{2!}p^{\sigma}_{mn}E_n + \frac{1}{2!}u^{\sigma}_{m}T$$
$$+\frac{1}{3!}k_{klpqm}\sigma_{kl}\sigma_{pq} + \frac{1}{3!}\beta_{pqnm}\sigma_{pq}E_n + \frac{1}{3!}\omega_{klm}\sigma_{kl}T \tag{23f}$$

In absence of higher order material constants (higher rank tensors), equations (19) to (21) reduce to

$$\varepsilon_{ij} = s^{E,T}_{ijkl}\sigma_{kl} + d^{T}_{ijn}E_n + \alpha^{E}_{ij}T \tag{24}$$

$$D_m = d^{T}_{klm}\sigma_{kl} + \eta^{\sigma,T}_{mn}E_n + p^{\sigma}_{m}T \tag{25}$$

$$S = \alpha^{E}_{kl}\sigma_{kl} + p^{\sigma}_{n}E_n + \xi^{\sigma,E}T \tag{26}$$

It is obvious that equations (24) to (26) are linear constitutive equations of thermopiezoelasticity.

Conclusion

Nonlinear thermopiezoelasticity problem has been studied. Employing thermodynamic Gibbs function and Taylor series, material constants are derived from zeroth to eighth rank tensors. If higher order material constants (higher rank tensors) are neglected in nonlinear equations, linear equations are obtained. Nonlinear and linear equations of thermopiezoelasticity constitute system of symmetry with basic effects and conjugate effects. All the derivations obtained in this analysis are convenient to be used in solving complicated problems in thermopiezoelastic materials. Further investigation may be carried out considering the work of Fumi (1952), Fieshi and Fumi (1953) and Abrahams (1994).

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