Yang-Mills theory on Nilmanifolds

Phenomenology of the Heisenberg manifold

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$\Rightarrow$ Can we do better ?

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$$[\mathfrak{g}, [\ \mathfrak{g}, \ldots, [\ \mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \ldots] = 0\ . \qquad (1)$$

ex : $[V_1, V_2] = -\mathtt{f} V_3\ ,\ [V_1, V_3] = [V_2, V_3] = 0\ .$

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- Consider the element of the algebra as tangent vectors of a manifold and find a coordinate system.
- Make identifications so that the manifold is compact (meaning, quotient by a lattice).

Heisenberg manifold

The simplest example, The Heisenberg algebra :

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Pick a coordinate system

$$e^1 = r^1 dx^1 \ ; \ \ e^2 = r^2 dx^2 \ ; \ e^3 = r^3 \left(dx^3 + N x^1 dx^2 \right) \ , \tag{4}$$

$$\text{where} \quad N = \frac{r^1 r^2}{r^3} \mathtt{f} \in \mathbb{N} \ . \tag{5}$$

Identifications

To make the manifold compact, we use

$$x^1 \sim x^1 + n^1 \; ; \; x^2 \sim x^2 + n^2 \; ; \; x^3 \sim x^3 + n^3 - n^1 N x^2 \; , \qquad (6)$$
$$n^1, n^2, n^3 \in \{0, 1\} \; .$$

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$\Rightarrow$ What about functions on this space ?

The Laplacian

Solve $\boxed{\Delta f = \lambda f}$

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$\Rightarrow$ Eigenfunctions form a complete set on the space, any function can be expanded on this basis (similarly to the Fourier basis) :

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Similarly, we solve for one-forms :

$$\Delta B_m = \lambda B_m \tag{8}$$

$\Rightarrow$ Eigenscalars and eigen-1-forms have **analytical** expressions.

"Low-lying" forms :

Scalars :

$$U_{I=1} = \frac{1}{\sqrt{V}} \; ; \quad \lambda_{U_1} = 0 \; , \tag{9}$$

One-forms :

$$\begin{aligned} B_{I=1} &= \frac{1}{\sqrt{V}} e^1 \; ; \quad \lambda_{B_1} = 0 \\ B_{I=2} &= \frac{1}{\sqrt{V}} e^2 \; ; \quad \lambda_{B_2} = 0 \\ B_{I=3} &= \frac{1}{\sqrt{V}} e^3 \; ; \quad \lambda_{B_3} = \mathtt{f}^2 \end{aligned} \tag{10}$$

Kaluza-Klein tower

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If we take the geometrical limit (known as the "large base, small fiber" limit)

$$|\mathtt{f}| \ll \frac{1}{r^i} \ , \ i = 1, 2, 3 \quad \Rightarrow \quad r^3 \ll r^{1,2} \ , \tag{12}$$

we effectively separate the low-lying masses from the rest of the tower.

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From 7D to 4D

The effective action is computed from the 7D action :

$$\mathcal{L}_{4D} = \int dy^3 \mathcal{L}_{7D} \ ; \ \mathcal{L}_{7D} = \frac{1}{2}\mathsf{Tr}\left(F_{MN}F^{MN}\right) \tag{13}$$

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Now we use the following decomposition

$$\mathcal{A}^a = \mathcal{A}^a_\mu(x^M)\text{d}x^\mu + \mathcal{A}^a_m(x^M)\text{d}y^m \tag{14}$$

where

$$\mathcal{A}^a_\mu(x^M) = A^a_\mu(x^\mu)U_1(y) \tag{15}$$

$$\mathcal{A}^a_m(x^M) = \sum_{i=1}^{3} \phi^{ai}(x^\mu)B_{im}(y) \; , \tag{16}$$

Inject $\mathcal{A}^a$ into the action and simplify

From 7D to 4D

The resulting action :

$$S = \int \mathrm{d}x^4 \mathrm{Tr}\Big(-\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{3} D_\mu \phi^i D^\mu \phi^i - M^2 (\phi^3)^2 - \mathcal{U} \Big) \quad (17)$$

where :

$$\mathcal{U} = \mathrm{Tr}\Big(-2i\mathrm{g}M[\phi^1, \phi^2]\phi^3 + \tfrac{1}{2}\mathrm{g}^2 \sum_{i,j=1}^{3} [\phi^i, \phi^j][\phi^i, \phi^j] \Big) \quad (18)$$

with $M = |\mathfrak{f}|$ and $\mathrm{g} = \frac{\mathrm{g}_{7D}}{\sqrt{V}}$.

$\Rightarrow$ 3 scalars in the **adjoint representation**

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The potential

We would like to find a minimum of

$$\frac{\mathcal{V}}{M^2} = (\phi^3)^2 - 2i\frac{\mathrm{g}}{M}\mathsf{Tr}\Big([\phi^1,\phi^2]\phi^3\Big) + \tfrac{1}{2}\frac{\mathrm{g}^2}{M^2}\sum_{i,j=1}^{3}\mathsf{Tr}\Big([\phi^i,\phi^j][\phi^i,\phi^j]\Big)\ . \tag{19}$$

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Compute the variation of the potential,

$$\begin{aligned}\frac{\delta\mathcal{V}}{M^2} \quad = \quad & \mathsf{Tr}\left(2\phi^3\delta\phi^3\right) - 2i\frac{\mathrm{g}}{M}\mathsf{Tr}\Big([\phi^1,\phi^2]\delta\phi^3 + [\phi^3,\phi^1]\delta\phi^2 + [\phi^2,\phi^3]\delta\phi^1\Big) \\ & + 2\frac{\mathrm{g}^2}{M^2}\mathsf{Tr}\Big(\sum_{I,J=1}^{3}[\phi^I,\phi^J][\phi^I,\delta\phi^J]\Big) = 0\ . \end{aligned} \tag{20}$$

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Solution : $\boxed{\phi^3 = 0\ ; \quad [\phi^1,\phi^2] = 0}$

The mass matrix

Next we compute de second order variation

$$\frac{\delta^2 \mathcal{V}}{2M^2} = \mathsf{Tr}(\delta\phi^3)^2 - 2\mathtt{i}\frac{g}{M}\mathsf{Tr}\left(\left[\delta\phi^1, \phi_0^2\right]\delta\phi^3 + \left[\phi_0^1, \delta\phi^2\right]\delta\phi^3\right) \tag{21}$$
$$+ \frac{g^2}{M^2}\mathsf{Tr}\Big(\left[\delta\phi^1, \phi_0^2\right]^2 + \left[\delta\phi^2, \phi_0^1\right]^2 + \left[\delta\phi^3, \phi_0^1\right]^2$$
$$+ \left[\delta\phi^3, \phi_0^2\right]^2 + 2\left[\delta\phi^1, \phi_0^2\right]\left[\phi_0^1, \delta\phi^2\right]\Big)$$

$\Rightarrow$ Naive approach : fix the gauge and compute the masses.

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$\Rightarrow$ Naive approach : fix the gauge and compute the masses.

$\Rightarrow$ General approach : write the Lie algebra in the Cartan basis.
(Note: All commutators are with the vacuum)

The Cartan basis

Take the Lie algebra $\mathfrak{g}$ in the basis

$$[H_i, H_j] = 0 \ , \ [H_i, E_\alpha] = \alpha_i E_\alpha \ , \ [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \tag{22}$$

H_i form the **Cartan subalgebra**, E_α are the **roots**.

<u>Vacuum condition</u> : $\phi^3 = 0 \ ; \quad [\phi^1, \phi^2] = 0$

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The mass matrix is **block diagonal** in root space.

$$\Rightarrow \frac{\delta^2 \mathcal{V}}{M^2} = \begin{pmatrix} A_\alpha & & & & \\ & A_\alpha^* & & & \\ & & \ddots & & \\ & & & \mathbb{J}_i & \\ & & & & \ddots \end{pmatrix} , \tag{23}$$

$$A_\alpha = \begin{pmatrix} -(b_2^\alpha)^2 & b_1^\alpha b_2^\alpha & b_2^\alpha \\ b_1^\alpha b_2^\alpha & -(b_1^\alpha)^2 & -b_1^\alpha \\ -b_2^\alpha & b_1^\alpha & 1-(b_1^\alpha)^2-(b_2^\alpha)^2 \end{pmatrix} , \quad \mathbb{J}_i = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \tag{24}$$

Mass of a root

Once the mass matrix is diagonalized, the masses for a given root E_α are :

$$0\ ,\ (m_\alpha^\pm)^2 = \frac{1}{2}M^2\left(1 + 2\big((b_1^\alpha)^2 + (b_2^\alpha)^2\big) \pm \sqrt{1 + 4\big((b_1^\alpha)^2 + (b_2^\alpha)^2\big)}\right)\ , \quad (25)$$

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For the gauge bosons, we have

$$\begin{aligned} m^2_{\alpha,gauge} &= g^2\sum_{I=1}^{2}\phi_0^{Ii}\alpha_i \\ &= M^2\left((b_1^\alpha)^2+(b_2^\alpha)^2\right) \end{aligned} \tag{26}$$

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The H_i directions are **not affected** by the vacuum.

An example : $SU(3)$

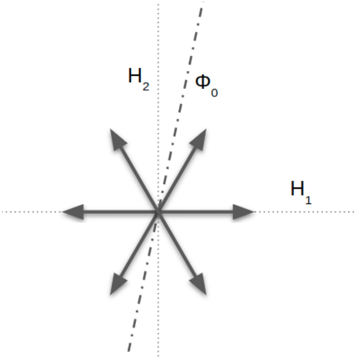


Figure: Root diagram of $\mathfrak{su}(3)$

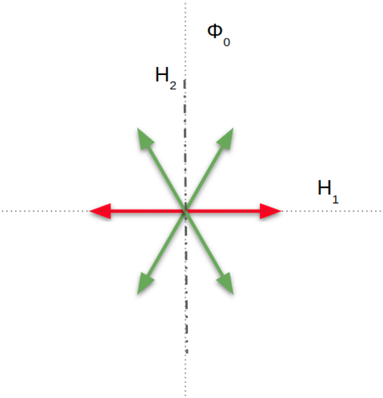
For a root E_α :

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{27}$$

We can think of the α_i as coordinates in a vector space. Here we have H_1 and H_2, corresponding to a 2-dimensional space :

$$E_\alpha \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \tag{28}$$

$\Rightarrow$ Here, $SU(3) \to U(1) \times U(1)$



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The root α is orthogonal to the vacuum. Therefore

$$m^2_{\alpha,gauge} = 0 \quad \text{and} \quad m^2_{-\alpha,gauge} = 0 \tag{29}$$

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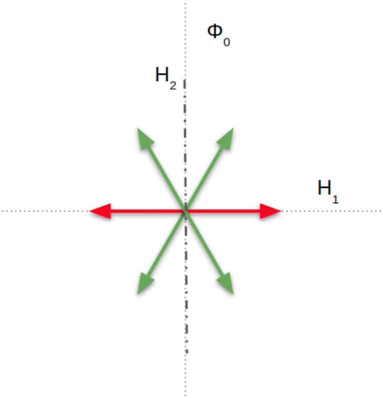

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The roots β and γ have same mass. They form a **fundamental representation** of the new gauge.

$SU(3) \to SU(2) \times U(1)$ one-loop renormalized masses

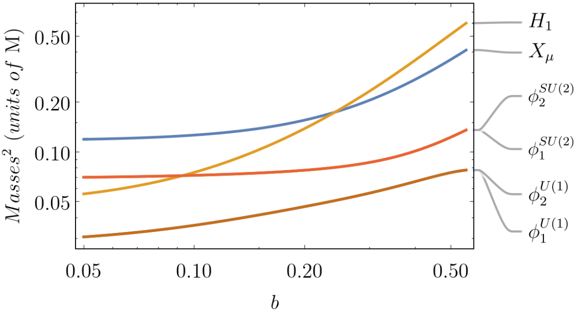

Figure: One-loop renormalized masses of the low-mass scalars for a the breaking pattern $SU(3) \to SU(2) \times U(1)$. H_1 and X_μ are in the fundamental representation of $SU(2) \times U(1)$, while $\phi_i^{SU(2)}$ is the adjoint of $SU(2)$ and $\phi_i^{U(1)}$ in the adjoint of $U(1)$. $\phi_i^{U(1)}$ does **not couple** to the gauge. The graph was realized for $\phi_{01} = \phi_{02}$.

Conclusion

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- Moduli of the metric on the Heisenberg manifold computed (here we used the flat metric)
- Laplacian spectrum for scalars with arbitrary metric solved
- Dirac operator with arbitrary metric solved (so fermions can be considered both in 7D and 4D)

Possible directions :

- Laplacian for one-forms with arbitrary metric not solved (yet), but solved for the first modes.
- More realistic models with fermions
- What about gravity in this context ?

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Thank you !