# (A)dS Dilatonic Black Holes

Carlo Branchina

LPTHE - Sorbonne Université

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Karim Benakli, C.B., Gaëtan Lafforgue-Marmet

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# Introduction & Motivation

...What we found in the literature...

- Dilatonic BH solution in flat space
  Gibbons, Maeda / Garfinkle, Horowitz, Strominger
- Extremality & Thermodynamics — Holzhey, Wilczek
- dS Reissner-Nordström & extremality — Romans/ Antoniadis, Benakli
- Dilatonic BH solution in (A)dS — Gao, Zhang/ Elvang, Friedmann, Liu/ Mignemi

...What we wanted to understand...

- Behaviour & extremality of dilatonic black holes in (A)dS

Interesting for application to WGC

# The Black Hole Solution

- Einstein-Maxwell-Dilaton action

$$\mathcal{S} = \int d^4x \; \mathcal{R} - 2(\partial\phi)^2 - e^{-2\alpha\phi}F^2 - V(\phi)$$

Asymptotically (A)dS solutions have been constructed for

Gao, Zhang/ Elvang, Friedmann, Liu/ Mignemi

$$V(\phi) = \frac{2}{3}\frac{\Lambda}{1+\alpha^2}\left((3\alpha^4-\alpha^2)e^{-2\frac{\delta\phi}{\alpha}} + (3-\alpha^2)e^{2\alpha\delta\phi} + 8\alpha^2 e^{\alpha\delta\phi - \frac{\delta\phi}{\alpha}}\right)$$

$\Lambda$: cosmological constant;
$\delta\phi \equiv \phi - \phi_0$ with $\phi_0$ asymptotic value of $\phi(r)$ for $r \to \infty$.

# The Black Hole Solution

- The solution takes the form

$$\begin{cases} ds^2 = & -f(r)dt^2 + f(r)^{-1}dr^2 + \boxed{r^2\left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}} d\Omega_2^2 \\ e^{2\alpha\phi} = & e^{2\alpha\phi_0}\left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \\ F = & \frac{1}{\sqrt{4\pi G}}\frac{Qe^{2\alpha\phi_0}}{r}\,\mathrm{d}t\wedge\mathrm{d}r \end{cases}$$

$$f(r) = -\left[\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2 r^2\left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

$$r_+ = M + \sqrt{M^2-(1-\alpha^2)Q^2e^{2\alpha\phi_0}}$$

$$r_- = \frac{(1+\alpha^2)Q^2e^{2\alpha\phi_0}}{M+\sqrt{M^2-(1-\alpha^2)Q^2e^{2\alpha\phi_0}}}$$

$H^2 \equiv |\Lambda|/3$: Hubble parameter

# Flat space: $\Lambda = 0$

- $\alpha = 0 \Rightarrow$ Reissner-Nordström Black Hole.

$$f(r) = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$$

Time-like singularity at $r = 0$

No naked singularity: $Q^2 \leq M^2 \leftrightarrow 2(1)$ horizons

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Time-like singularity at $r = 0$

No naked singularity: $Q^2 \leq M^2 \leftrightarrow 2(1)$ horizons

- $\alpha \neq 0$

$$f(r) = -\left[\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}\right]$$

Garfinkle, Horowitz, Strominger / Gibbons, Maeda

Space-like singularity at $r_-$

No naked singularity:

$$r_+ > r_- \leftrightarrow Q^2 e^{2\alpha\phi_0} < (1+\alpha^2)M^2 \leftrightarrow 1\,\text{horizon}$$

# dS space: $\Lambda > 0$

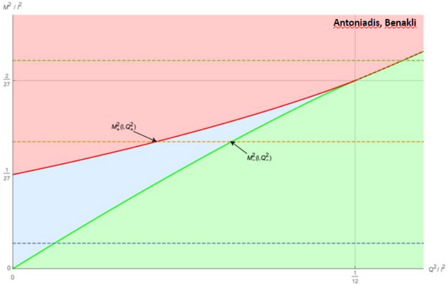
$\alpha = 0 \Rightarrow$ Reissner-Nordström dS Black Hole.

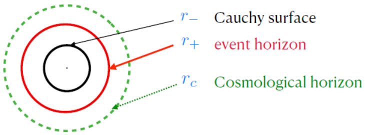
$$f(r) = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} + H^2 r^2\right)$$

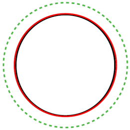
Time-like singularity at $r = 0$.

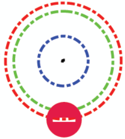
- 3 horizons
- Only Cosmological horizon
- dS patch eaten

Extremality:
$Q^2 = M^2 + M^4 H^2 + \mathcal{O}(M^6 H^4)$



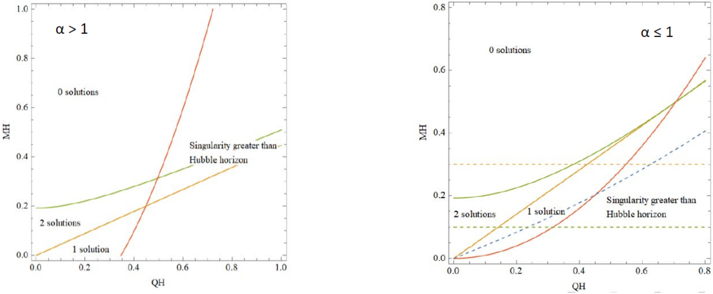


# dS space: $\Lambda > 0$

$$f(r) = -\left[\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} - H^2 r^2 \left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

$$\alpha > \alpha_c \equiv \frac{1}{\sqrt{3}}$$

- $\alpha \to \infty \leftrightarrow Q = 0$: Schwarzschild dS




- Extremality: $Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2$. Same as in flat space

# dS space: $\Lambda > 0$

$$f(r) = -\left[\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} - H^2 r^2 \left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

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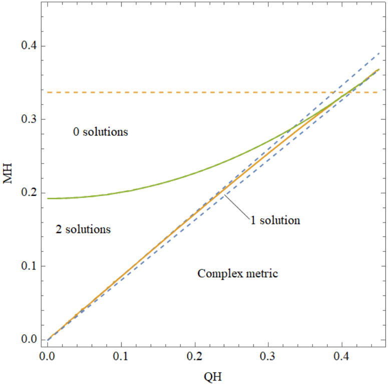


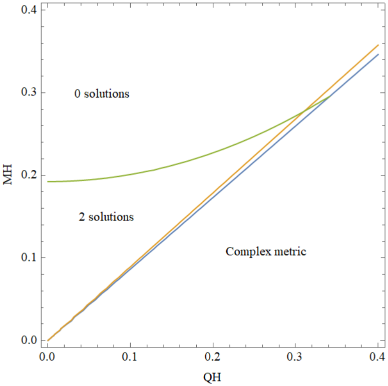

- New extremal solution

$$Q^2 e^{\frac{2}{\sqrt{3}}\phi_0} = \frac{4}{3}M^2 + \frac{4^3}{3^4}M^4H^2 + \mathcal{O}(M^6H^4)$$

# dS space: $\Lambda > 0$

$$f(r) = -\left[\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} - H^2 r^2 \left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

$$0 < \alpha < \alpha_c \equiv \frac{1}{\sqrt{3}}$$




- Obstruction to extremal:

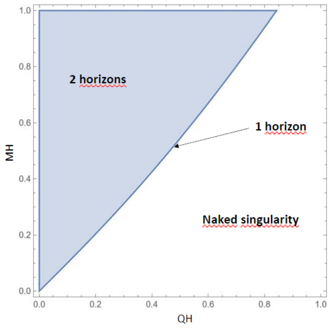
$$(1-\alpha^2)Q^2 e^{2\alpha\phi_0} = M^2$$

Complex metric:
extremality never reached

# AdS space: $\Lambda < 0$

$\alpha = 0 \Rightarrow$ Reissner-Nordström AdS Black Hole.

$$f(r) = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} + H^2 r^2\right)$$




- Time-like singularity at $r = 0$
- 2(1) horizons

# AdS space: $\Lambda < 0$

$$f(r) = -\left[\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} + H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

- $\alpha > \alpha_c \equiv \frac{1}{\sqrt{3}}$:
  - $\alpha \to \infty$ Schwarzschild AdS
  - 1 horizon $\leftrightarrow$ Space-like singularity at $r = r_-$
  - Extremality: $Q^2 e^{2\alpha\phi_0} = (1 + \alpha^2) M^2 \Longrightarrow$ Same as in flat space
- $\alpha = \alpha_c \equiv \frac{1}{\sqrt{3}}$
  - 1 horizon + New extremal solution

$$Q^2 e^{\frac{2}{\sqrt{3}}\phi_0} = \frac{4}{3} M^2 - \frac{4^3}{3^4} M^4 H^2 + \mathcal{O}(M^6 H^4)$$

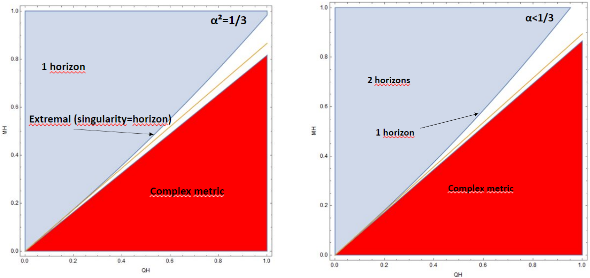
# AdS space: $\Lambda < 0$

$$f(r) = -\left[\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} + H^2 r^2 \left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

$$0 < \alpha < \alpha_c \equiv \frac{1}{\sqrt{3}}$$

- 2(1) horizons ⇔ Singularity changes nature to "emulate" RN AdS
- RN-type extremality: Cauchy surface = event horizon

$$Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2 + \alpha^2(1+\alpha^2)^{\frac{2}{1-\alpha^2}} c\, M^{\frac{3-\alpha^2}{1-\alpha^2}} H^{\frac{1+\alpha^2}{1-\alpha^2}} + o(M^{\frac{3-\alpha^2}{1-\alpha^2}} H^{\frac{1+\alpha^2}{1-\alpha^2}})$$



# Thermodynamics

$\Lambda = 0$

**Hawking temperature**

Schwarzschild

$$T = \frac{1}{8\pi M}$$

"Extremality": $T \to \infty$

Reissner-Nordström

$$T = \frac{1}{2\pi}\frac{\sqrt{M^2 - Q^2}}{\left(M + \sqrt{M^2 - Q^2}\right)^2}$$

Extremality: $T \to 0$

# Thermodynamics

$\Lambda = 0$

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Extremality: $T \to 0$

Dilatonic Black holes: $T = \frac{1}{2\pi r_+} \left(1 - \frac{r_-}{r_+}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}$

Extremality $(r_+ \to r_-)$ Holzhey, Wilczek

$\alpha > 1$ diverges $\boxed{\alpha = 1}$ finite $0 < \alpha < 1$ vanishes

**Hawking-Beckenstein entropy**

$$S = \pi r_h^2 \left(1 - \frac{r_-}{r_h}\right)^{\frac{2\alpha^2}{1+\alpha^2}}$$

Extremality: $S \to 0$ $\forall\ \alpha \neq 0$

# Thermodynamics

$\Lambda \neq 0$

**Hawking temperature**

Schwarzschild

$$T = \frac{1}{4\pi}\frac{1 \mp 3H^2 r_h^2}{r_h}$$

"Extremality": $T \to \infty$

Reissner-Nordström

$$T = \frac{1}{4\pi}\frac{1 - \frac{Q^2}{r_h^2} \mp 3H^2 r_h^2}{r_h}$$

Extremality: $T \to 0$

Dilatonic Black holes:

$$T = \frac{1}{4\pi}\Bigg[\frac{r_+}{r_h^2}\left(1 - \frac{r_-}{r_h}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} + \frac{1-\alpha^2}{1+\alpha^2}\left(1 - \frac{r_+}{r_h}\right)\left(1 - \frac{r_-}{r_h}\right)^{-\frac{2\alpha^2}{1+\alpha^2}}\frac{r_-}{r_h^2}$$
$$\mp 2H^2 r_h\left(1 - \frac{r_-}{r_h}\right)^{\frac{2\alpha^2}{1+\alpha^2}} \mp 2\frac{\alpha^2}{1+\alpha^2}H^2 r_-\left(1 - \frac{r_-}{r_h}\right)^{-\frac{1-\alpha^2}{1+\alpha^2}}\Bigg]$$

# Thermodynamics

**Hawking temperature**

- $\alpha > \alpha_c$ Extremal limit $r_h \to r_-(=r_+)$

1. $\alpha > 1$ $T \underset{r_h \to r_-}{\sim} \frac{1}{4\pi r_h} \frac{2}{1+\alpha^2} \left(1 - \frac{r_-}{r_h}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}$ diverges
2. $\alpha_c < \alpha < 1$ $T \underset{r_h \to r_-}{\sim} \frac{1}{2\pi} \frac{\alpha^2}{1+\alpha^2} H^2 r_- \left(1 - \frac{r_-}{r_h}\right)^{-\frac{1-\alpha^2}{1+\alpha^2}}$ diverges
3. $\alpha = 1$ $T = \frac{1}{4\pi}\left(\frac{1}{2M} \mp 2MH^2\right)$ always finite $\geq 0$

In dS the extremal black holes with singularity the size of the Hubble horizon have $T = 0$

# Thermodynamics

**Hawking temperature**

- $\alpha = \alpha_c$ Extremal limit $r_h \to r_- \neq r_+$

$$T \underset{r_h \to r_-}{\to} \frac{1}{8\pi}\frac{r_-}{r_h^2}\left(1 - \frac{r_+}{r_h} \mp H^2 r_h^2\right)\left(1 - \frac{r_-}{r_h}\right)^{-\frac{1}{2}} = 0$$

$\Longrightarrow$ New discontinuity in the $\alpha$ dependence of $T_{\rm extr}$ for $\alpha = \alpha_c$

- $0 < \alpha < \alpha_c$
  - NO extremality in dS
  - New extremality in AdS $\Rightarrow r_h \neq r_- \Rightarrow \begin{cases} T \text{ finite} \\ S \neq 0 \end{cases}$

# Conclusions: $\Lambda = 0$ vs $\Lambda \neq 0$

...Similarities...

- Above $\alpha_c \equiv \frac{1}{\sqrt{3}}$ singularity space-like with 1 less horizon than RN
- Same extremality bound above $\alpha_c$
- Interpolation between a Sc-like to RN-like behaviour with turning point at $\alpha = 1$ ($\Lambda = 0$) or $\alpha_c$ ($\Lambda \neq 0$)

...Differences...

- Existence of a transition value $\alpha_c$ stronger than turning point $\alpha = 1$
- New extremality ($\alpha = \alpha_c$);
  $\alpha < \alpha_c$: different singularity ($\Lambda < 0$) or obstruction ($\Lambda > 0$)
- $\alpha_c < \alpha < 1$: $T$ close to extremality driven by $\Lambda$
- Trivial endpoint of Hawking evaporation at $\alpha = 1$ ($\Lambda > 0$)
- $S_{\text{extr}} \neq 0$ below $\alpha_c$ ($\Lambda < 0$)

# Thank you for your attention!

# Geometrized units

$$M = \frac{\kappa^2 \tilde{M}}{8\pi} \qquad Q^2 = \frac{\kappa^2 \tilde{Q}^2}{32\pi^2} \qquad \Rightarrow \qquad \frac{M^2}{Q^2} = \frac{\kappa^2}{2} \frac{\tilde{M}^2}{\tilde{Q}^2}$$

$\kappa^2 = 1/M_P^2 = 8\pi G \equiv 8\pi$ and $G$ Newton's constant

# Bijection $(r_+, r_-) \Leftrightarrow (M, Q)$

$$\begin{cases} 2M = r_+ + \frac{1-\alpha^2}{1+\alpha^2} r_-, \\ Q^2 e^{2\alpha\phi_0} = \frac{r_+ r_-}{1+\alpha^2}, \end{cases} \Leftrightarrow \begin{cases} r_+ = M \pm \sqrt{M^2 - (1-\alpha^2) Q^2 e^{2\alpha\phi_0}} \\ r_- = \frac{(1+\alpha^2) Q^2 e^{2\alpha\phi_0}}{M \pm \sqrt{M^2 - (1-\alpha^2) Q^2 e^{2\alpha\phi_0}}}, \end{cases}$$

- For $Q = 0$ to correspond to Schwarzschild $\Rightarrow$ Upper sign
- $\alpha \geq 1$: $(r_+, r_-)$ plane covers whole $(M, Q)$ one;
  1. For $r_+ < \left[(\alpha^2 - 1)/(\alpha^2 + 1)\right] r_- \Longrightarrow M < 0$ : unphysical
  2. Bijection defined between $r_+ \geq \left[(\alpha^2 - 1)/(\alpha^2 + 1)\right] r_-$ and $(M, Q)$

# Bijection $(r_+, r_-) \Leftrightarrow (M, Q)$

- $0 < \alpha < 1$: for $M^2 < (1-\alpha^2)Q^2 e^{2\alpha\phi_0}$ complex metric. Inaccessible part of the $(M, Q)$ plane manifests

$$M^2 - (1-\alpha^2)Q^2 e^{2\alpha\phi_0} = \left(\frac{r_+}{2} - \frac{1-\alpha^2}{1+\alpha^2}\frac{r_-}{2}\right)^2 \geq 0.$$

Writing $r_- = r_+ \tan\theta$

$$\frac{Q^2 e^{2\alpha\phi_0}}{M^2} = \frac{4}{1+\alpha^2} \frac{\tan\theta}{\left(1 + \frac{1-\alpha^2}{1+\alpha^2}\tan\theta\right)^2}$$

1. Increases from 0 to $1/(1-\alpha^2)$ for $\theta \in \left[0, \arctan\frac{1+\alpha^2}{1-\alpha^2}\right]$
2. Decreases to 0 for $\theta \in \left[\arctan\frac{1+\alpha^2}{1-\alpha^2}, \frac{\pi}{2}\right]$.

In (2) $\left(1-\alpha^2\right) Q^2 e^{2\alpha\phi_0} < M^2$, $Q = 0$ for $r_+ = 0$, but metric does not reduce to Schwarzschild.
$\Rightarrow$ Bijection defined between the $r_+ \geq \left[(1-\alpha^2)/(1+\alpha^2)\right] r_-$ and the $M^2 \geq (1-\alpha^2)Q^2 e^{2\alpha\phi_0}$ portions of the planes.

# Maximal masses

- $\alpha = 1$: $(Qe^{\phi_0} H, MH) = (\frac{1}{2}, \frac{1}{\sqrt{2}})$
- $\alpha = \frac{1}{\sqrt{3}}$: $(Qe^{\frac{\phi_0}{\sqrt{3}}} H, MH) = (\frac{1}{\sqrt{6}}, \frac{7}{12\sqrt{3}})$
- $0 < \alpha < \frac{1}{\sqrt{3}}$: $M_{\max} = \frac{1}{2\sqrt{2}H} \left( \frac{1-3\alpha^2}{2(1-\alpha^2)} \right)^{\frac{1-3\alpha^2}{2(1+\alpha^2)}}$

# Technique

$$f(r) = -\left[\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2 r^2 \left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]$$

$$\Rightarrow F(r) \equiv \left[r - r_+ \mp H^2 r^3 \left(1-\frac{r_-}{r}\right)^{\frac{3\alpha^2-1}{1+\alpha^2}}\right] \equiv A(r) + B(r)$$

- Find the intersection points of $A(r)$ and $B(r)$
- Extremal Black Holes found at change in behaviour of one of the two curves (depending on $\alpha - \alpha_c$)
- Nariai Black Holes obtained for the combined solution

$$\begin{cases} F(r) = 0 \\ F'(r) = 0 \end{cases}$$

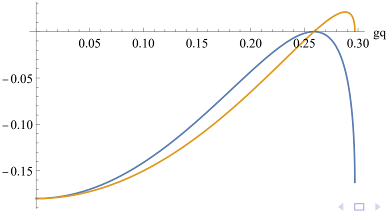
# Point particle reduction

$$S_m = \int \mathrm{d}\tau \left( -m(\phi)\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + \sqrt{4\pi G} gqA_\mu \dot{x}^\mu \right)$$

Scalar charge and properties of BH are encoded in the function $m(\phi)$

$$m(\phi) = m(\bar{\phi}) \left( 1 + \gamma(\bar{\phi})(\phi - \bar{\phi}) + \frac{1}{2}\left(\gamma^2(\bar{\phi}) + \beta(\bar{\phi})\right)(\phi - \bar{\phi})^2 + \mathcal{O}\left((\phi - \bar{\phi})^3\right) \right)$$

$$\begin{cases} \gamma(\phi) = \frac{\alpha}{1-\alpha^2}\left(1 - \sqrt{1 - (1-\alpha^2)\frac{q^2}{m^2(\phi)} e^{2\alpha\phi}}\right) \\ \beta(\phi) = \frac{\alpha^2}{1-\alpha^2}\frac{q^2 e^{2\alpha\phi}}{m^2(\phi)}\left(1 - \frac{\alpha^2}{\sqrt{1-(1-\alpha^2)\frac{q^2}{m^2(\phi)} e^{2\alpha\phi}}}\right) \end{cases}$$



# Identifications from compactification

- Identifying $g = e^{\alpha\phi}$ and $g_0 = e^{\alpha\phi_0}$

$$V(\phi) = \frac{2}{3}\frac{\Lambda}{1+\alpha^2}\left((3\alpha^4 - \alpha^2)\left(\frac{g}{g_0}\right)^{-\frac{2}{\alpha^2}} + (3-\alpha^2)\left(\frac{g}{g_0}\right)^2 + 8\alpha^2\left(\frac{g}{g_0}\right)^{1-\frac{1}{\alpha^2}}\right)$$

$\alpha = 1$

$$V(\phi) = \frac{\Lambda}{3}\left(\left(\frac{g_0}{g}\right)^2 + \left(\frac{g}{g_0}\right)^2 + 4\right)$$