Quantum diffusion during cosmic inflation

Vincent Vennin

PLANCK 2022

Paris, May 30 - June 3, 2022

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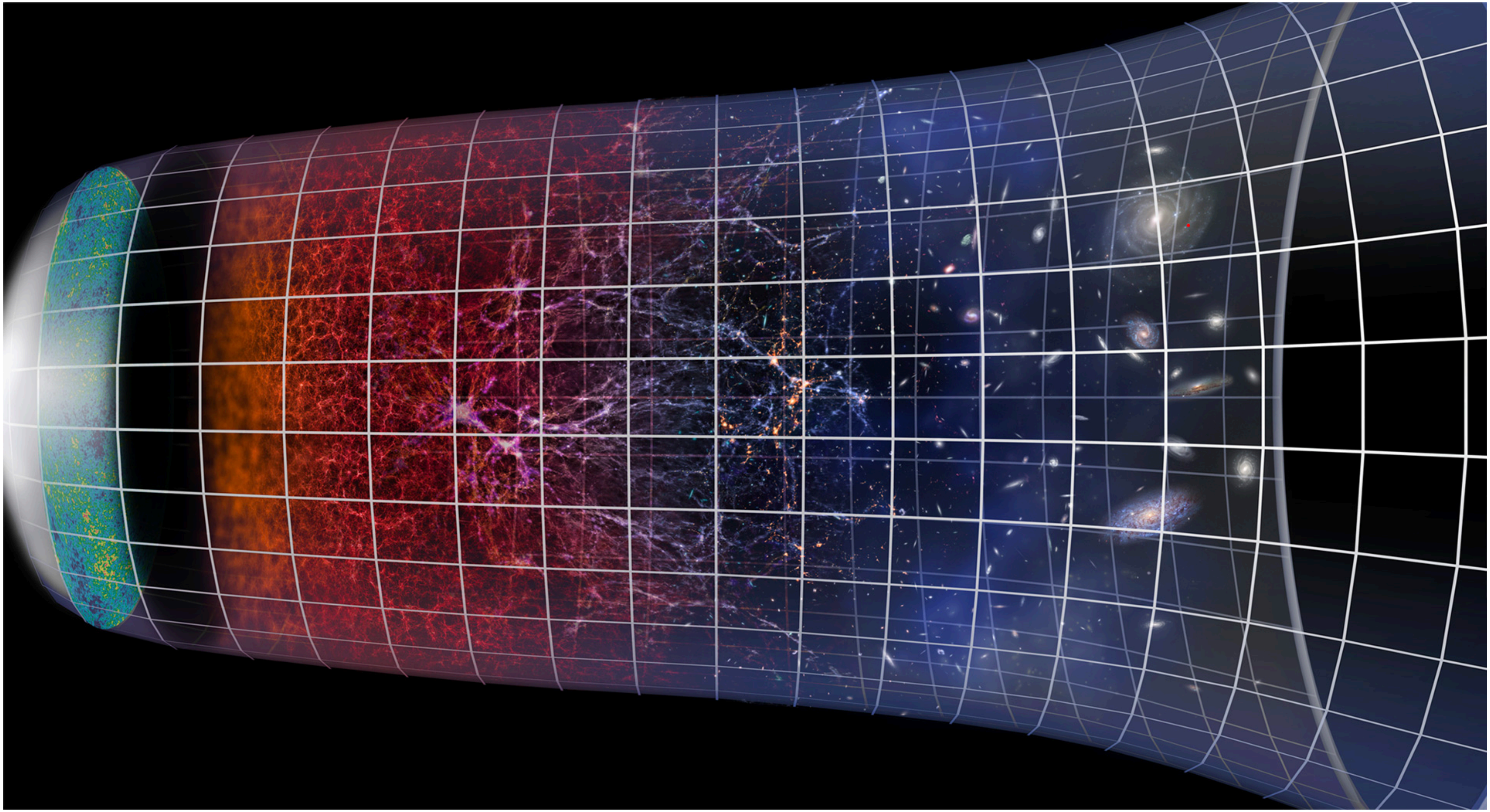
Cosmic Inflation

$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\vec{x}^2$ with $\ddot{a} > 0$ and $(10\,\mathrm{MeV})^4 < \rho < \left(10^{16}\,\mathrm{GeV}\right)^4$



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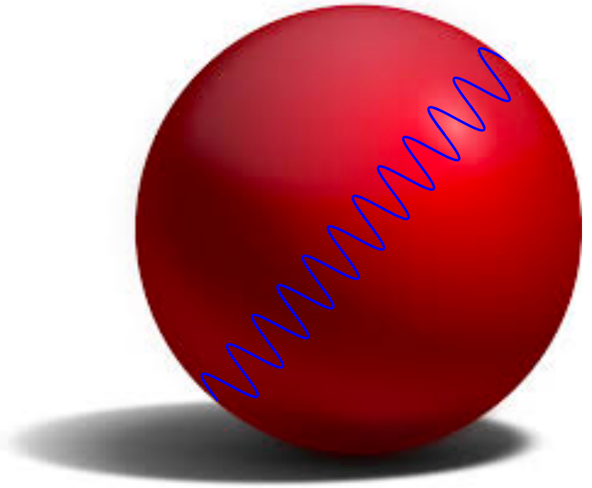
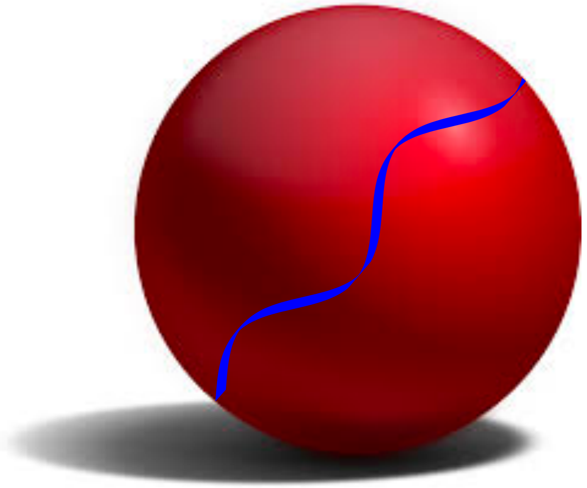


$\lambda \ll H^{-1}$

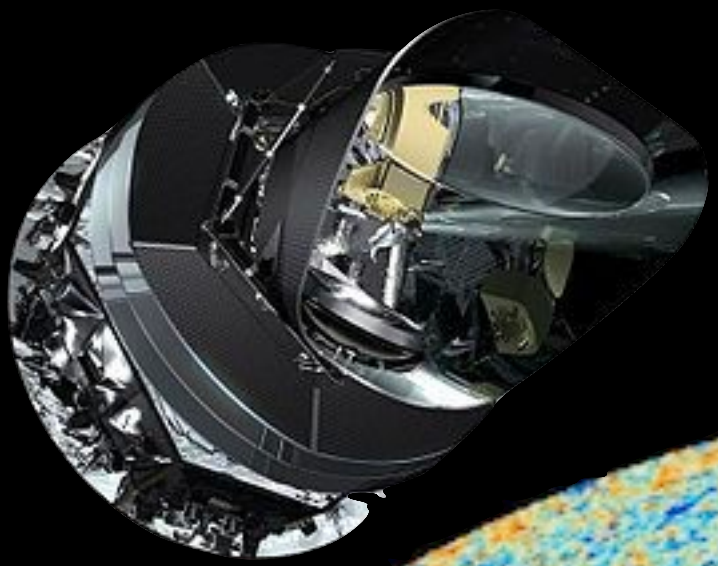
Insensitive to space-time curvature



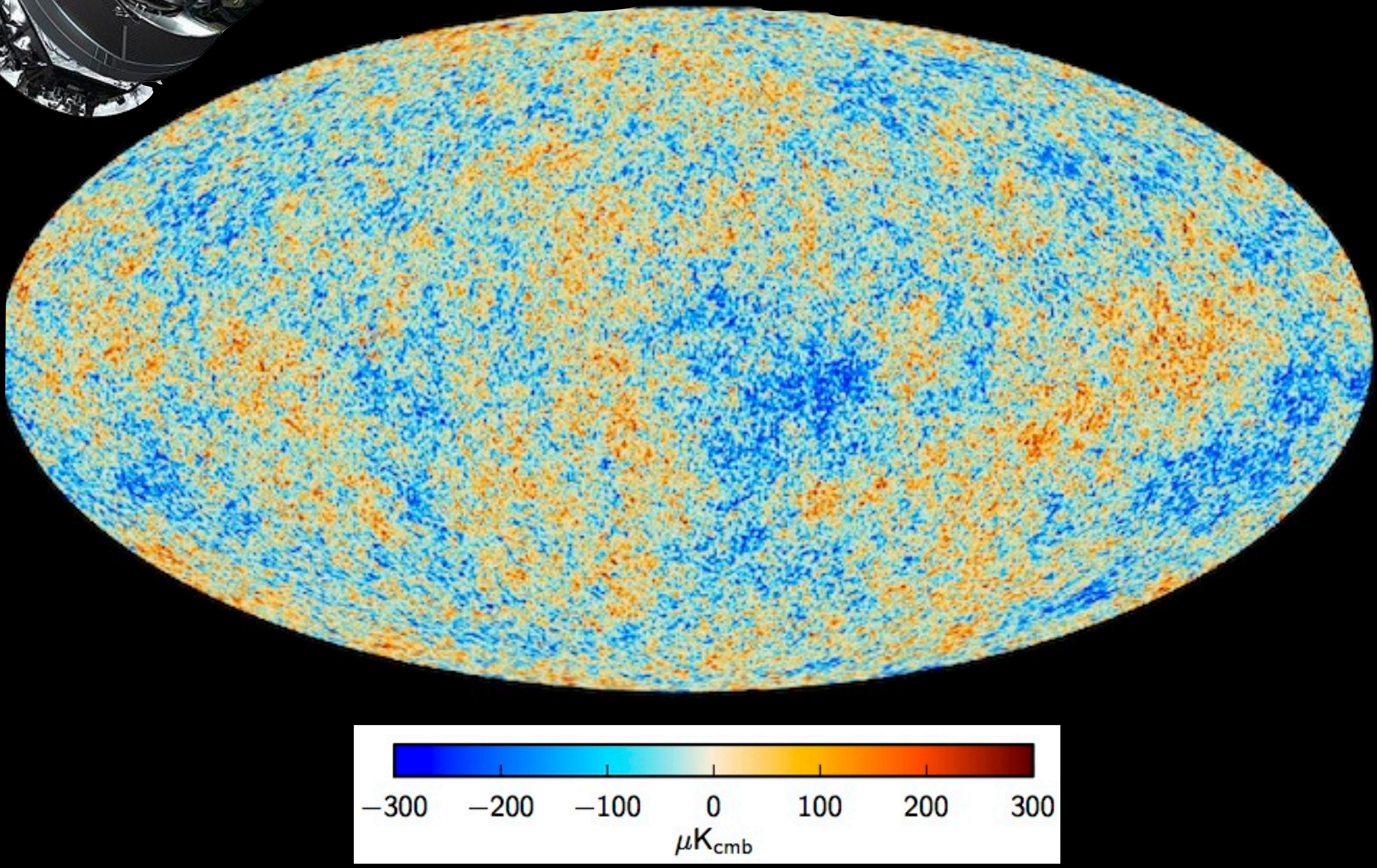
$\lambda \gtrsim H^{-1}$

Feels space-time curvature

Planck satellite



$$\frac{\delta T}{T} \sim 10^{-5} \ll 1$$



Cosmological Perturbation Theory

Density fluctuations are small at CMB scales ⟶ Perturbation Theory

Cosmological Perturbation Theory

Density fluctuations are small at CMB scales → Perturbation Theory

$$g_{\mu\nu}(\overrightarrow{x},t) = \bar{g}_{\mu\nu}(t) + \widehat{\delta g}_{\mu\nu}(\overrightarrow{x},t)$$

$$\phi(\overrightarrow{x},t) = \bar{\phi}(t) + \widehat{\delta\phi}\,(\overrightarrow{x},t)$$

Homogeneous and isotropic solution of the classical problem

Quantised fluctuation

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—> Quantum-field-theory on curved space-time

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—> Quantum-field-theory on curved space-time

Strong assumption: universe is quasi homogeneous and isotropic at all scales

This may be broken at:

- Larger scales: space-time structure beyond the observable universe
- Smaller scales: formation of extreme objects such as primordial black holes, heavy clusters, large voids etc

Separate Universe

$$\mathrm{d}s^2 = a^2\left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2\right)$$

de-Sitter universe: $a = -1/(H\eta), \quad -\infty < \eta < 0$

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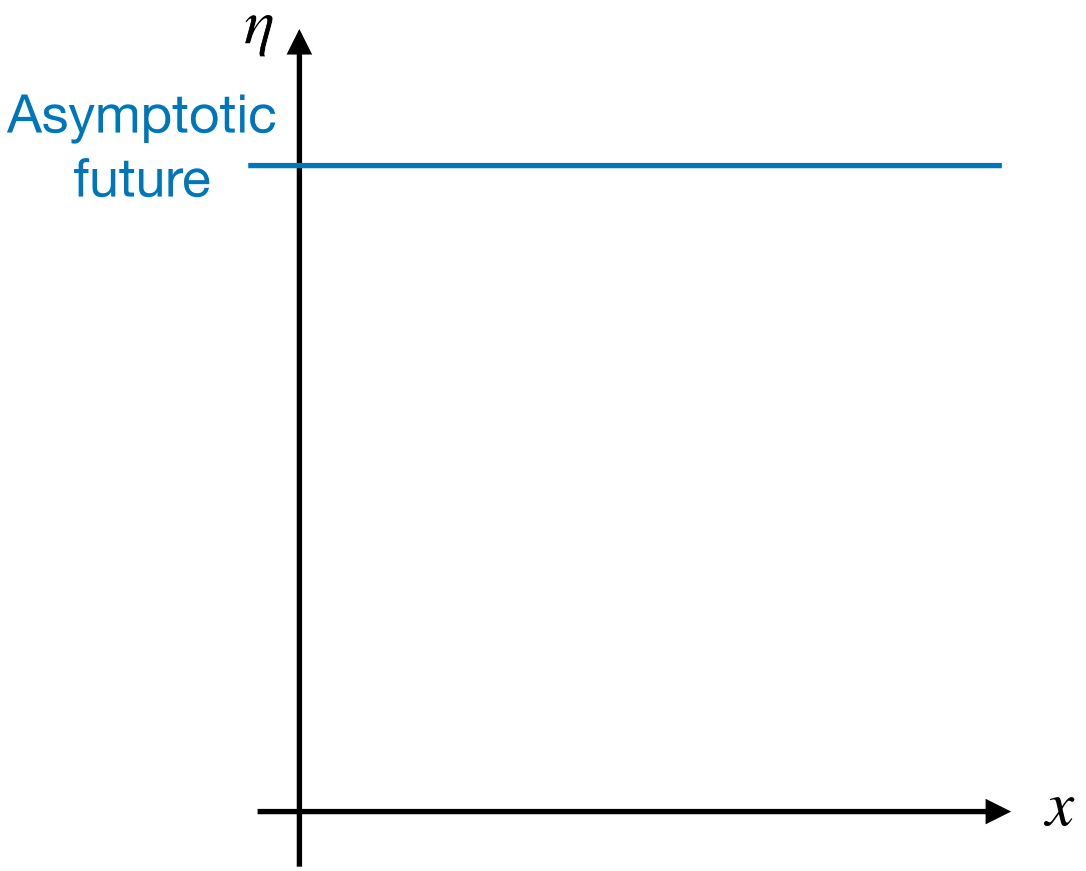

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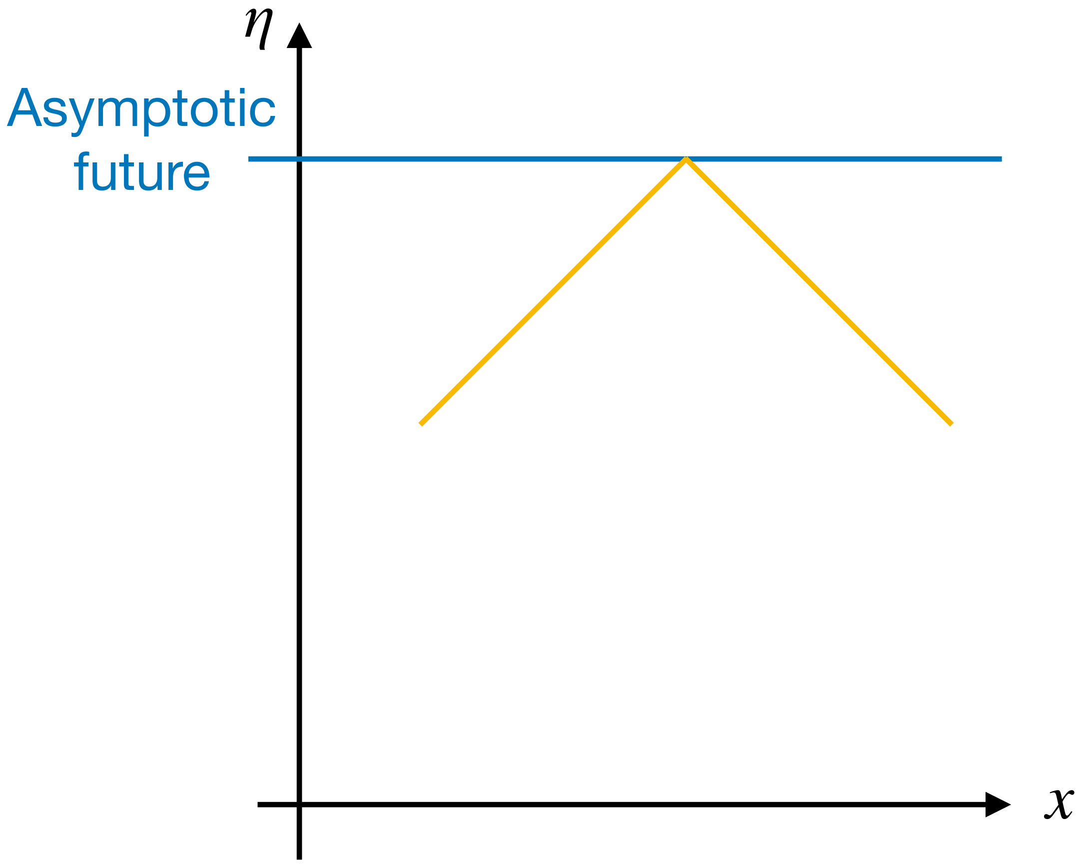

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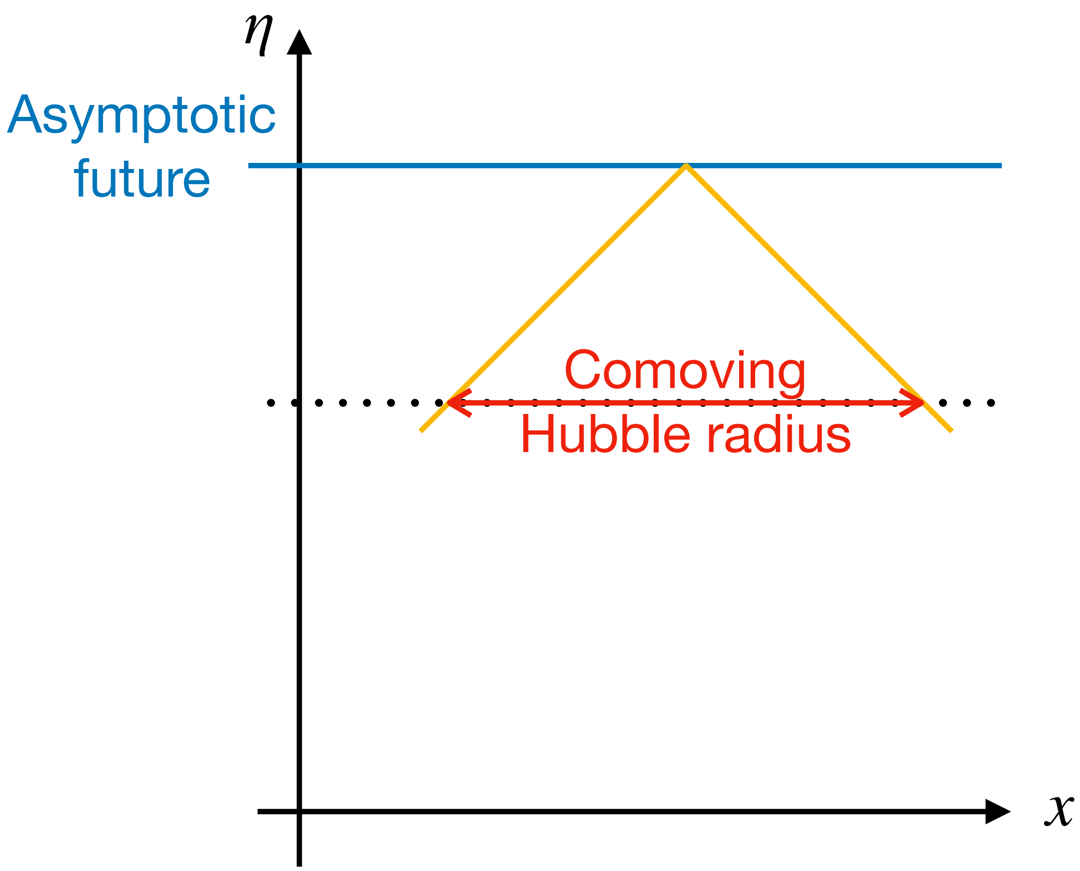

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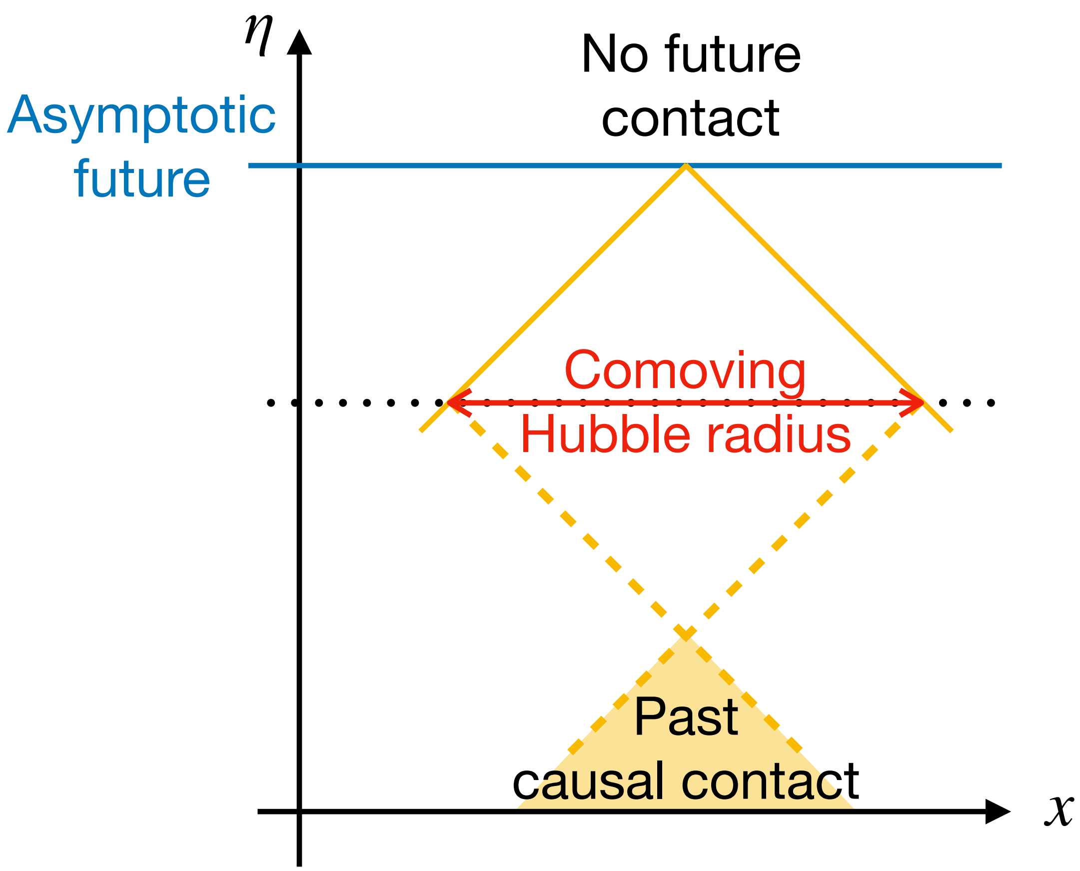

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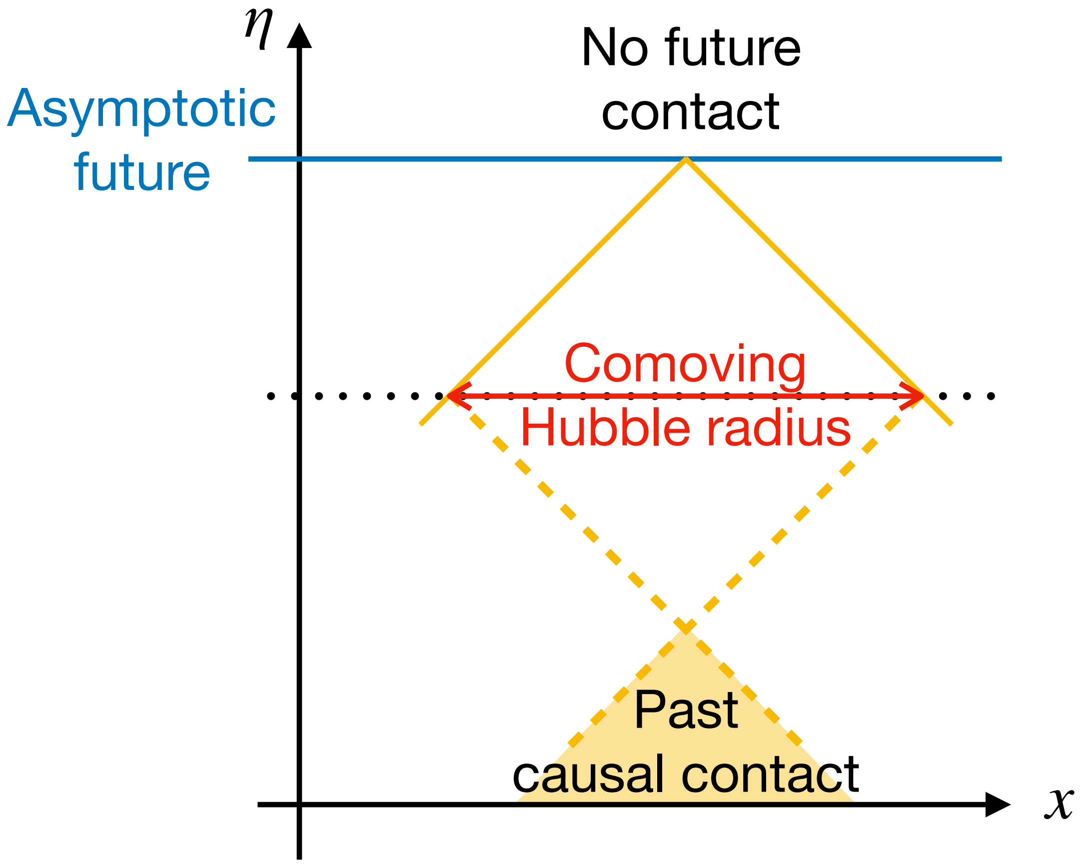

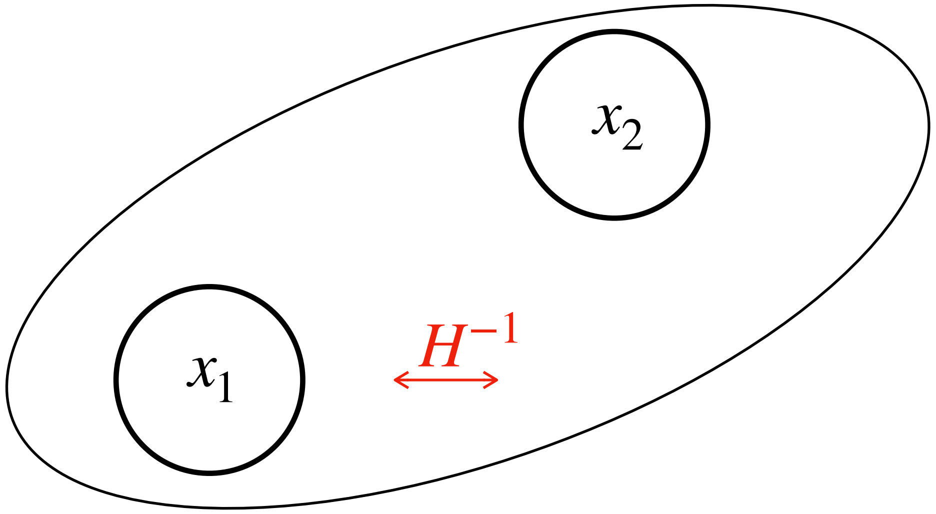

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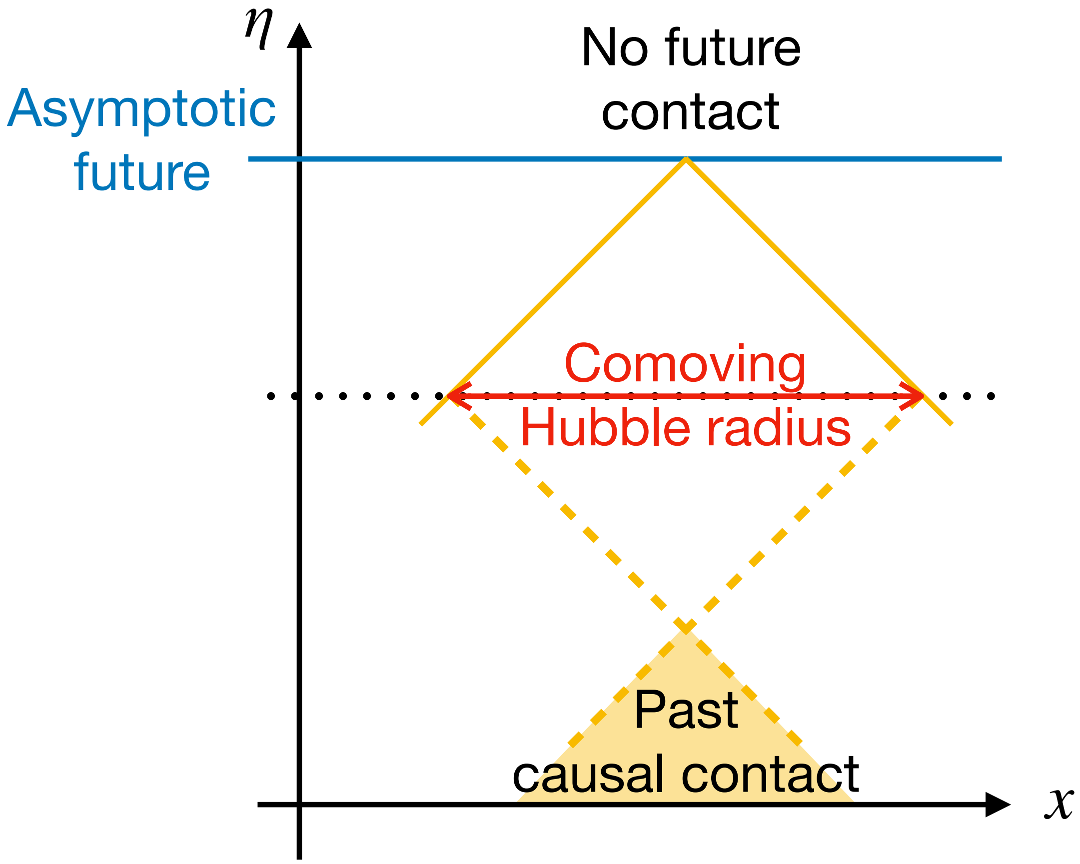
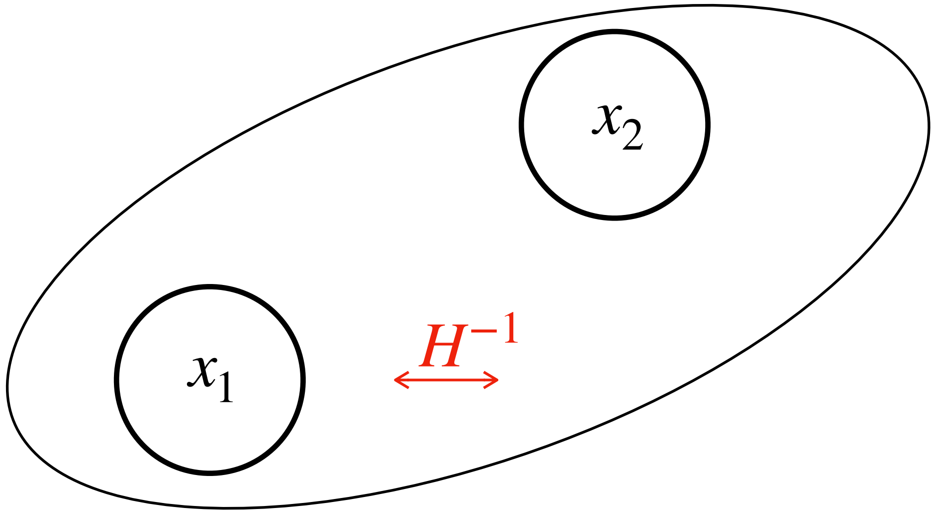


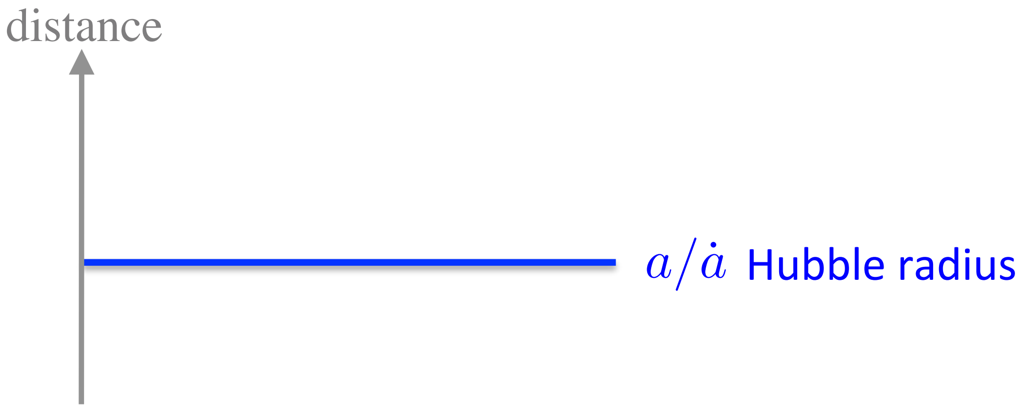


If a large fluctuation develops at $\boldsymbol{x}_1$, this cannot affect the local geometry at $\boldsymbol{x}_2$

Separate universe: On large scales, the universe can be described by an ensemble of independent, locally homogeneous and isotropic patches

Salopek & Bond; Sasaki & Stewart; Wands, Malik, Lyth & Liddle

Stochastic Inflation



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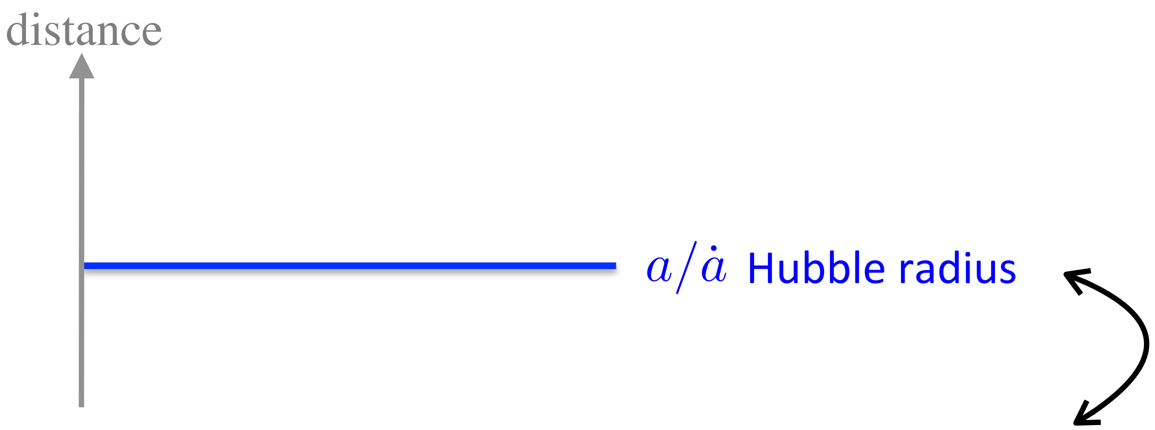

sub-Hubble scales —> CPT

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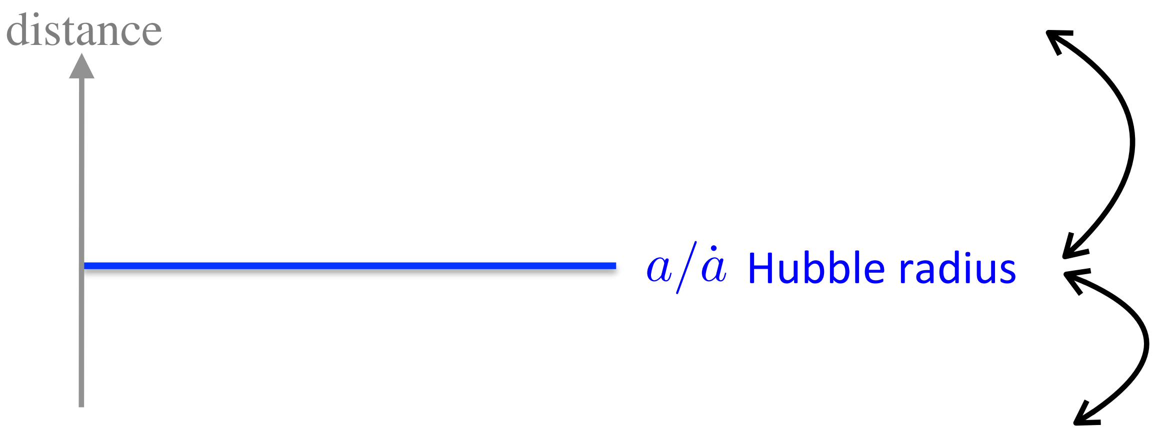

coarse-grained fields —> stochastic inflation

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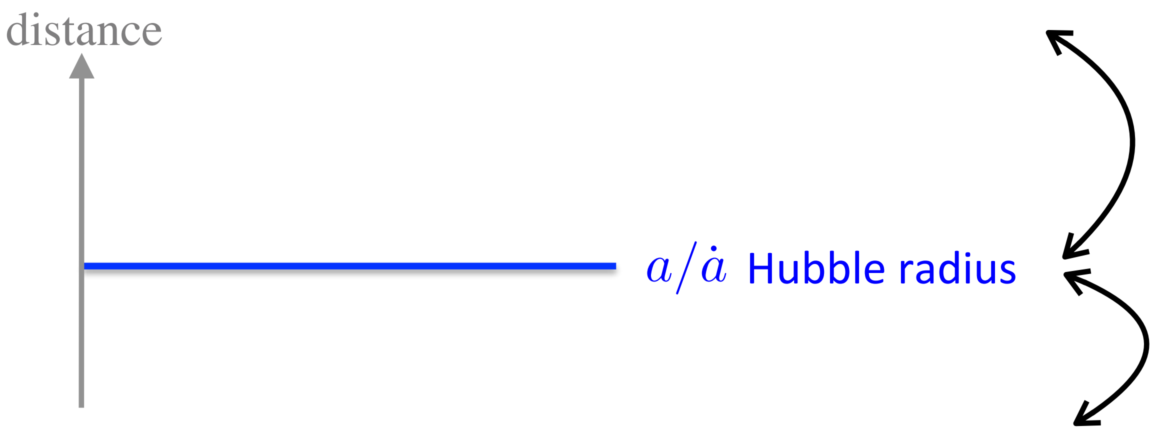

coarse-grained fields —> stochastic inflation

sub-Hubble scales —> CPT

Coarse-grained field

$N = \ln(a)$

$$\hat{\mathbf{\Phi}}_{\mathrm{cg}}(N, \vec{x}) = \int_{k<\sigma H a(N)} \mathrm{d}\vec{k} \left[\mathbf{\Phi}_{\vec{k}}(N) e^{-i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + \mathbf{\Phi}^{\star}_{\vec{k}}(N) e^{i\vec{k}\cdot\vec{x}} \hat{a}^{\dagger}_{\vec{k}}\right]$$

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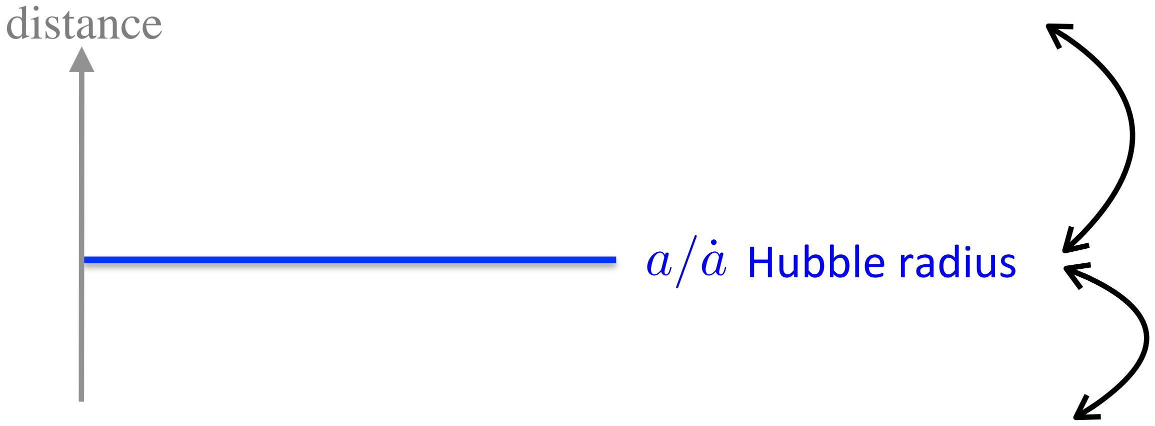

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Equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}N}\mathbf{\Phi}_{\mathrm{cg}} = \mathscr{D}_{\mathrm{background}}(\mathbf{\Phi}_{\mathrm{cg}}) + \xi$$

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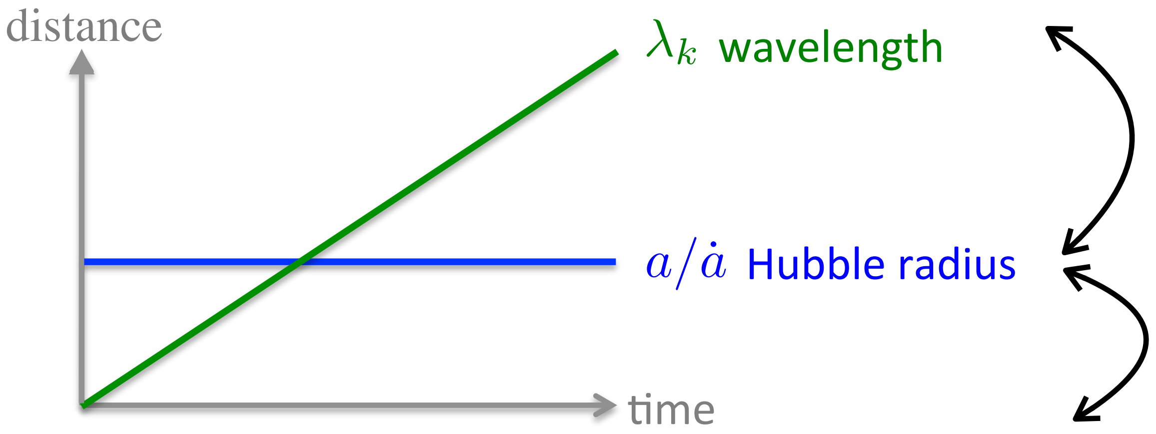

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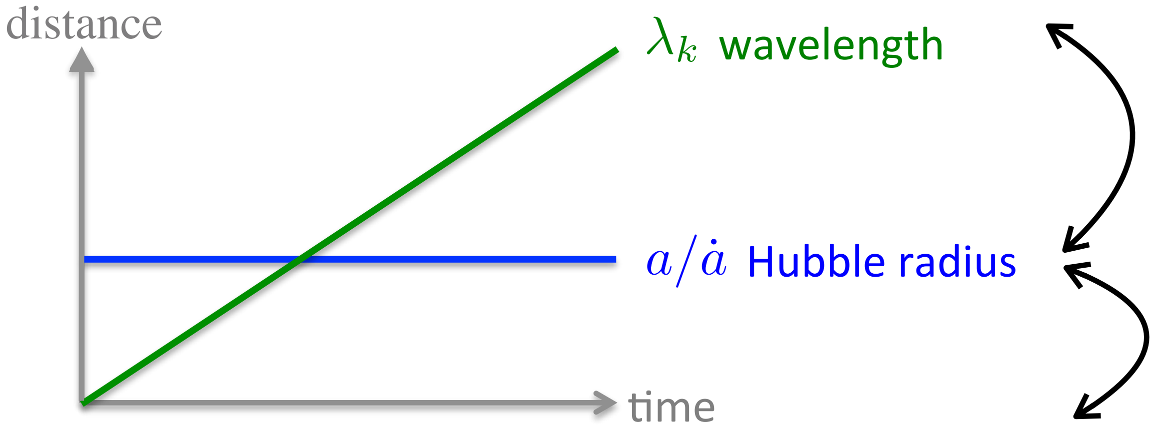

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Quantum fluctuations source the **background**

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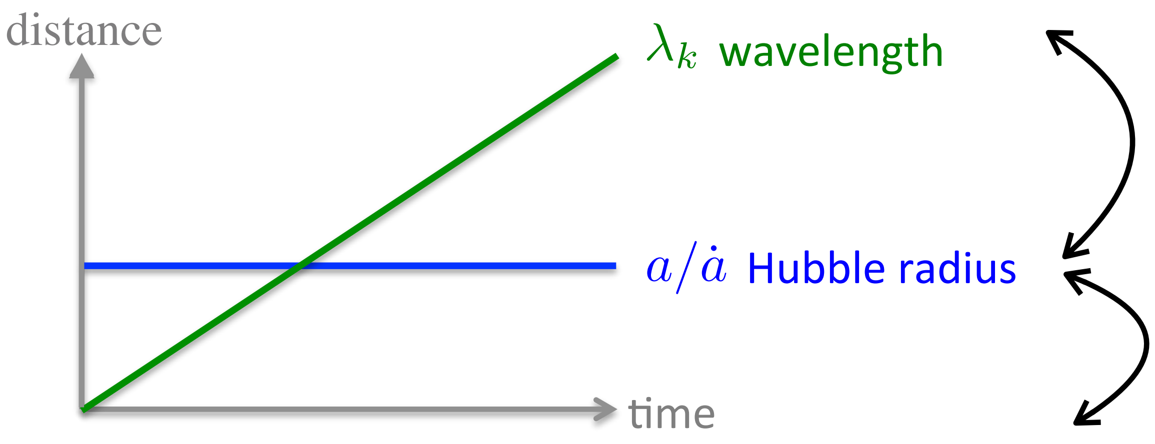

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Starobinsky, (1982) 1986

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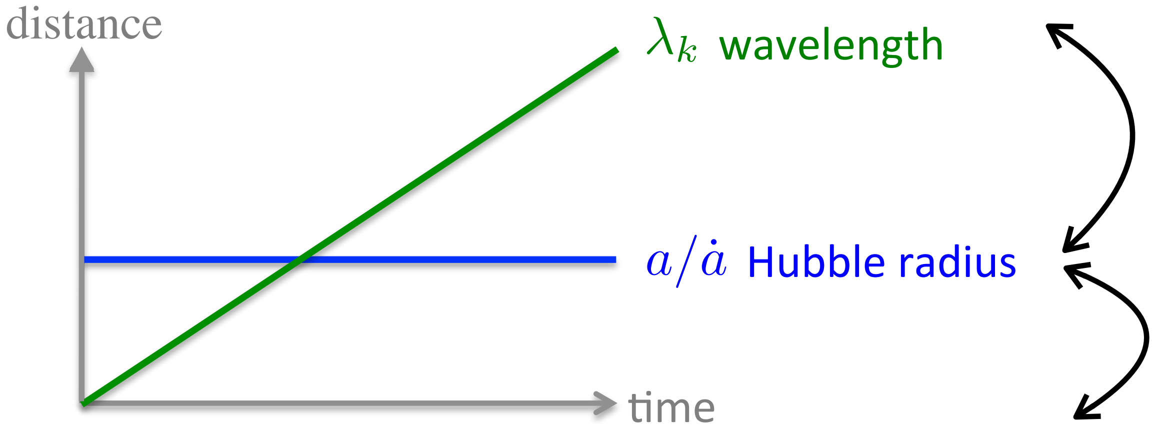

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Over one e-fold: $\dfrac{\Delta\phi_{\rm quant}}{\Delta\phi_{\rm classical}} \sim \zeta_{\rm classical}$

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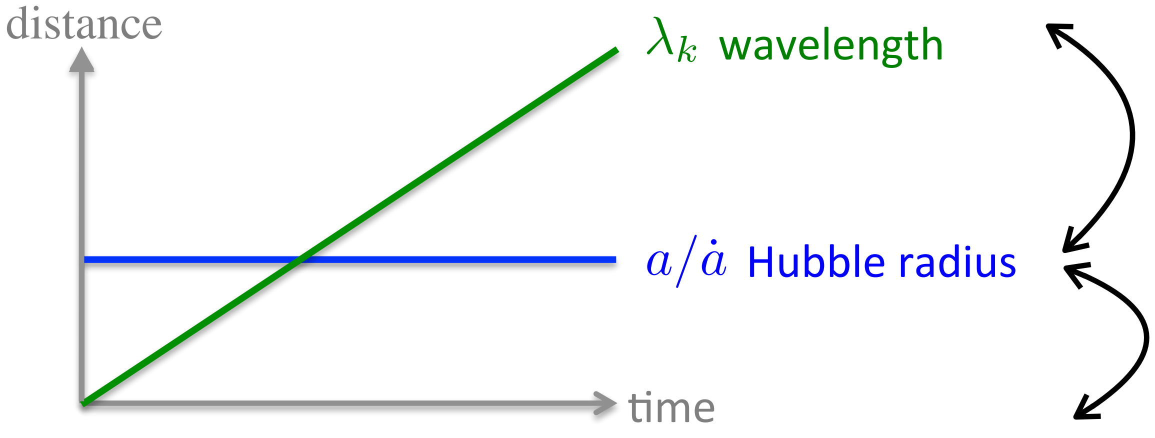

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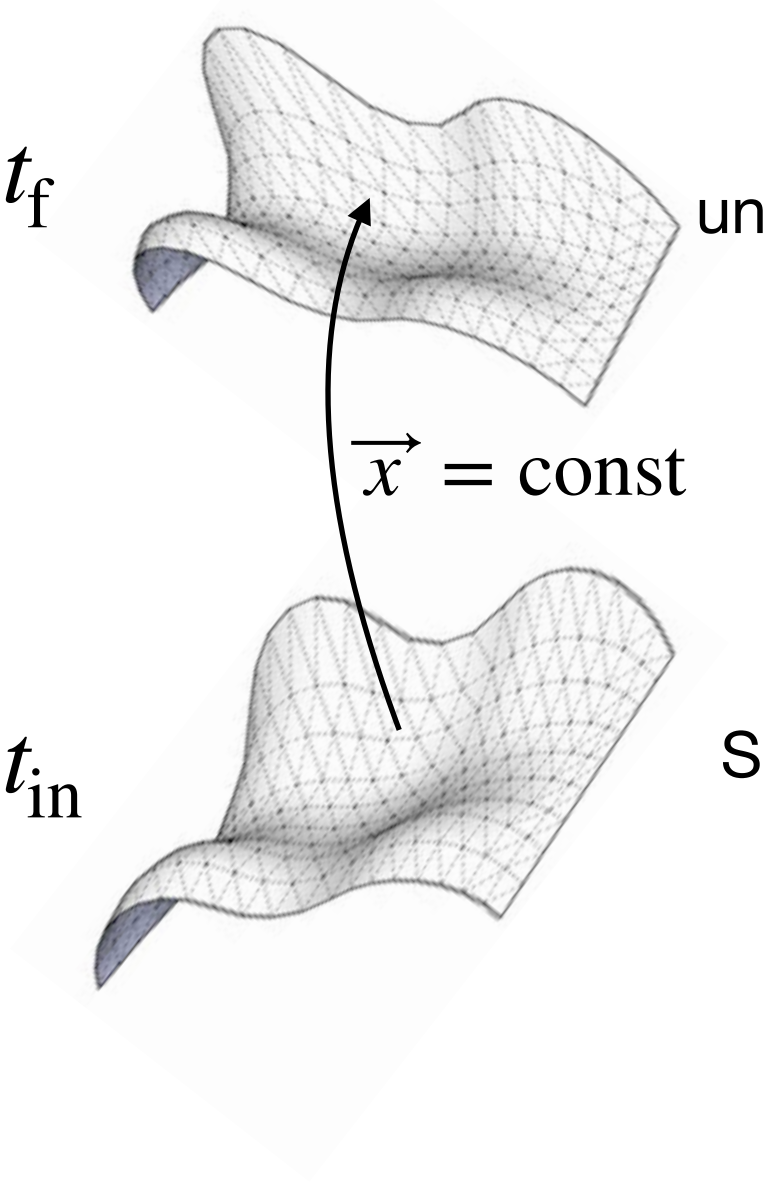
What about far from the classical regime?

What about tail effects?

Stochastic-δ*N* formalism



t_{f}

uniform density slice
$\delta\rho = 0$

$\overrightarrow{x} = \mathrm{const}$

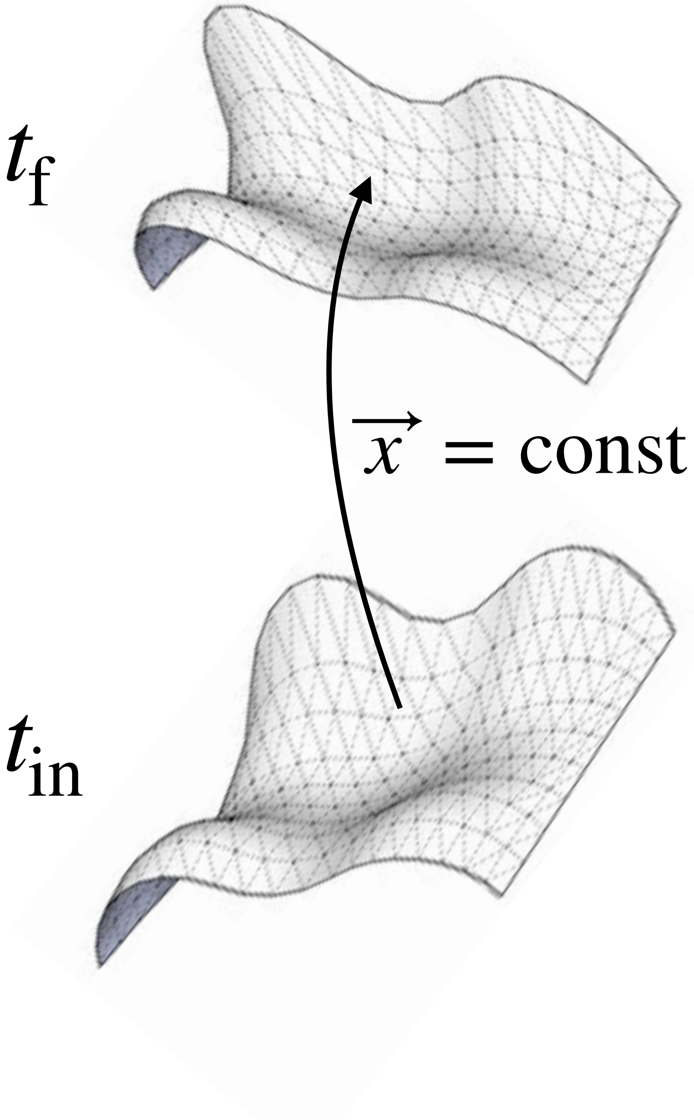
t_{in}

Spatially flat slice
$\psi = 0$

$$\zeta(t, x) = N(t, x) - N_0(t) \equiv \delta N$$

Lifshitz, Khalatnikov (1960)
Starobinsky (1983)
Wands, Malik, Lyth, Liddle (2000)

Stochastic-δ*N* formalism



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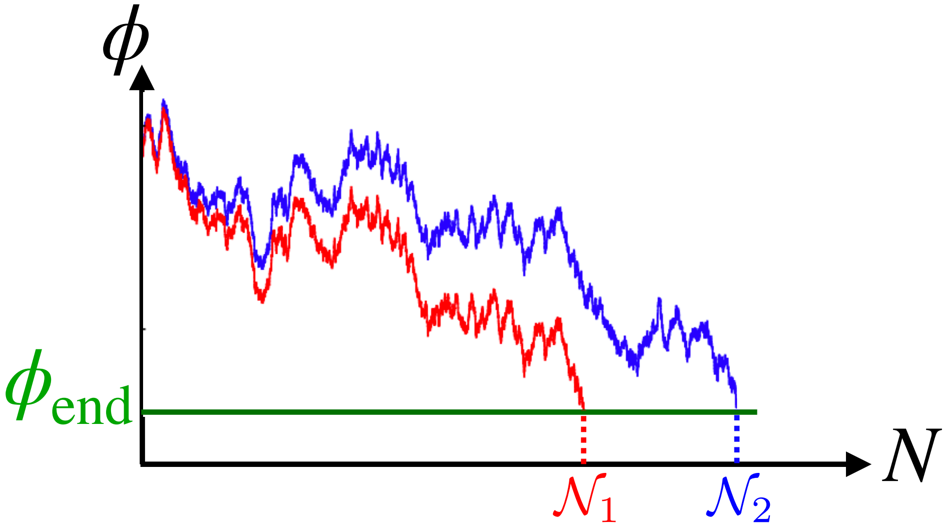

The realised number of e-folds is a stochastic quantity:

$$\zeta_\text{coarse grained} = \mathcal{N} - \langle \mathcal{N} \rangle$$

Enqvist, Nurmi, Podolsky, Rigopoulos (2008) ; Fujita, Kawasaki, Tada, Takesako (2014); VV, Starobinsky (2015)

Stochastic-δN formalism

Stochastic-δ*N* formalism

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi\left(N\right)$$

Langevin equation

Stochastic-δ*N* formalism

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N) \quad \longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}N}P(\phi;N) = \frac{\partial}{\partial\phi}\left(\frac{V'}{3H^2}P\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{H^2}{8\pi^2}P\right)$$

Langevin equation

Fokker-Planck equation

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Langevin equation

Fokker-Planck equation

Equation for the PDF of the first passage time

$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{N}}P_{\mathrm{FPT}}(\mathcal{N};\phi) = \mathscr{L}_\phi^\dagger \cdot P_{\mathrm{FPT}}$$

VV, Starobinsky (2015)
Pattison, VV, Assadullahi, Wands (2017)

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Fokker-Planck equation

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Computational program:

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Computational program:

- Solve the first passage time problem
- This gives the one-point PDF of curvature perturbation coarse-grained at H_{end}

Stochastic-δ*N* formalism

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi(N) \longrightarrow \frac{\mathrm{d}}{\mathrm{d}N}P(\phi; N) = \frac{\partial}{\partial\phi}\left(\frac{V'}{3H^2}P\right) + \frac{\partial^2}{\partial\phi^2}\left(\frac{H^2}{8\pi^2}P\right) = \mathcal{L}_\phi \cdot P$$

Langevin equation

Fokker-Planck equation

Equation for the PDF of the first passage time

VV, Starobinsky (2015)
Pattison, VV, Assadullahi, Wands (2017)

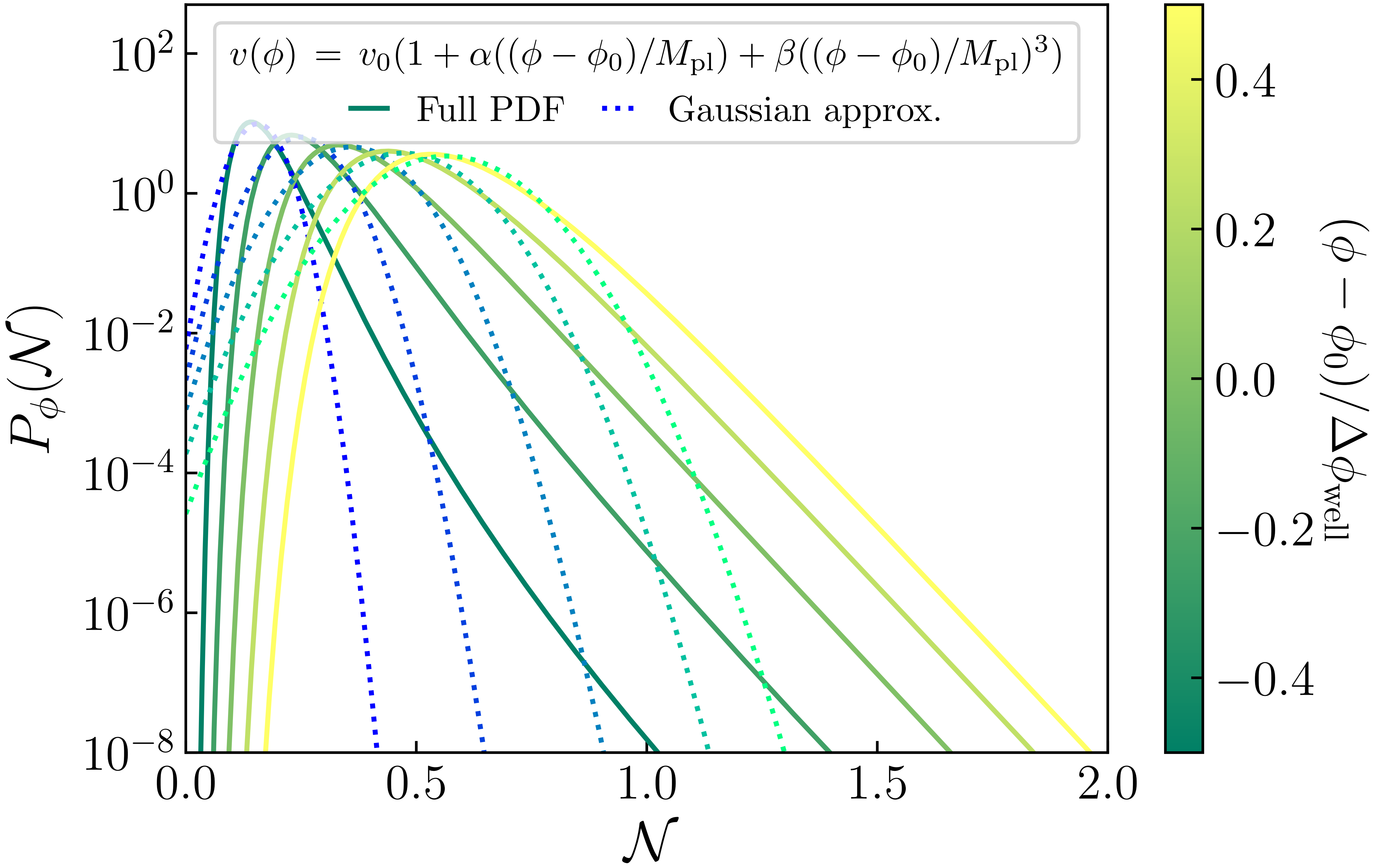
$$\frac{\mathrm{d}}{\mathrm{d}\mathcal{N}}P_{\mathrm{FPT}}(\mathcal{N}; \phi) = \mathcal{L}_\phi^\dagger \cdot P_{\mathrm{FPT}}$$

Computational program:

- Solve the first passage time problem
- This gives the one-point PDF of curvature perturbation coarse-grained at H_{end}
- Extract cosmologically relevant quantities (power spectrum, mass functions, etc)

Exponential tails

Pattison, VV, Assadullahi, Wands (2017)
Ezquiaga, Garcia-Bellido, VV (2020)



Exponential tails

Pattison, VV, Assadullahi, Wands (2017)
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$$P_{\mathrm{FPT}}(\mathcal{N}, \phi) = \sum a_n(\phi) e^{-\Lambda_n \mathcal{N}}$$

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Flat well

$$\Lambda_n = \frac{\pi^2}{\mu^2}\left(n + \frac{1}{2}\right)^2$$

Exponential tails

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$$P_{\mathrm{FPT}}(\mathcal{N}, \phi) = \sum a_n(\phi) e^{-\Lambda_n \mathcal{N}}$$

$$v = V/(24\pi^2 M_{\mathrm{Pl}}^4)$$

Flat well

$$\Lambda_n = \frac{\pi^2}{\mu^2}\left(n + \frac{1}{2}\right)^2 \qquad \mu^2 = \frac{\Delta\phi_{\mathrm{well}}^2}{v_0 M_{\mathrm{Pl}}^2}$$

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Constant slope well $v = v_0\left(1 + \alpha\frac{\phi}{M_{\mathrm{Pl}}}\right)$

$$\Lambda_n \simeq \frac{\alpha^2}{4v_0} + \frac{\pi^2}{\mu^2}\left(n + \frac{1}{2}\right)^2$$

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Cubic inflection point $v = v_0\left(1 + \alpha\frac{\phi^3}{M_{\mathrm{Pl}}^3}\right)$

$$\Lambda_n \simeq \left(\frac{3}{2}\right)^{2/3} \pi^2 \left(v_0\alpha\right)^{1/3}\left(n + \frac{1}{2}\right)^2$$

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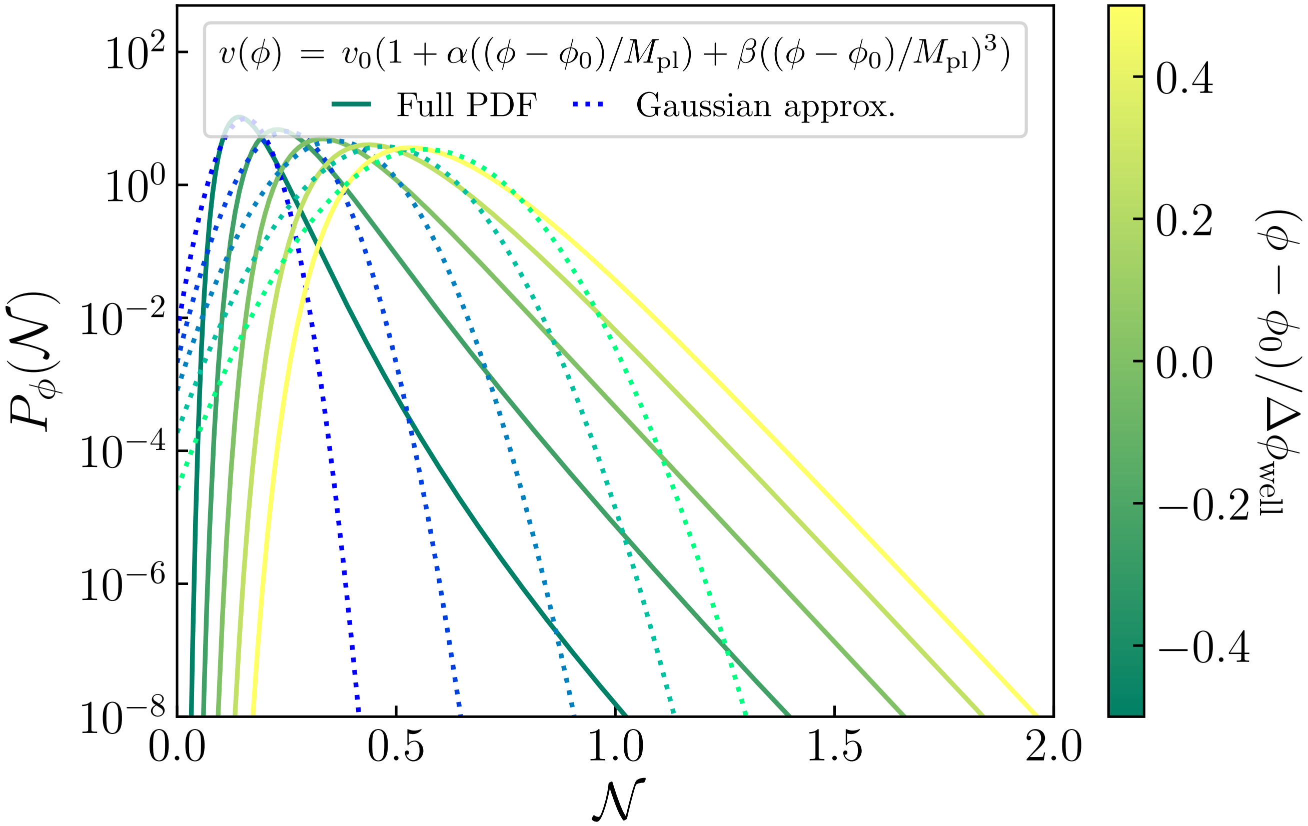

$v_0 = 10^{-3}$, $\alpha = 0.24$, $\beta = 9$, $\phi_{\mathrm{end}} = 9$

$$v = V/(24\pi^2 M_{\mathrm{Pl}}^4)$$

Flat well

$$\Lambda_n = \frac{\pi^2}{\mu^2}\left(n+\frac{1}{2}\right)^2 \qquad \mu^2 = \frac{\Delta\phi_{\mathrm{well}}^2}{v_0 M_{\mathrm{Pl}}^2}$$

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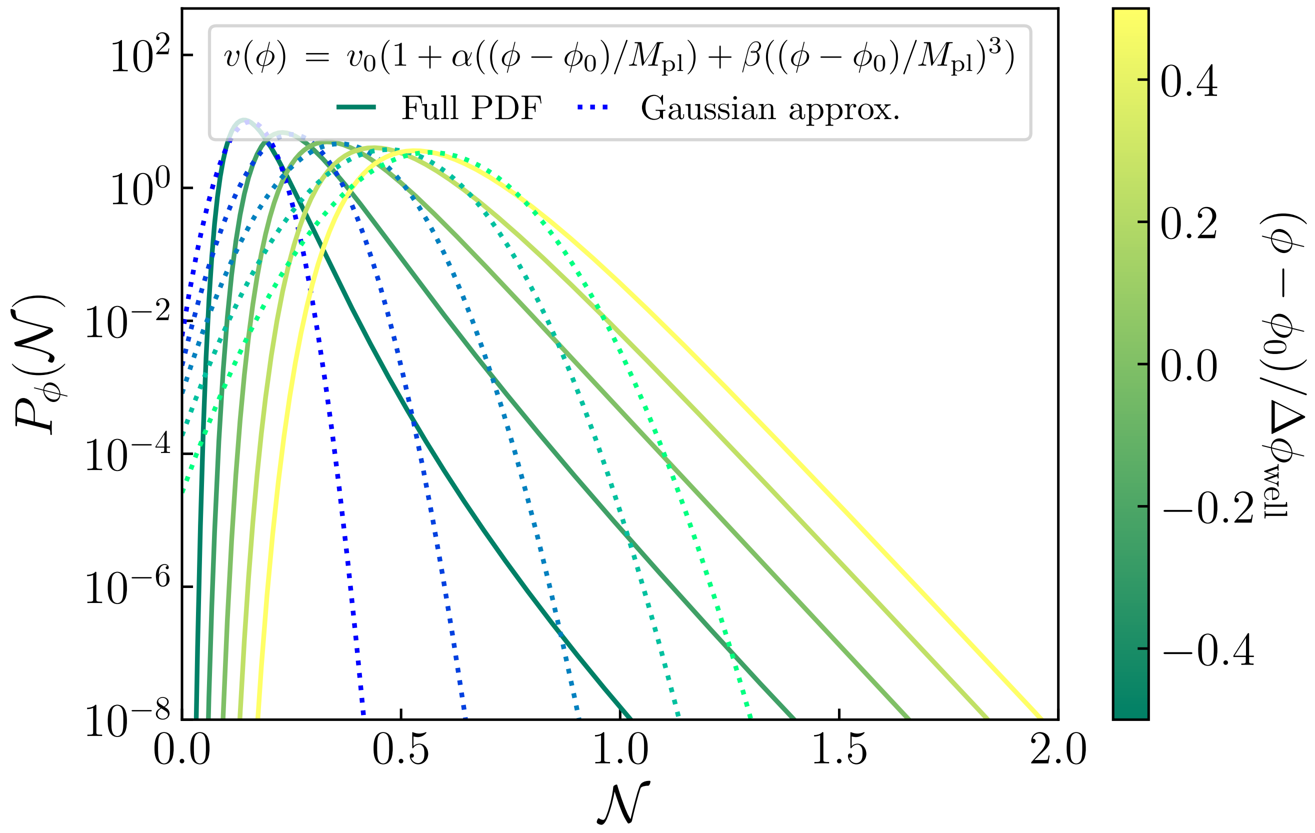

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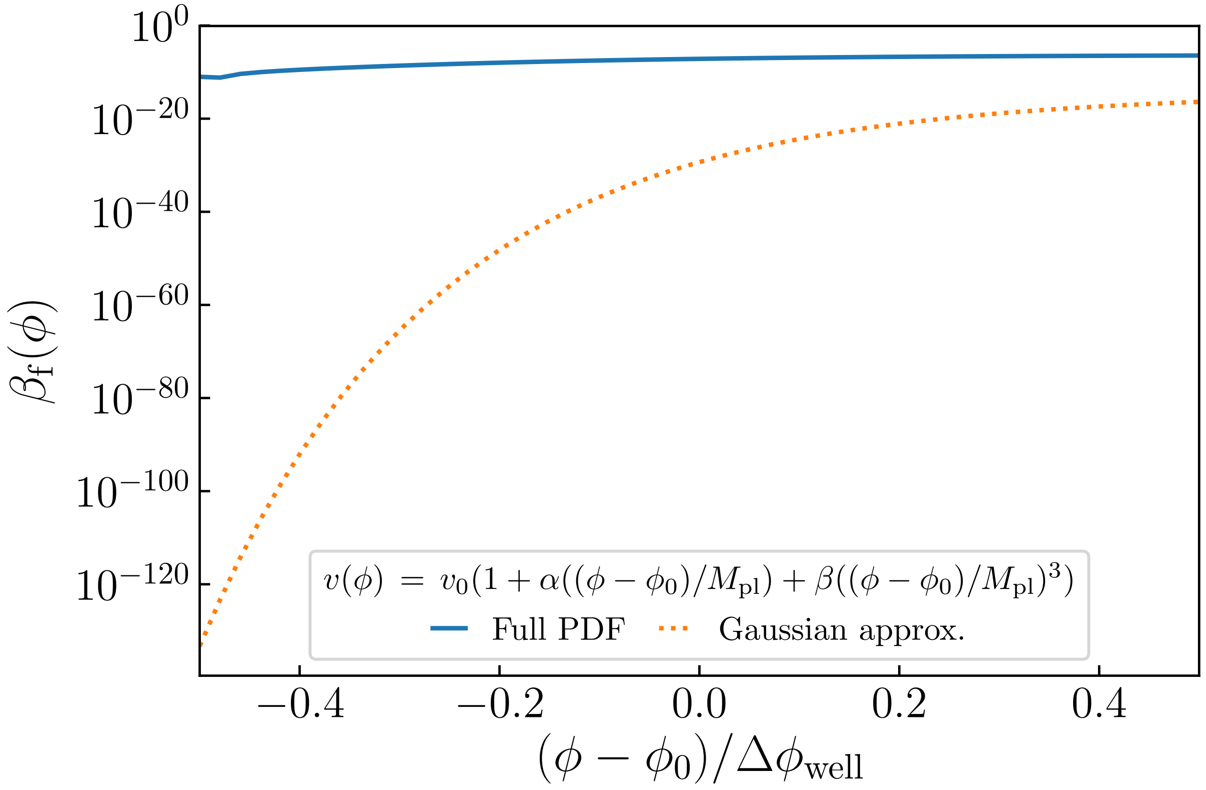

Exponential tails

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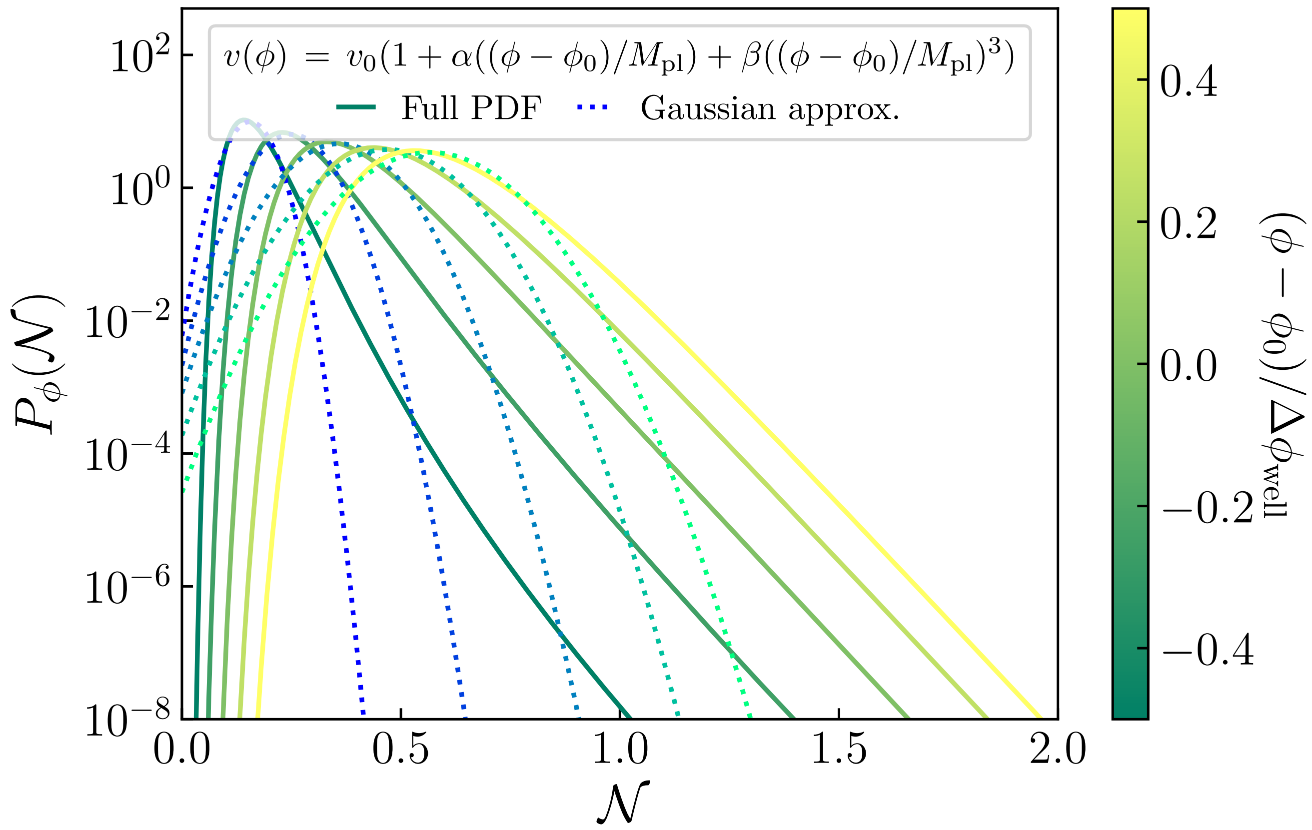

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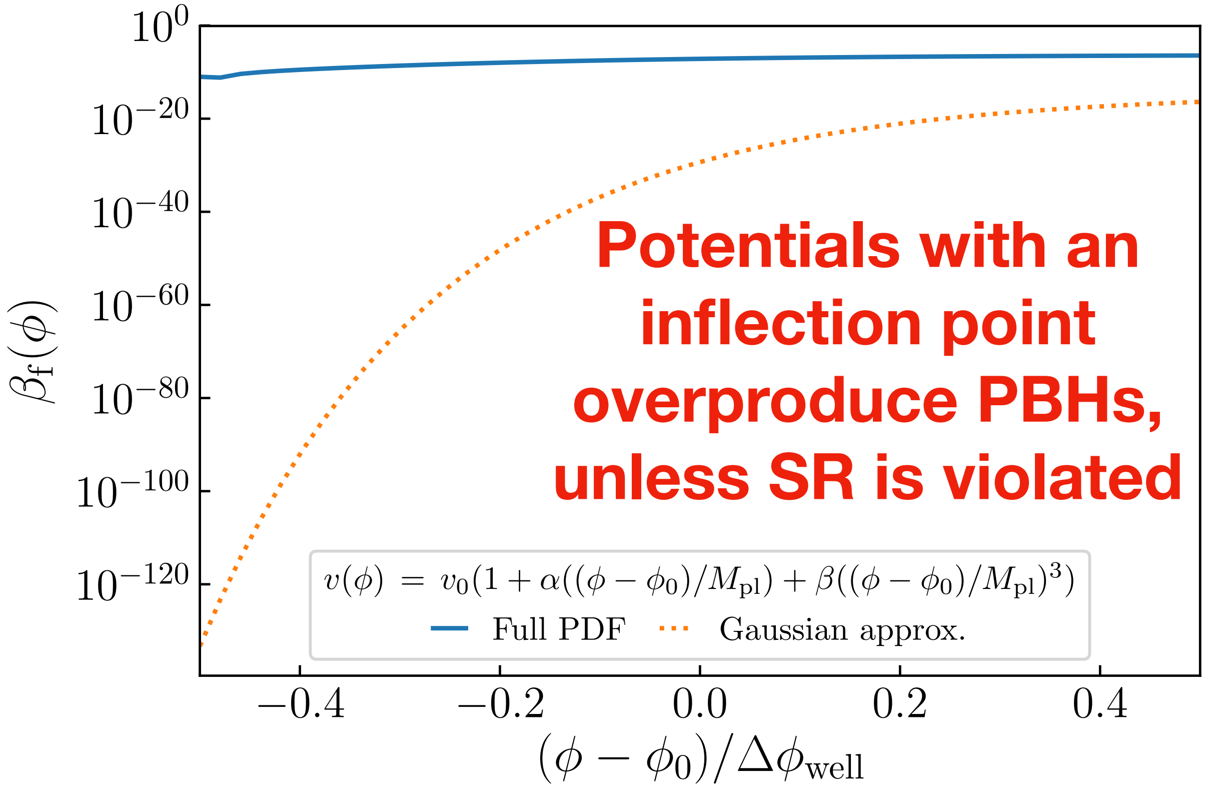
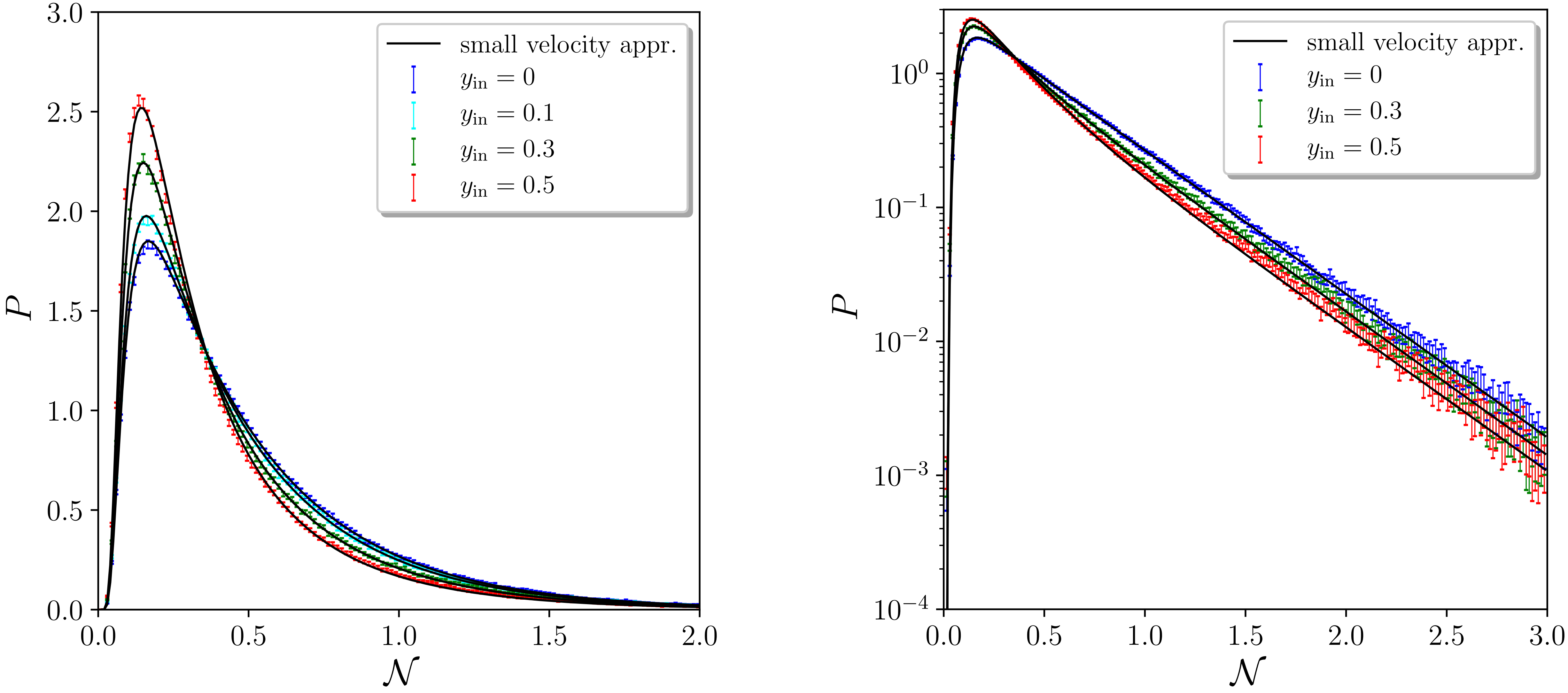


Potentials with an inflection point overproduce PBHs, unless SR is violated

Exponential tails

in ultra slow roll models

Pattison, Vennin, Wands, Assadullahi (2021)



See also Figueroa, Raatikainen, Rasanen, Tomberg 2021

Extracting cosmological observables

Scale k $\longrightarrow$ Hubble-crossing time $\longrightarrow$ Hubble-crossing field ϕ_*

$$N_* = N_{\rm end} - N_{\rm bw}(k)$$

$$N_{\rm bw}(k) = \ln(a_{\rm end}H/k)$$

Extracting cosmological observables

Scale k ⟶ Hubble-crossing time ⟶ Hubble-crossing field ϕ_*

$$N_* = N_{\rm end} - N_{\rm bw}(k)$$

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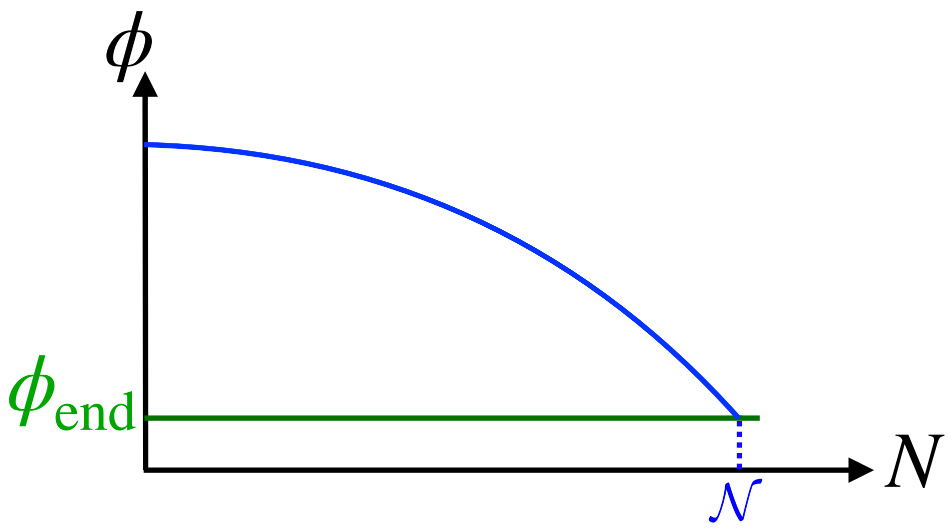
<u>Classical picture</u>



Extracting cosmological observables

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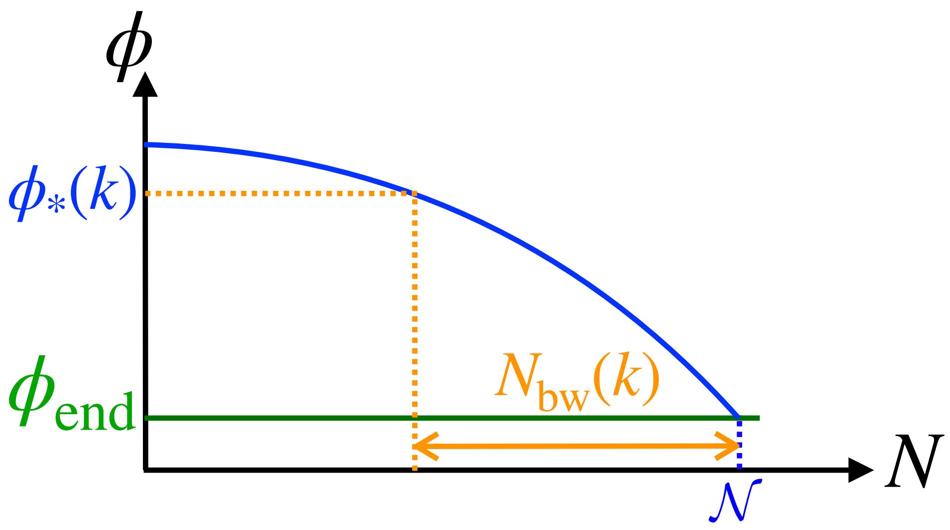
Classical picture



Extracting cosmological observables

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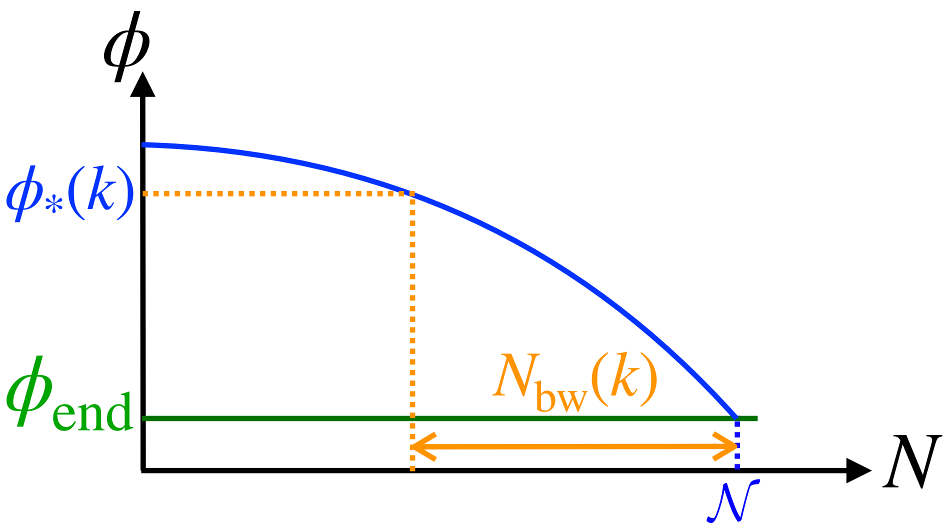

Observables (power spectrum etc) at scale *k* depend on **local properties** of the potential at location $\phi_*(k)$

Extracting cosmological observables

Scale k → Hubble-crossing time → Hubble-crossing field ϕ_*

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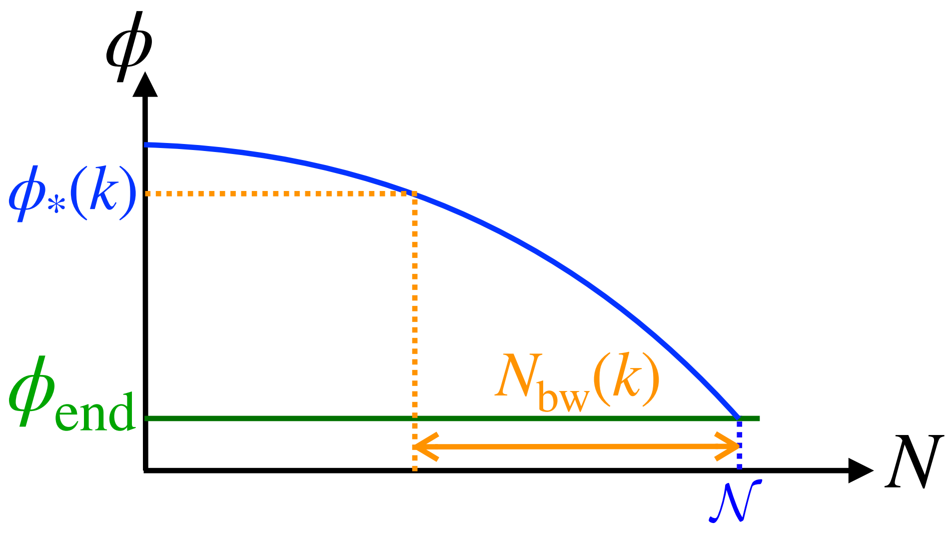

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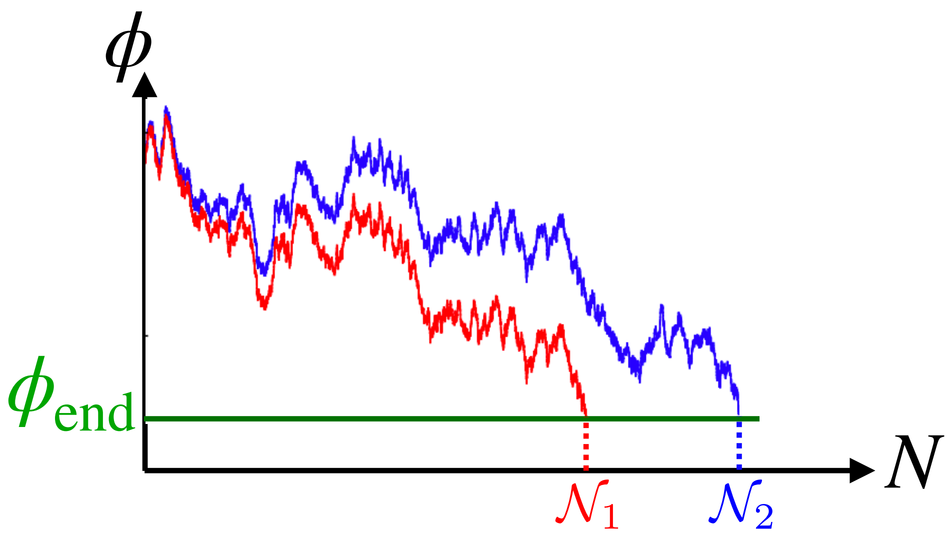

Extracting cosmological observables

Scale k → Hubble-crossing time → Hubble-crossing field ϕ_*

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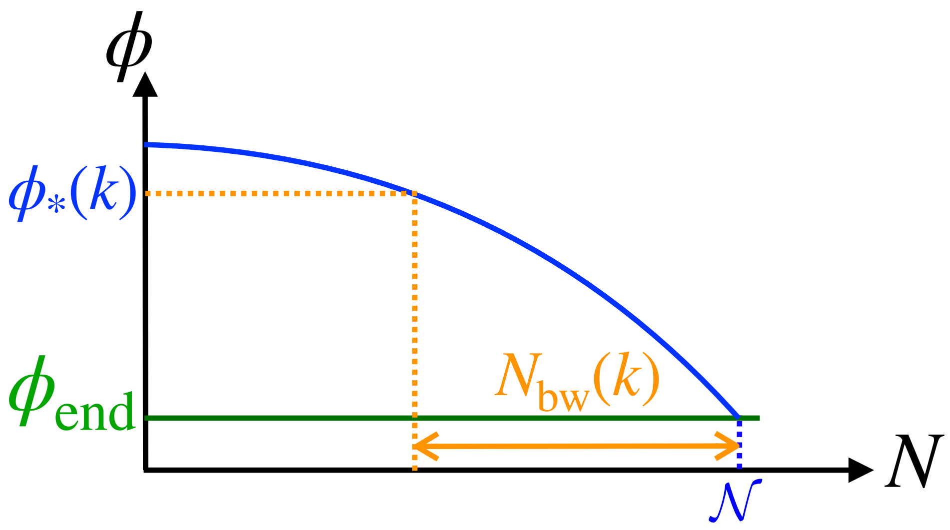
$$N_{\rm bw}(k) = \ln(a_{\rm end}H/k)$$

Classical picture



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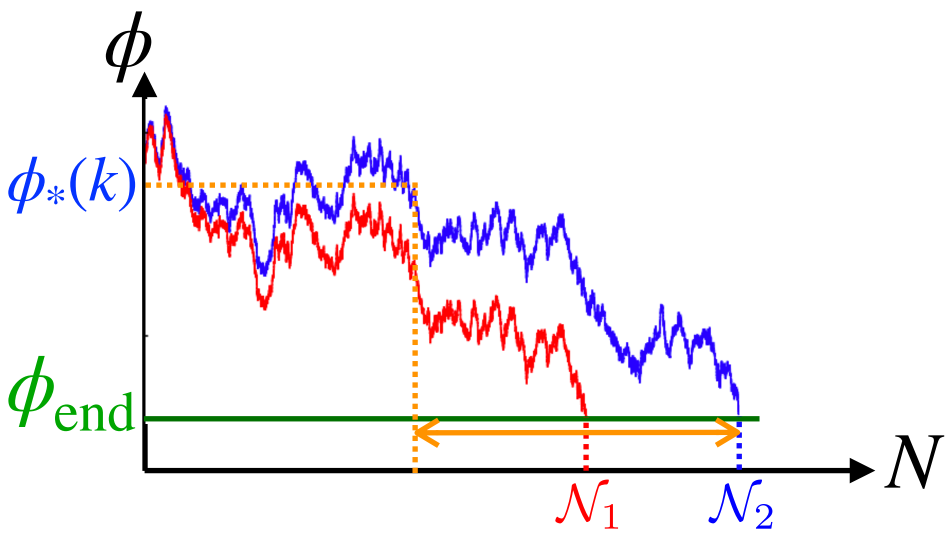


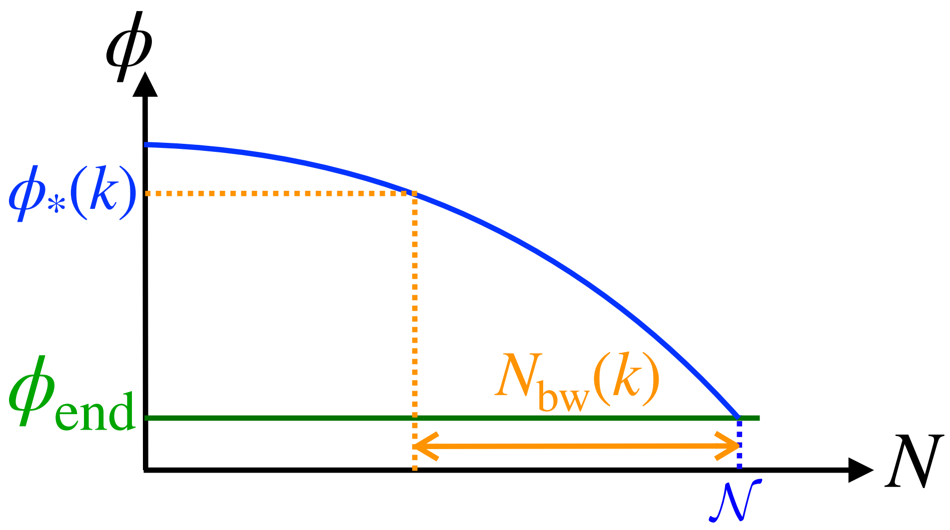
Extracting cosmological observables

Scale k ⟶ Hubble-crossing time ⟶ Hubble-crossing field ϕ_*

$$N_* = N_{\mathrm{end}} - N_{\mathrm{bw}}(k)$$

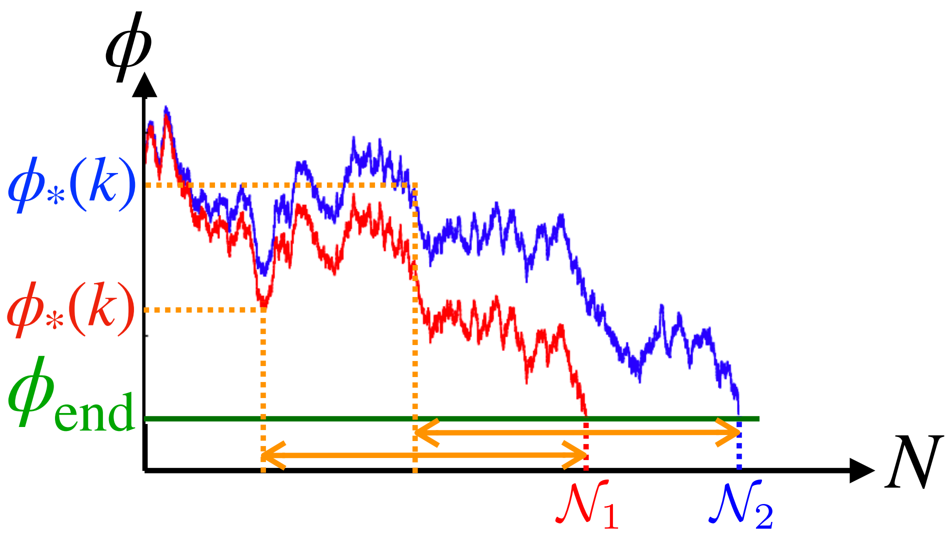
$$N_{\mathrm{bw}}(k) = \ln(a_{\mathrm{end}} H/k)$$

<u>Classical picture</u>



Observables (power spectrum etc) at scale k depend on **<u>local properties</u>** of the potential at location $\phi_*(k)$

<u>Stochastic picture</u>



Extracting cosmological observables

Scale k ⟶ Hubble-crossing time ⟶ Hubble-crossing field ϕ_*

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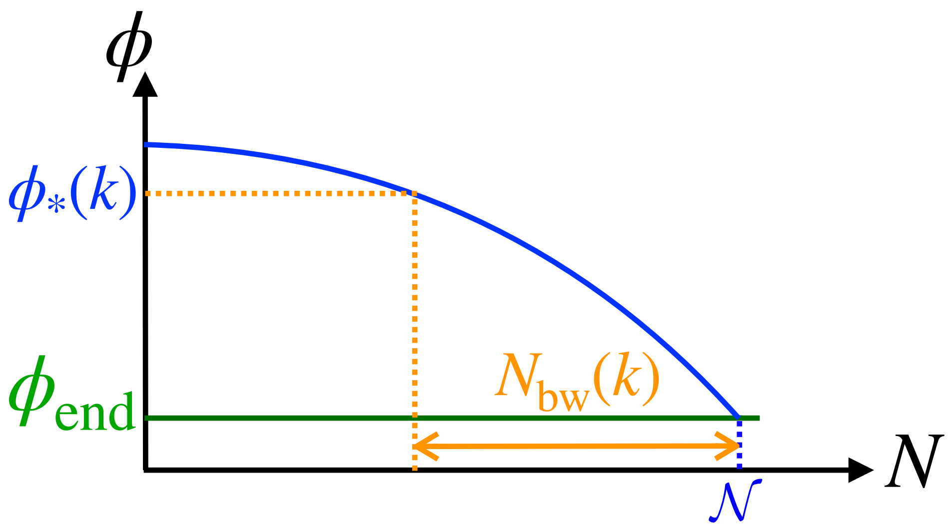
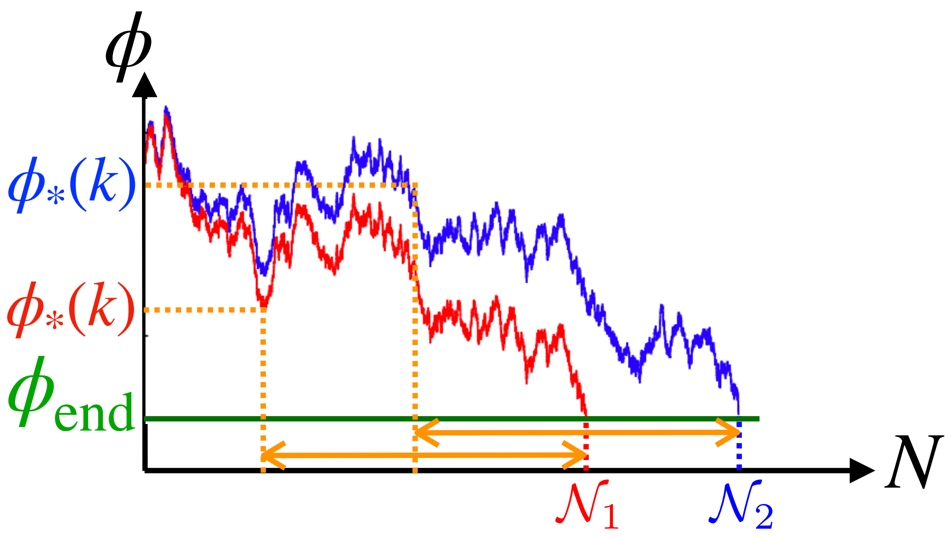
Classical picture



Observables (power spectrum etc) at scale *k* depend on **local properties** of the potential at location $\phi_*(k)$

Stochastic picture



$$P_{\rm bw}(\phi_*;k) = P_{\rm FPT}\left[N_{\rm bw}(k);\phi_*\right] \frac{\int_0^\infty P(\phi_*;N)\,\mathrm{d}N}{\int_{N_{\rm bw}(k)}^\infty P_{\rm FPT}(\mathcal{N};\phi_{\rm in})\,\mathrm{d}\mathcal{N}}$$

Kenta Ando, VV (2020)

Extracting cosmological observables

Scale k ⟶ Hubble-crossing time ⟶ Hubble-crossing field ϕ_*

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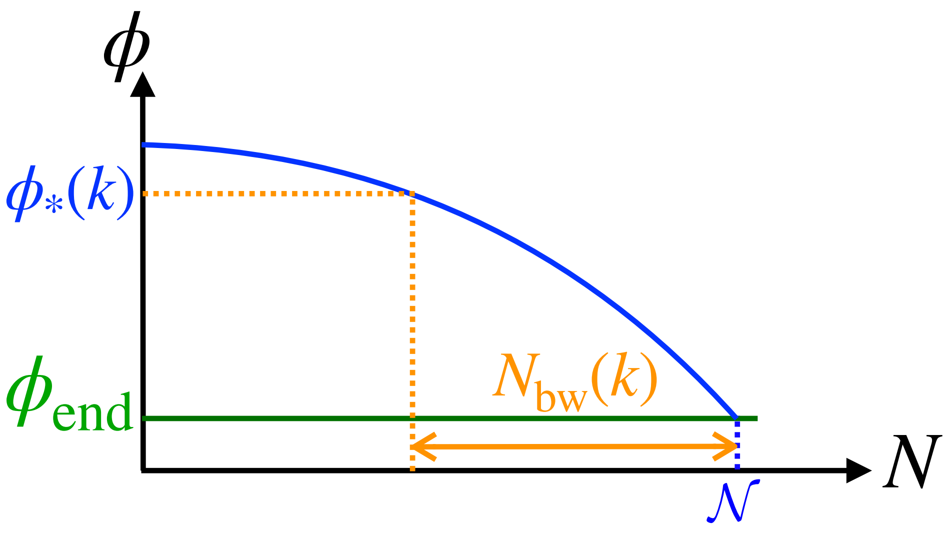
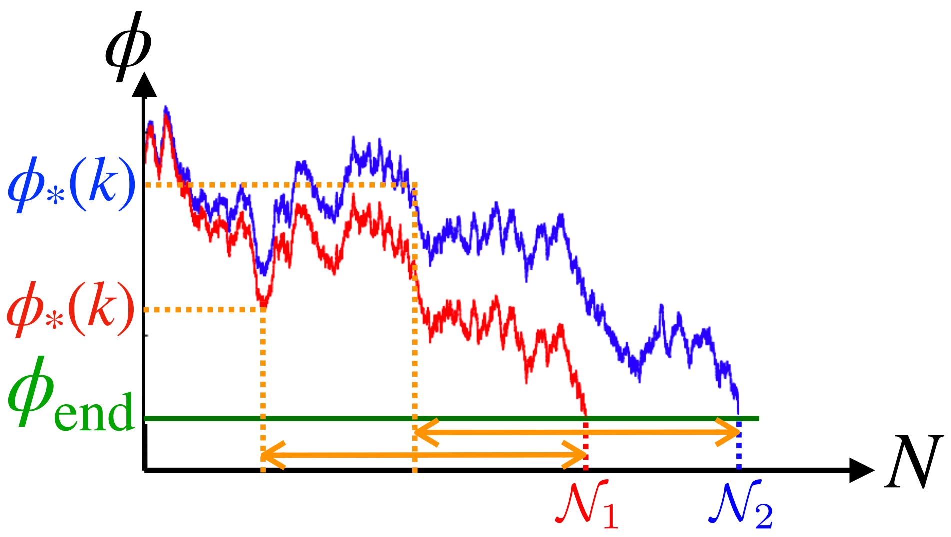


Observables (power spectrum etc) at scale *k* depend on **local properties** of the potential at location $\phi_*(k)$

Stochastic picture



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Kenta Ando, VV (2020)

Observables at scale *k* depend on the **whole potential** and on the **initial condition**

Extracting cosmological observables

Power Spectrum

Kenta Ando, VV (2020)

$$\mathscr{P}_\zeta(k) = -\int_\Omega \mathrm{d}\mathbf{\Phi}_* \left.\frac{\partial P_{\mathrm{bw}}(\mathbf{\Phi}_*; N_{\mathrm{bw}})}{\partial N_{\mathrm{bw}}}\right|_{N_{\mathrm{bw}}(k)} \left\langle \delta\mathcal{N}^2(\mathbf{\Phi}_0 \to \mathbf{\Phi}_*)\right\rangle$$

Extracting cosmological observables

Power Spectrum

Kenta Ando, VV (2020)

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Integration over the full inflating domain

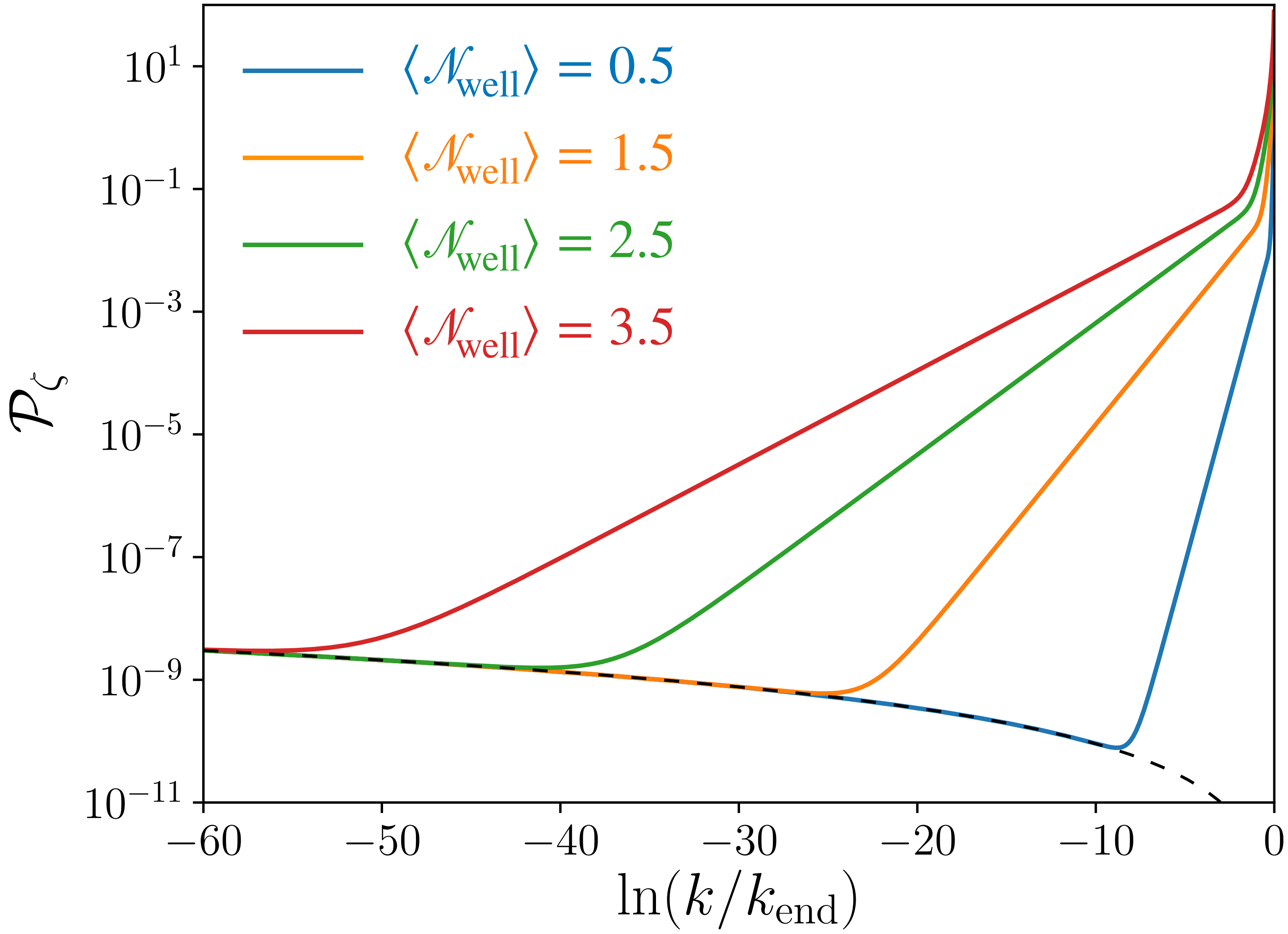
Extracting cosmological observables

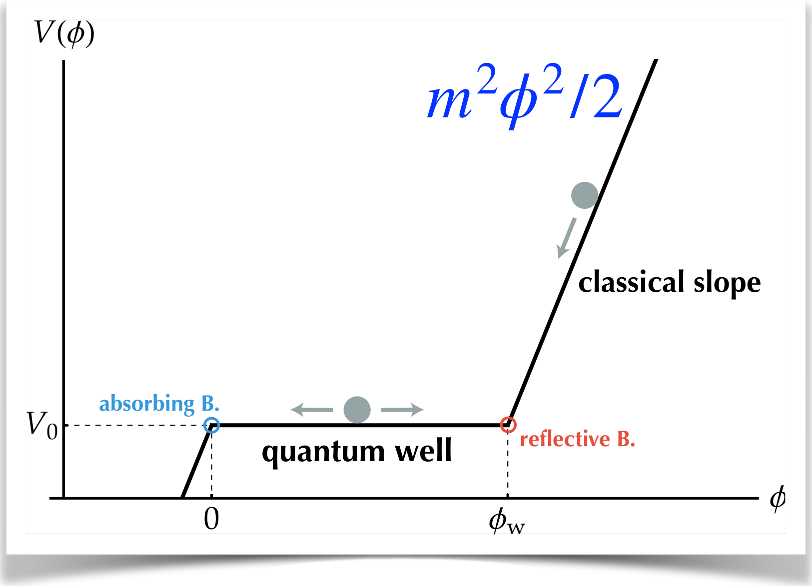
Power Spectrum

Kenta Ando, VV (2020)

$$\mathscr{P}_\zeta(k) = -\boxed{\int_\Omega \mathrm{d}\boldsymbol{\Phi}_*} \frac{\partial P_{\mathrm{bw}}(\boldsymbol{\Phi}_*; N_{\mathrm{bw}})}{\partial N_{\mathrm{bw}}}\Bigg|_{N_{\mathrm{bw}}(k)} \left\langle \delta\mathcal{N}^2(\boldsymbol{\Phi}_0 \to \boldsymbol{\Phi}_*)\right\rangle$$

Integration over the full inflating domain





Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

$$P\left(\zeta_R\right) = \int_\Omega \mathrm{d}\boldsymbol{\Phi}_* P_{\mathrm{bw}}\left[\boldsymbol{\Phi}_* \mid N_{\mathrm{bw}}(R)\right] P_{\mathrm{FPT},\boldsymbol{\Phi}_0\to\boldsymbol{\Phi}_*}\left[\zeta_R - \langle\mathcal{N}(\boldsymbol{\Phi}_*)\rangle + \langle\mathcal{N}(\boldsymbol{\Phi}_0)\rangle\right]$$

R

Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

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Extracting cosmological observables

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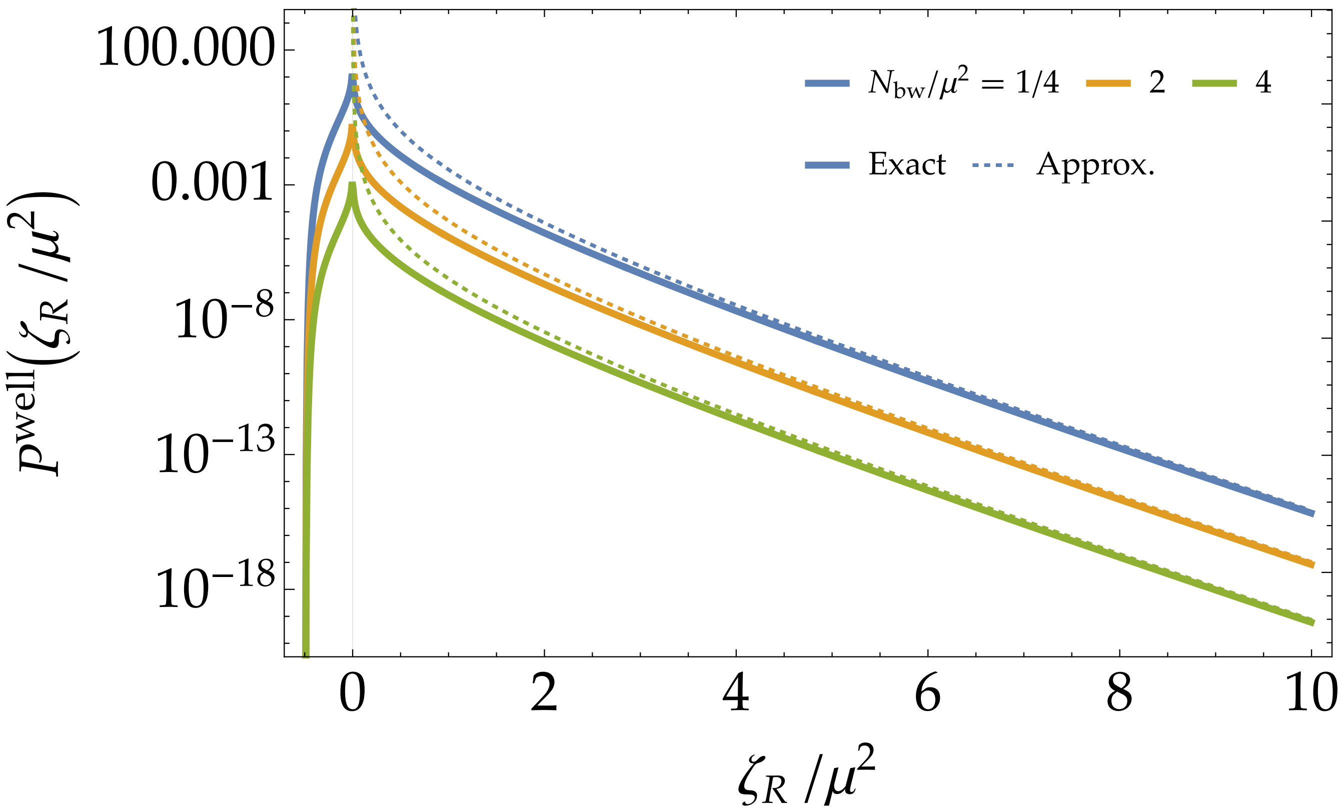
Ⓡ



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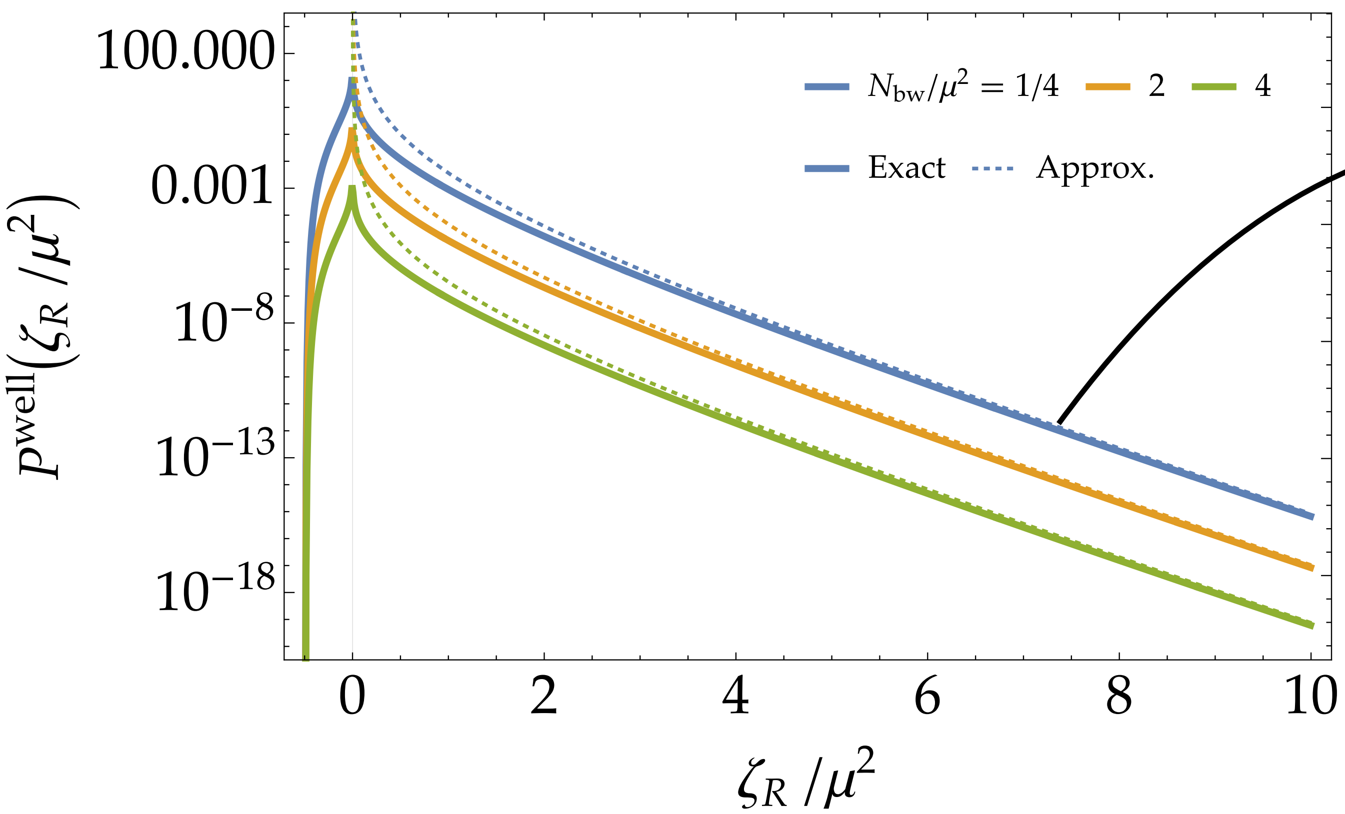
Ⓡ



$$P\left(\zeta_R\right)\propto\frac{e^{-\frac{\pi^2}{4}\frac{\zeta_R}{\mu^2}}}{\left(\zeta_R/\mu^2\right)^3}$$

Quasi-exponential tail

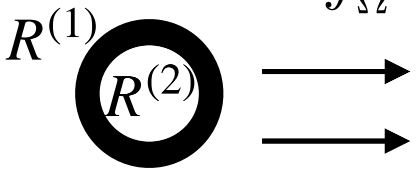
Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

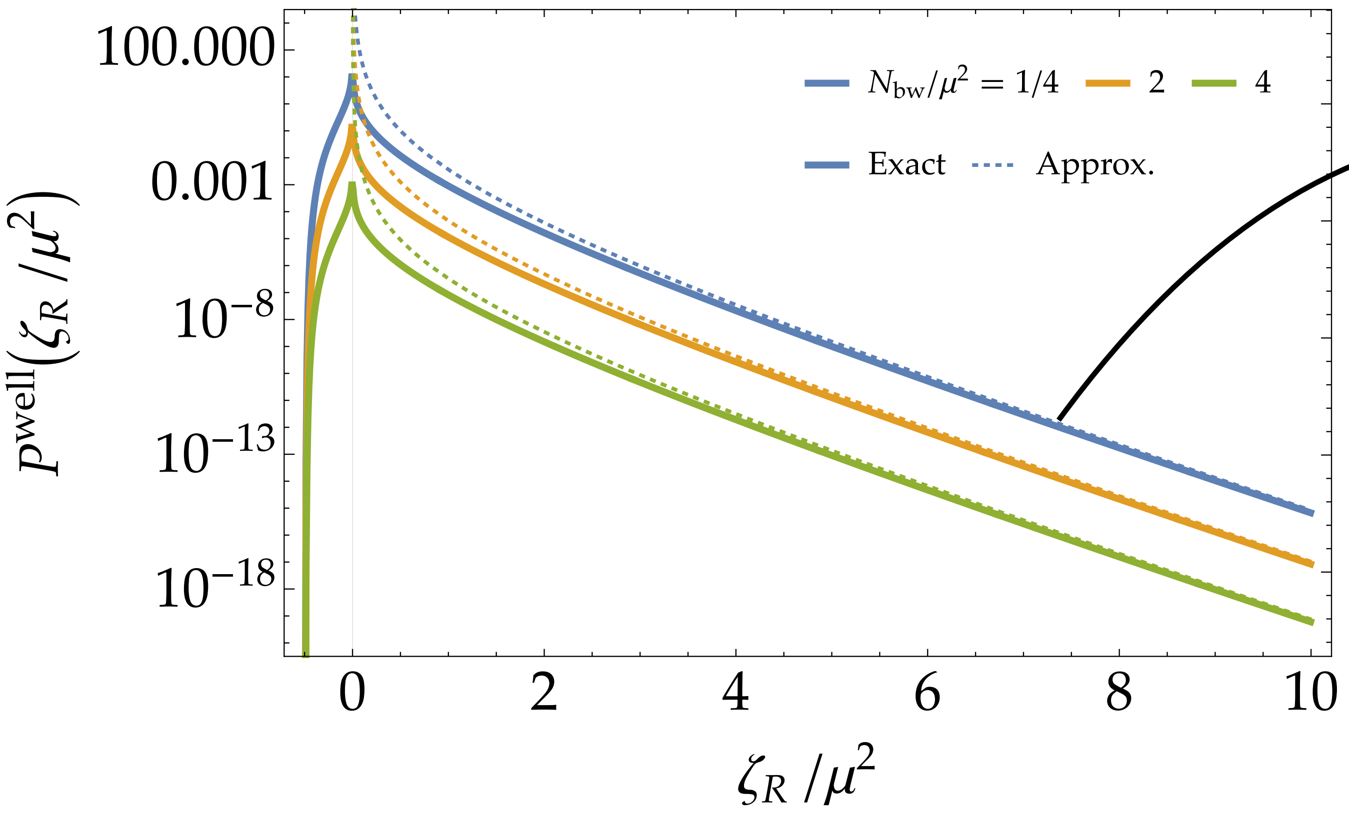
$$P\left(\zeta_R\right) = \boxed{\int_\Omega \mathrm{d}\boldsymbol{\Phi}_*} P_{\mathrm{bw}}\left[\boldsymbol{\Phi}_* \mid N_{\mathrm{bw}}(R)\right] P_{\mathrm{FPT},\boldsymbol{\Phi}_0\to\boldsymbol{\Phi}_*}\left[\zeta_R - \langle\mathcal{N}(\boldsymbol{\Phi}_*)\rangle + \langle\mathcal{N}(\boldsymbol{\Phi}_0)\rangle\right]$$

$$P(\Delta\zeta) = \int_\Omega \mathrm{d}\boldsymbol{\Phi}_*^{(1)} \mathrm{d}\boldsymbol{\Phi}_*^{(2)} P_{\mathrm{bw}}\left(\boldsymbol{\Phi}_*^{(1)}, \boldsymbol{\Phi}_*^{(2)} \middle| N_{\mathrm{bw}}^{(1)}, N_{\mathrm{bw}}^{(2)}\right) \delta\left[\Delta\zeta + \left\langle \mathcal{N}\left(\boldsymbol{\Phi}_*^{(1)}\right)\right\rangle - \left\langle \mathcal{N}\left(\boldsymbol{\Phi}_*^{(2)}\right)\right\rangle - \ln(1+\beta)\right]$$



Comoving density contrast

Compaction function



$$P\left(\zeta_R\right) \propto \frac{e^{-\frac{\pi^2}{4}\frac{\zeta_R}{\mu^2}}}{\left(\zeta_R/\mu^2\right)^3}$$

Quasi-exponential tail

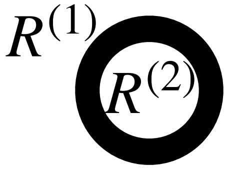
Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

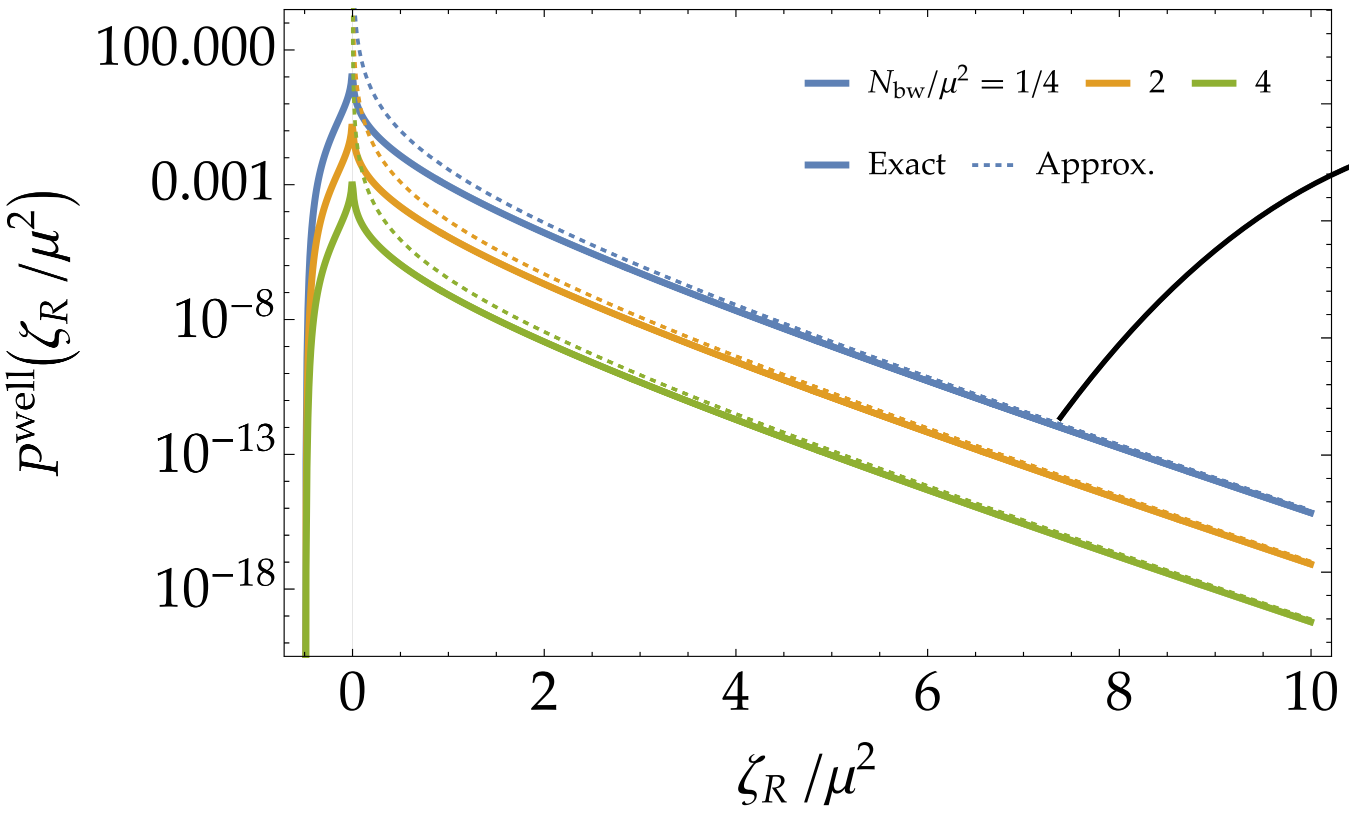
$$P\left(\zeta_R\right) = \boxed{\int_\Omega \mathrm{d}\mathbf{\Phi}_*} P_{\mathrm{bw}}\left[\mathbf{\Phi}_* \mid N_{\mathrm{bw}}(R)\right] P_{\mathrm{FPT},\mathbf{\Phi}_0\to\mathbf{\Phi}_*}\left[\zeta_R - \langle\mathcal{N}(\mathbf{\Phi}_*)\rangle + \langle\mathcal{N}(\mathbf{\Phi}_0)\rangle\right]$$

$$P(\Delta\zeta) = \boxed{\int_\Omega \mathrm{d}\mathbf{\Phi}_*^{(1)} \mathrm{d}\mathbf{\Phi}_*^{(2)}} P_{\mathrm{bw}}\left(\mathbf{\Phi}_*^{(1)}, \mathbf{\Phi}_*^{(2)} \middle| N_{\mathrm{bw}}^{(1)}, N_{\mathrm{bw}}^{(2)}\right) \delta\left[\Delta\zeta + \left\langle\mathcal{N}\left(\mathbf{\Phi}_*^{(1)}\right)\right\rangle - \left\langle\mathcal{N}\left(\mathbf{\Phi}_*^{(2)}\right)\right\rangle - \ln\left(1+\beta\right)\right]$$



→ Comoving density contrast

→ Compaction function



$$P\left(\zeta_R\right) \propto \frac{e^{-\frac{\pi^2}{4}\frac{\zeta_R}{\mu^2}}}{\left(\zeta_R/\mu^2\right)^3}$$

Quasi-exponential tail

Extracting cosmological observables

One-point function at arbitrary scale

Yuichiro Tada, VV (2021)

$$P\left(\zeta_R\right) = \boxed{\int_\Omega \mathrm{d}\boldsymbol{\Phi}_*} P_{\mathrm{bw}}\left[\boldsymbol{\Phi}_* \mid N_{\mathrm{bw}}(R)\right] P_{\mathrm{FPT},\boldsymbol{\Phi}_0\to\boldsymbol{\Phi}_*}\left[\zeta_R - \langle\mathcal{N}(\boldsymbol{\Phi}_*)\rangle + \langle\mathcal{N}(\boldsymbol{\Phi}_0)\rangle\right]$$

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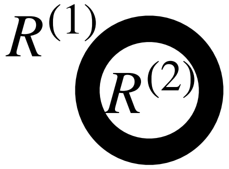
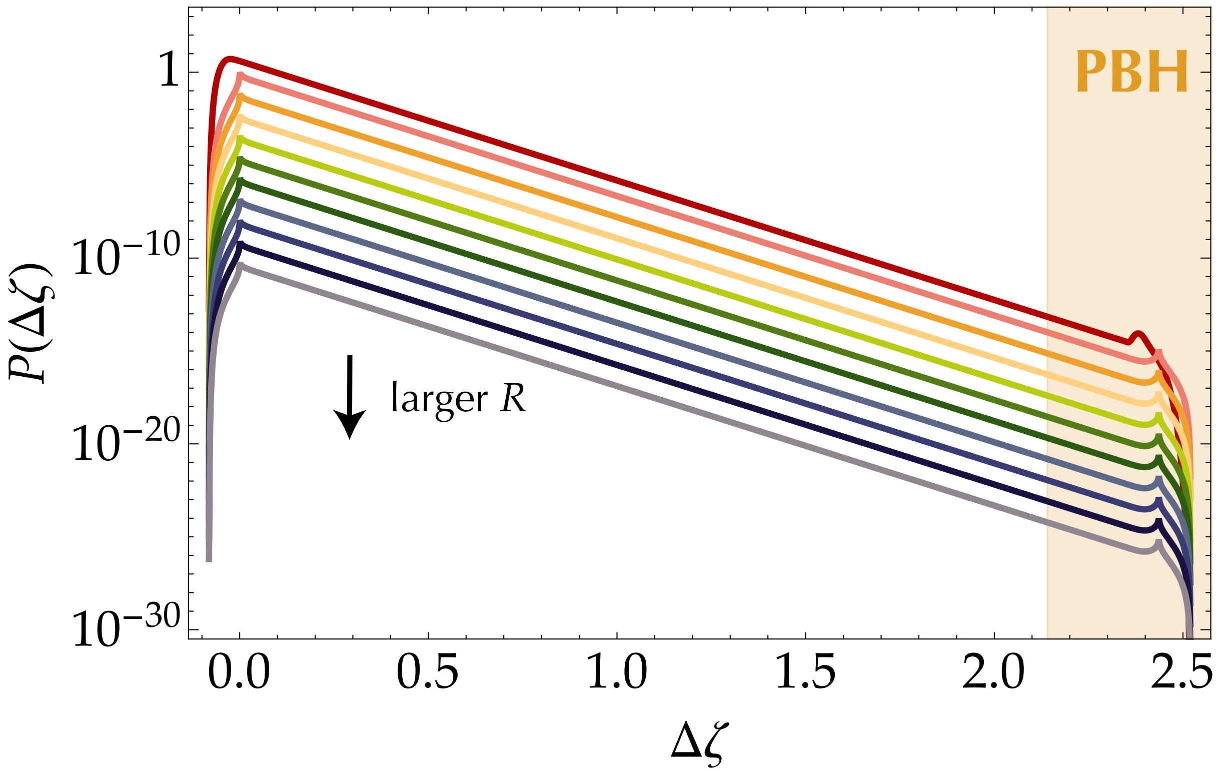
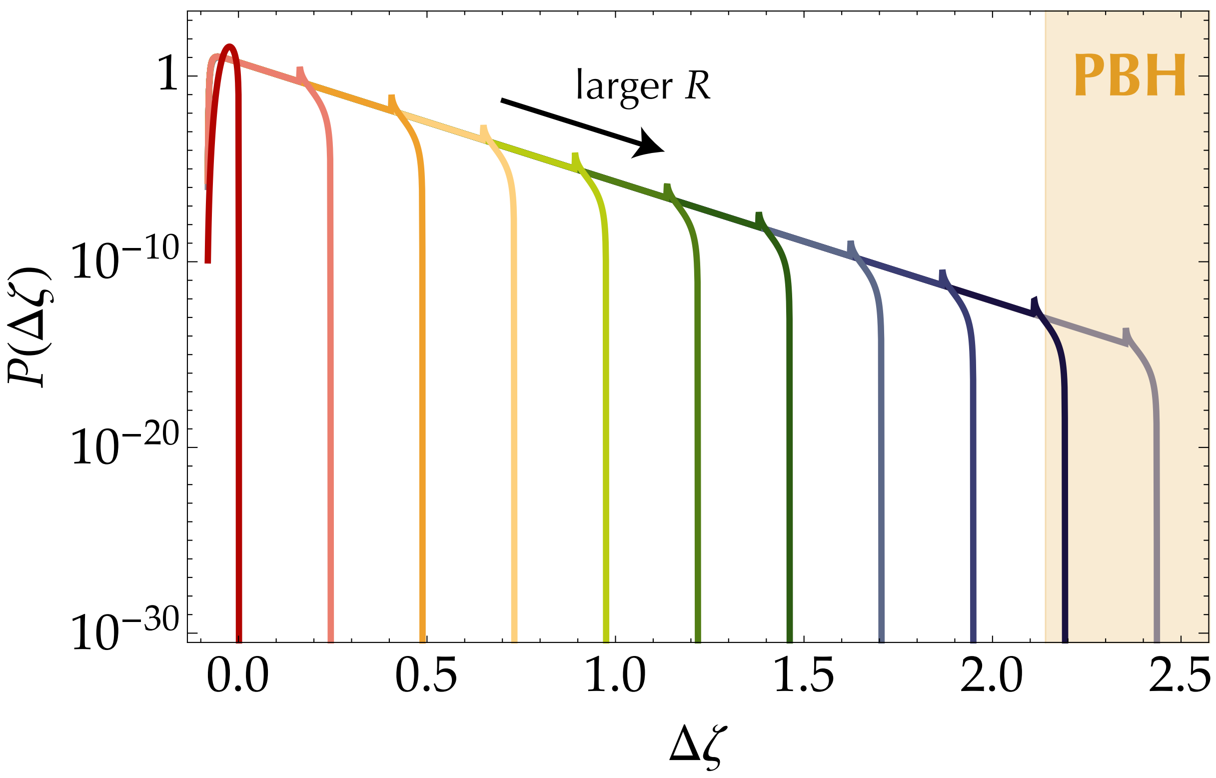


$\mu = \frac{1}{\sqrt{6}}, N_{\mathrm{bw}}^{(2)} > 0$



R_2 exits within the quantum well

$\mu = \frac{1}{\sqrt{6}}, N_{\mathrm{bw}}^{(2)} < 0$



R_2 exits below the quantum well

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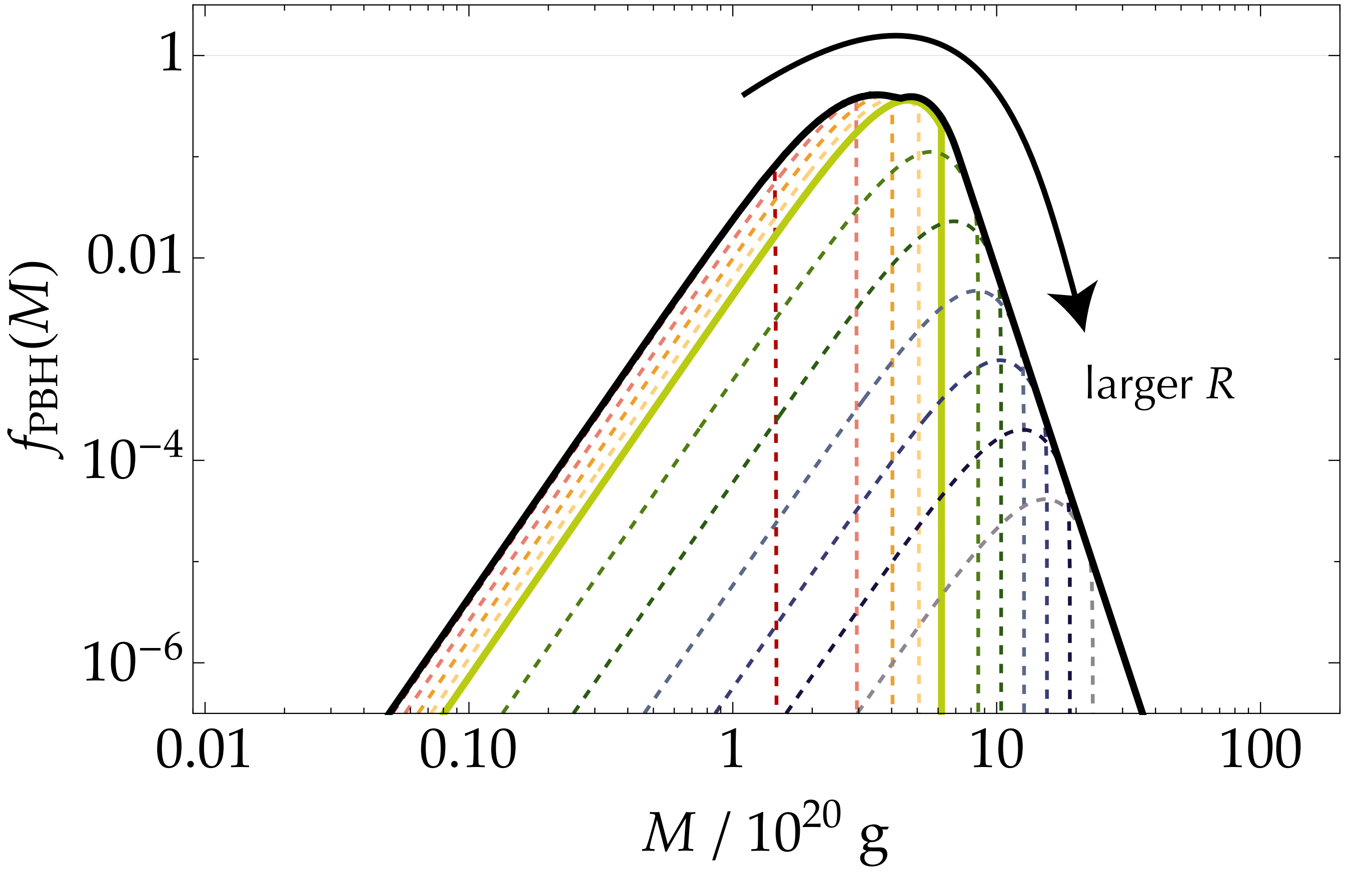


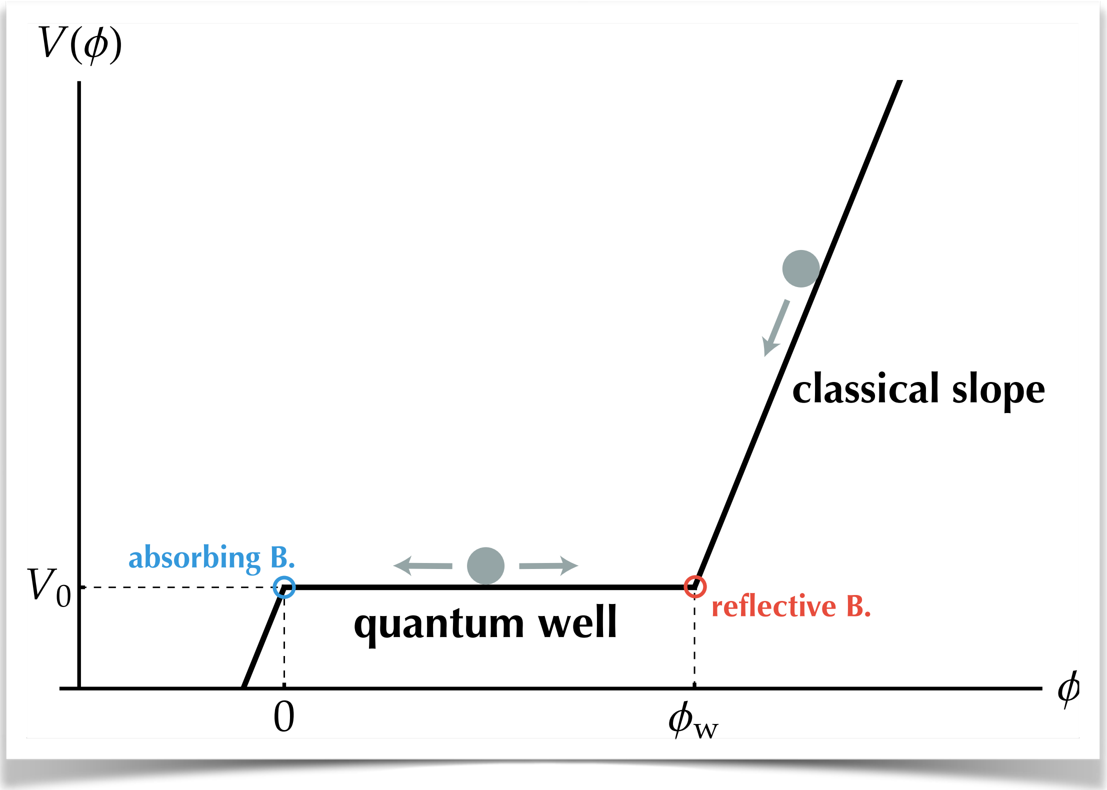
Conclusions

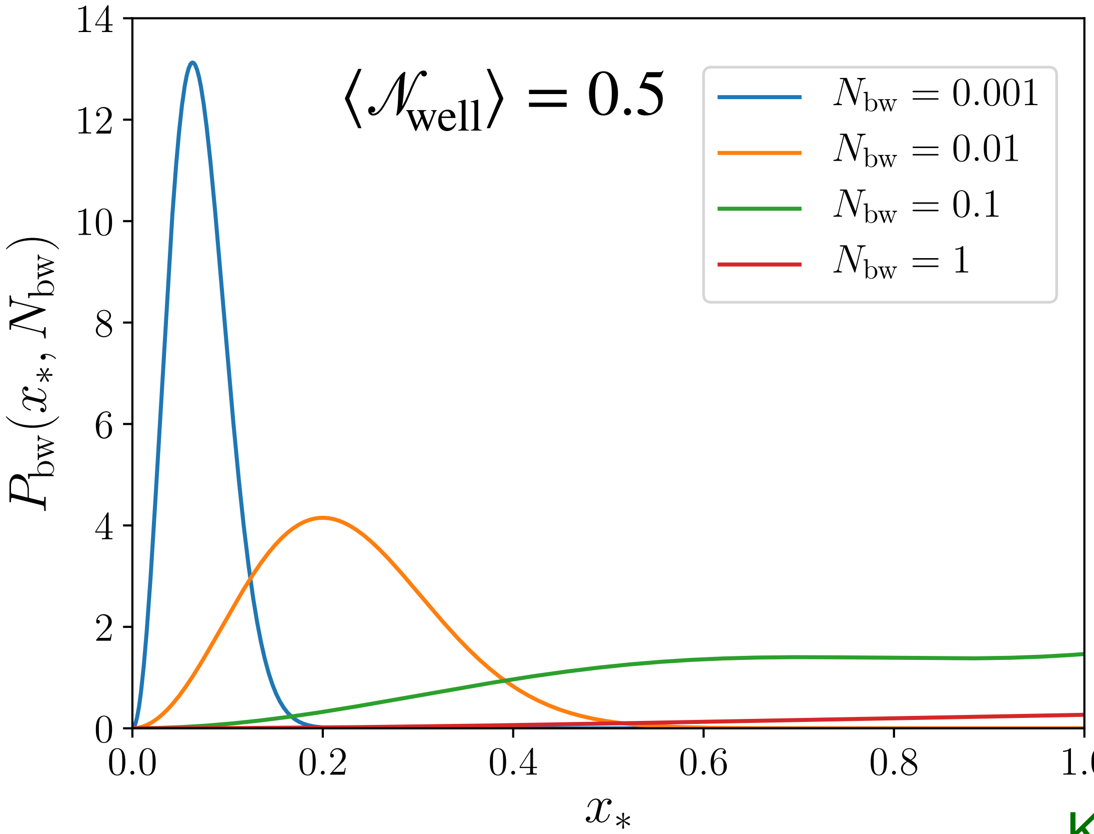
- The back-reaction of vacuum quantum fluctuations on the background dynamics can be incorporated within the formalism of stochastic inflation
- This is necessary to describe regimes leading to large fluctuations, such as those yielding primordial black holes
- Quantum diffusion leads to exponential tails: non-perturbative breakdown of Gaussian statistics
- Most cosmological observables can be reconstructed from first-passage time analysis (power spectrum, mass functions, n-point functions?)
- Quantum diffusion makes the CMB probe the whole potential: models leading to PBHs are constrained by the CMB, even if those two sets of scales are well separated

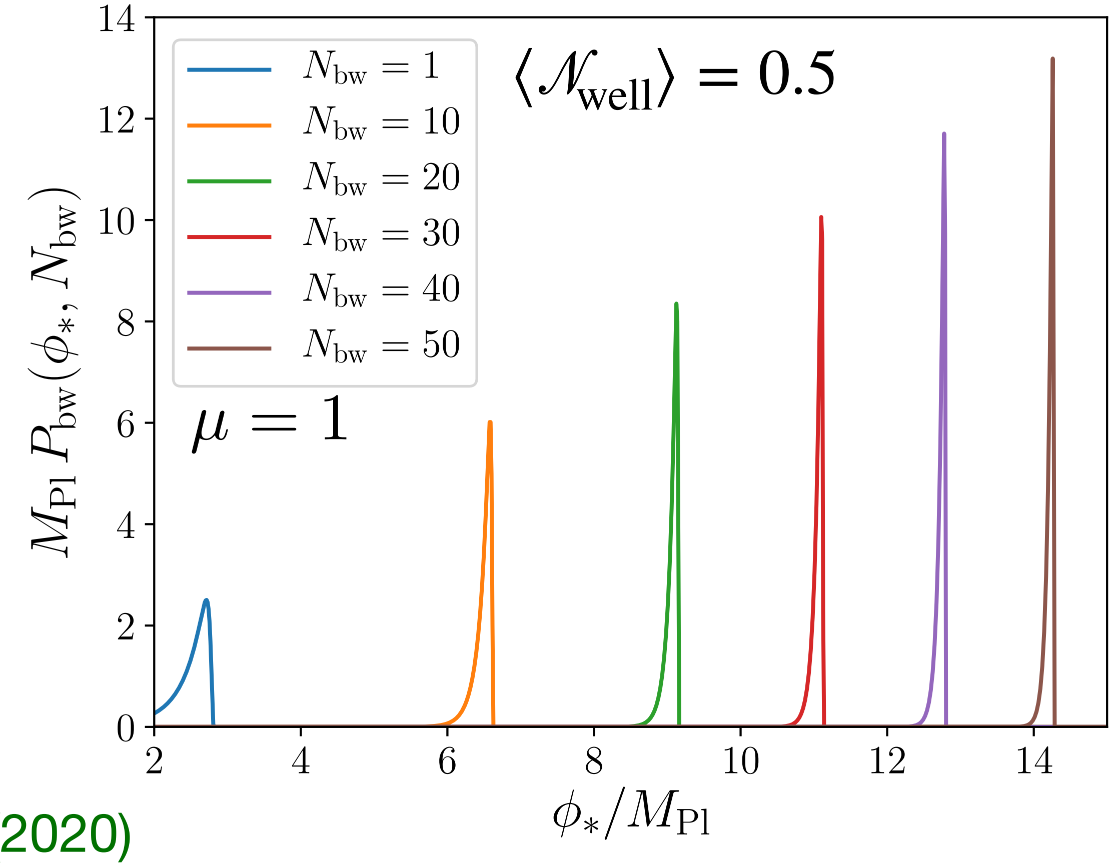
Thank you for your attention

Back-up slides

CMB probes the full potential







Kenta Ando, VV (2020)

Stochastic-δ*N* formalism

Moments obey an interactive equation VV, Starobinsky (2015) $v = V/(24\pi^2 M_{\mathrm{Pl}}^4)$

$$\langle \mathcal{N}^n \rangle''(\phi) - \frac{v'}{v^2} \langle \mathcal{N}^n \rangle'(\phi) = -\frac{n}{v M_{\mathrm{Pl}}^2} \left\langle \mathcal{N}^{n-1} \right\rangle(\phi)$$

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Mean number of e-folds $\langle \mathcal{N} \rangle(\phi) = \int_{\phi_{\rm end}}^{\phi} \frac{{\rm d}x}{M_{\rm Pl}} \int_x^{\phi_{\rm UV}} \frac{{\rm d}y}{M_{\rm Pl}} \frac{e^{\frac{1}{v(y)} - \frac{1}{v(x)}}}{v(y)}$

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Involves the full inflationary domain

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Saddle-point expansion

$v \ll 1, |v^2 v''/v'^2| \ll 1$

$$\simeq \int_{\phi_{\rm end}}^{\phi}\frac{\mathrm{d}x}{M_{\rm Pl}^2}\frac{v(x)}{v'(x)}\left[1+v(x)-\frac{v''(x)v^2(x)}{v'^2(x)}+\cdots\right]$$

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classical result

first-order correction

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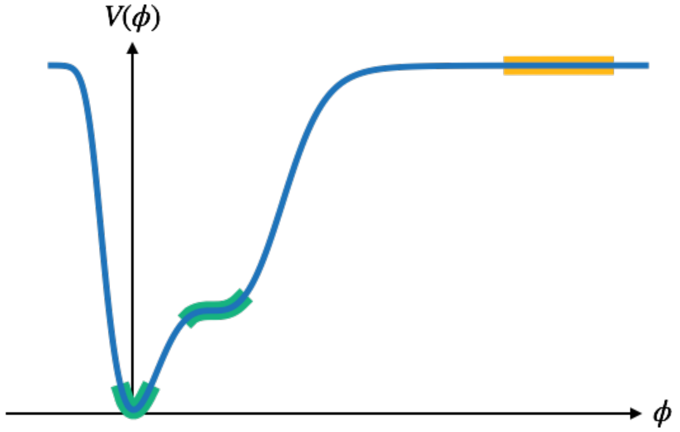

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Stochastic-δN formalism

Stochastic-δ*N* formalism

Second moment and power spectrum VV, Starobinsky (2015)

$$\mathscr{P}_\zeta\left(\phi\right) = 2\frac{\int_\phi^{\phi_\mathrm{UV}} \frac{\mathrm{d}x}{M_\mathrm{Pl}} \left\{ \int_x^{\phi_\mathrm{UV}} \frac{\mathrm{d}y}{M_\mathrm{Pl}} \frac{1}{v\left(y\right)} \exp\left[\frac{1}{v\left(y\right)} - \frac{1}{v(x)}\right] \right\}^2 \exp\left[\frac{1}{v(x)} - \frac{1}{v\left(\phi\right)}\right]}{\int_\phi^{\phi_\mathrm{UV}} \frac{\mathrm{d}x}{M_\mathrm{Pl}} \frac{1}{v(x)} \exp\left[\frac{1}{v(x)} - \frac{1}{v\left(\phi\right)}\right]}$$

Stochastic-δ*N* formalism

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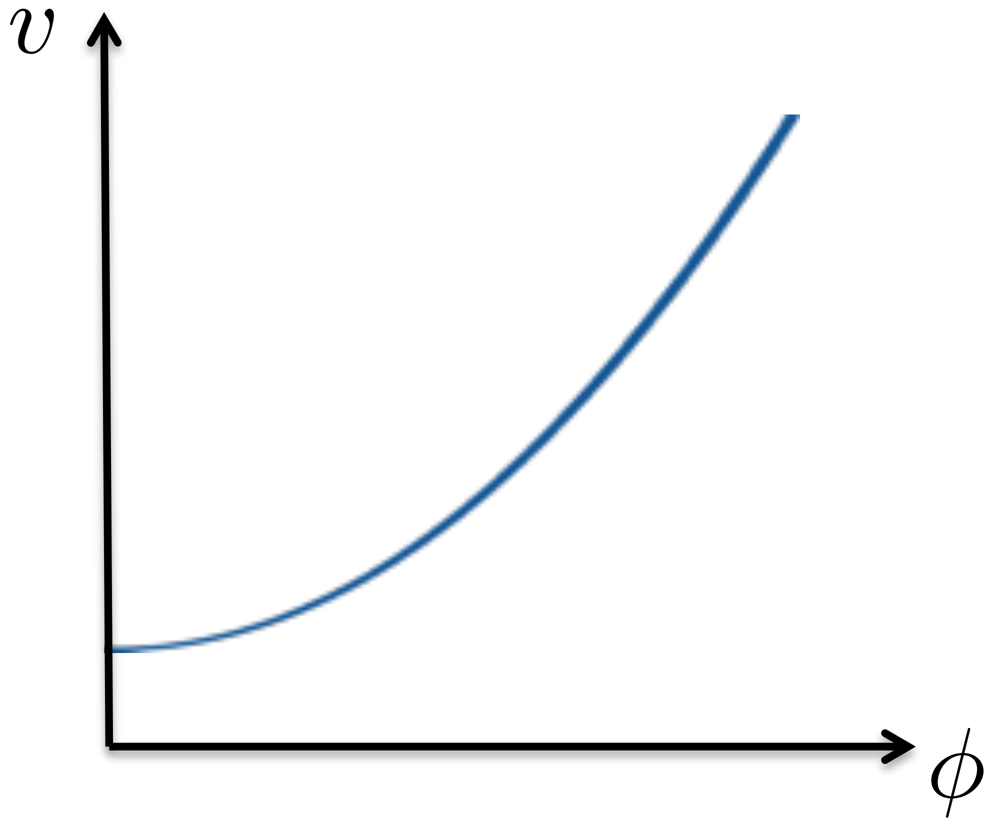
Third moment and local non-Gaussianity

$$f_{\mathrm{NL}} = \frac{5}{24}M_{\mathrm{Pl}}^2\left[6\frac{v'^2}{v^2}-4\frac{v''}{v}+v\left(11\frac{v'^2}{v^2}-158\frac{v''}{v}-9\frac{v'''}{v'}+118\frac{v''^2}{v'^2}\right)+\cdots\right]$$

Example

$$v(\phi) = v_0\left[1 + \left(\frac{\phi}{\phi_0}\right)^p\right]$$

Pattison, VV, Assadullahi, Wands (2017)



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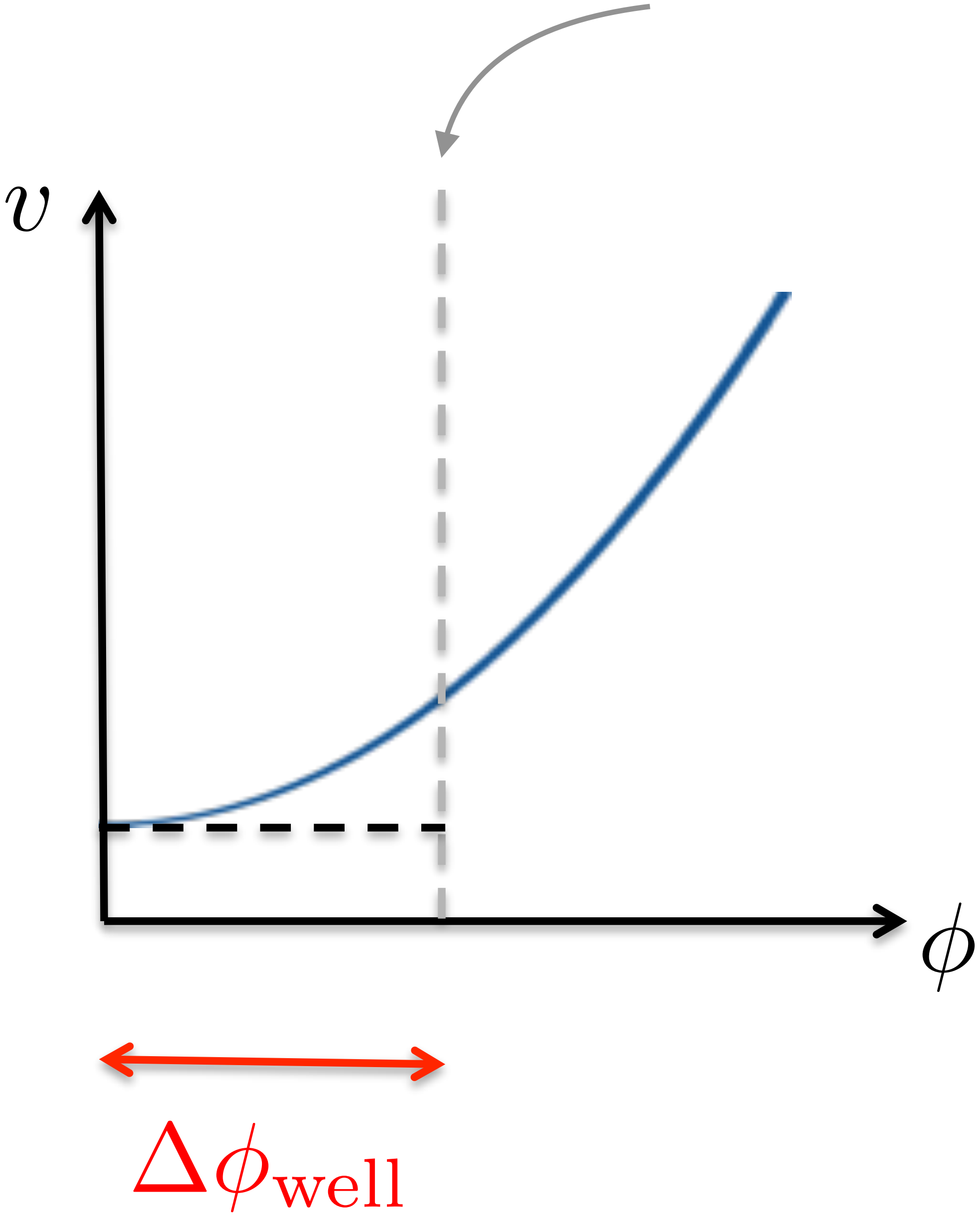
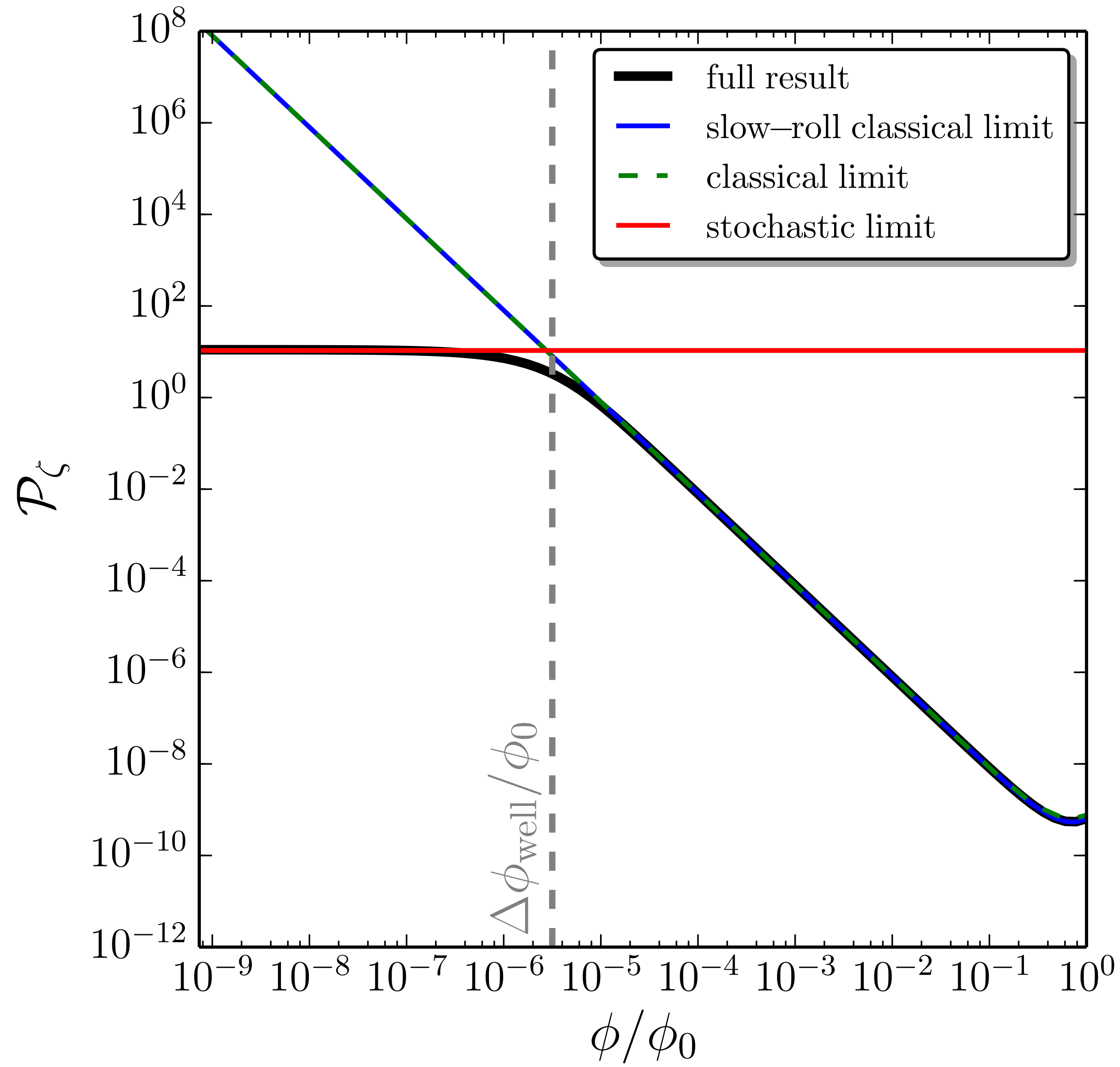
classical criterion $v^2 v'' \sim v'^2$





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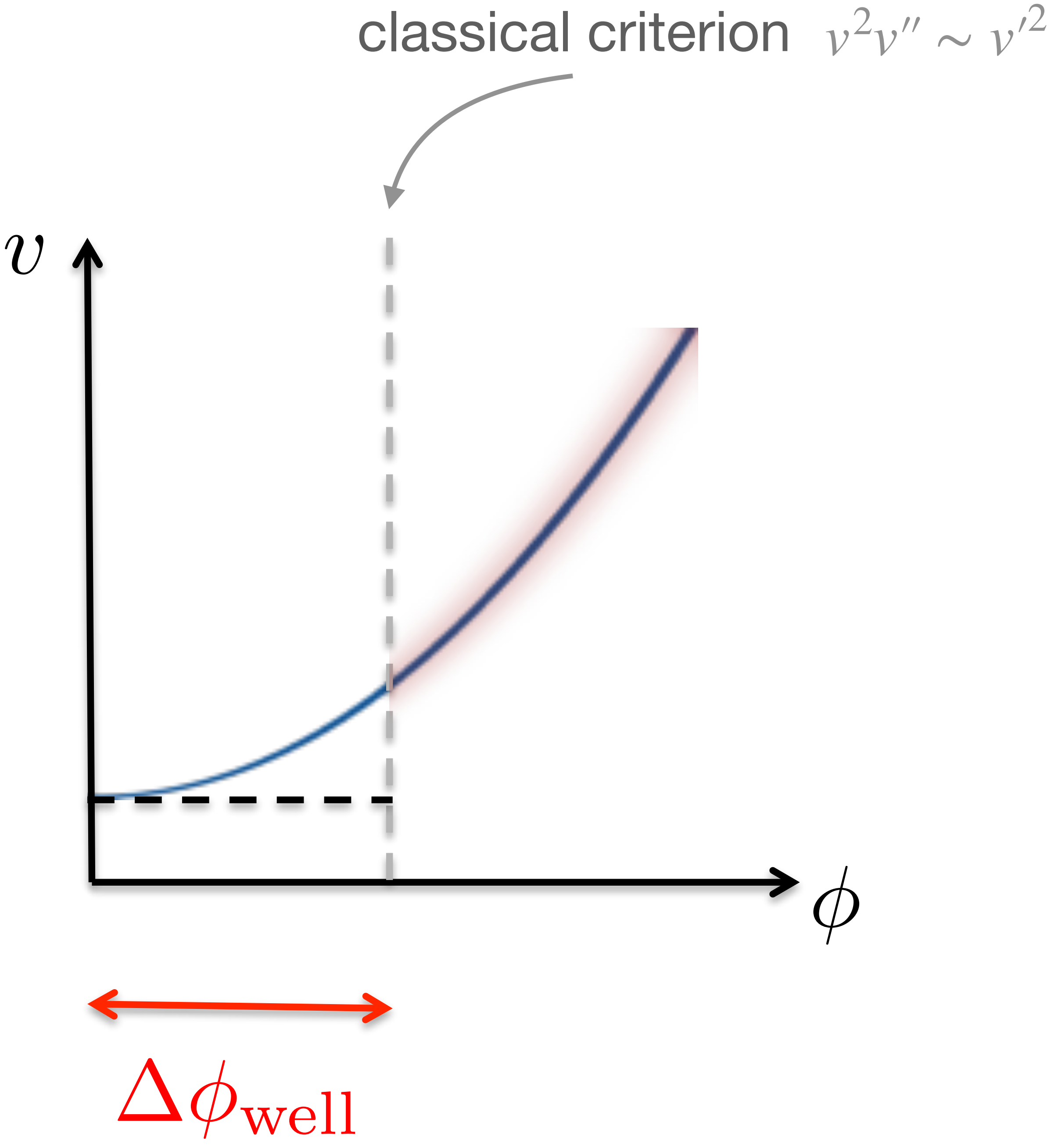

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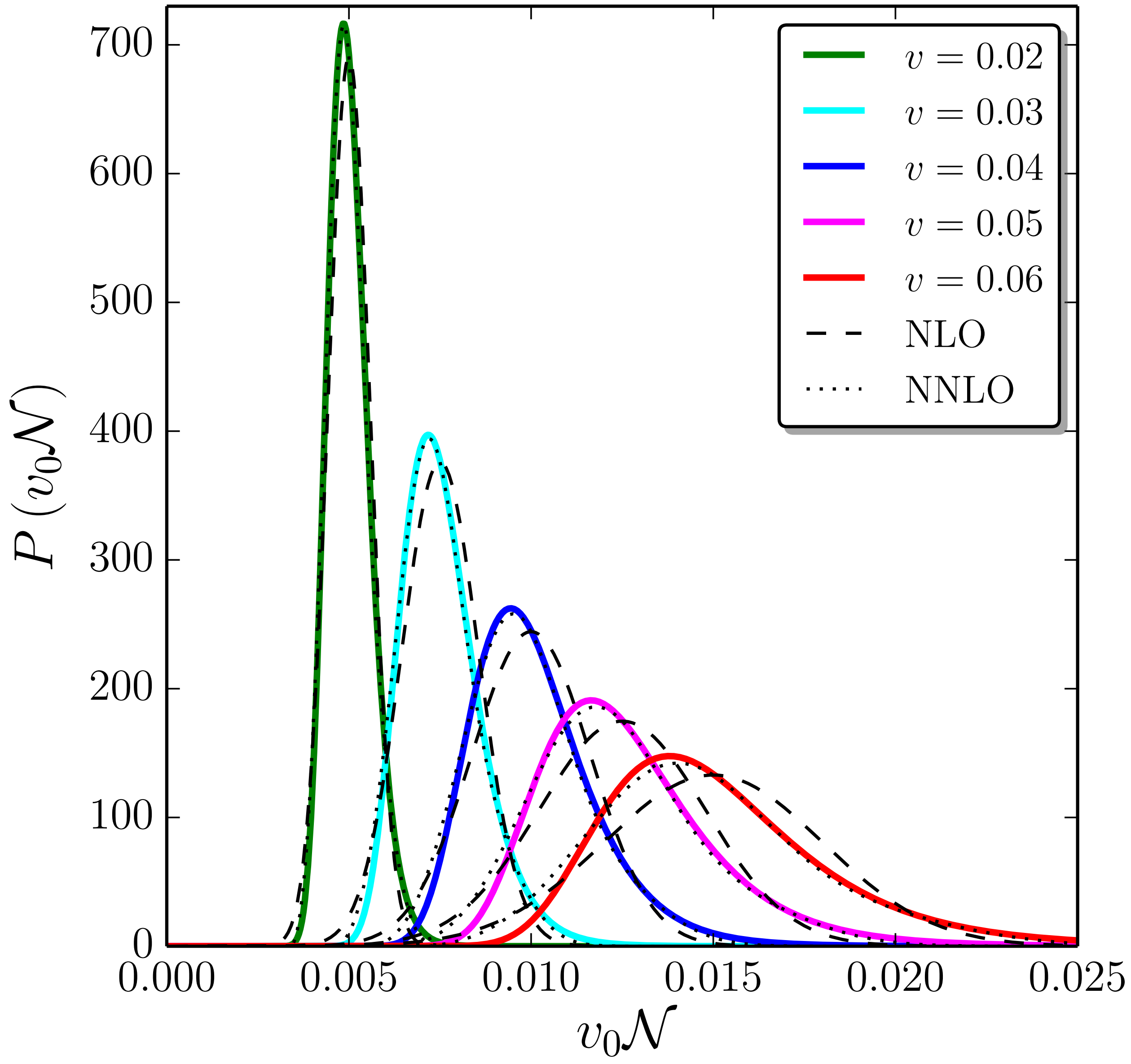
“classical” regime

Is the Gaussian approximation sufficient?



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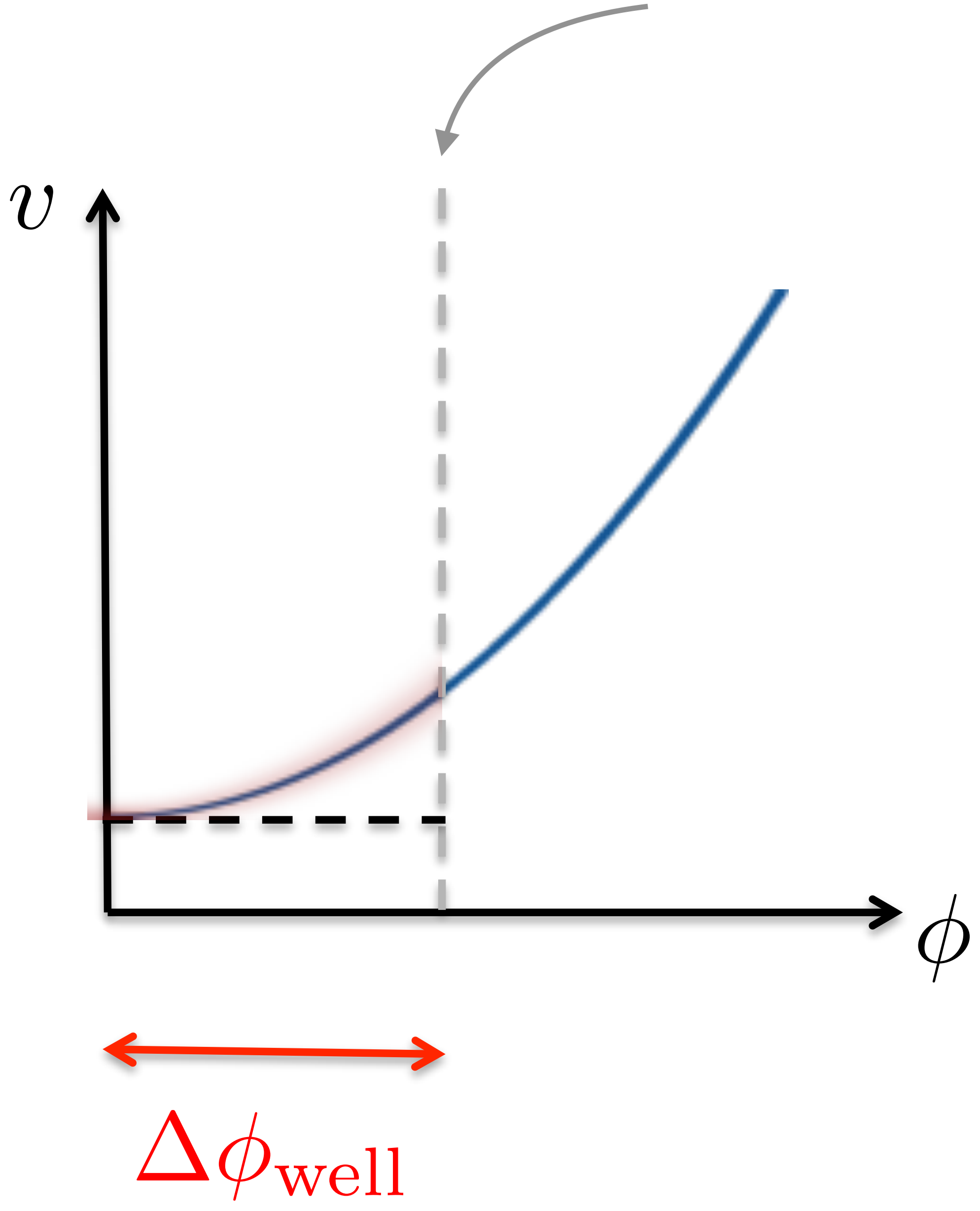
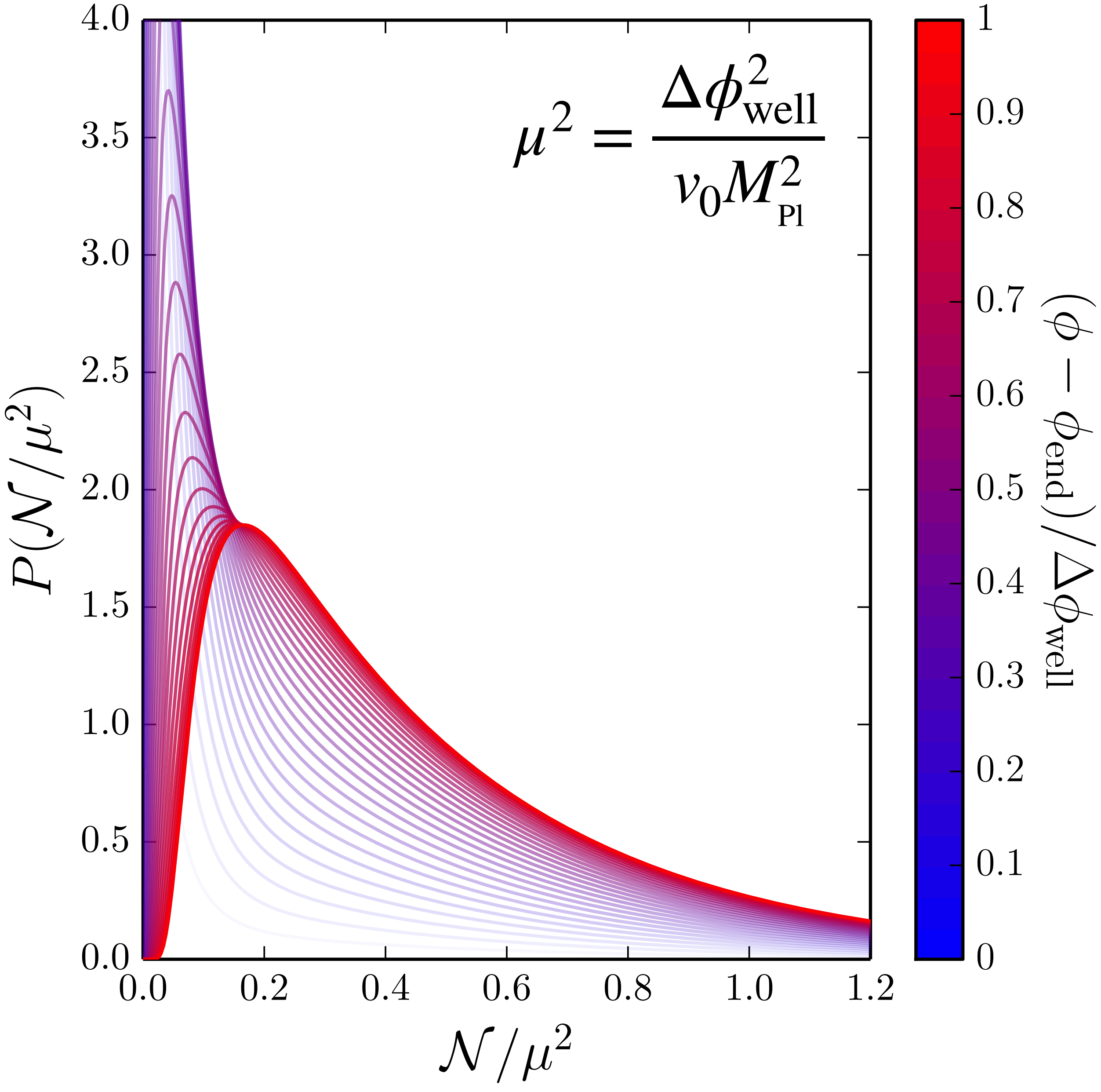


“stochastic” regime



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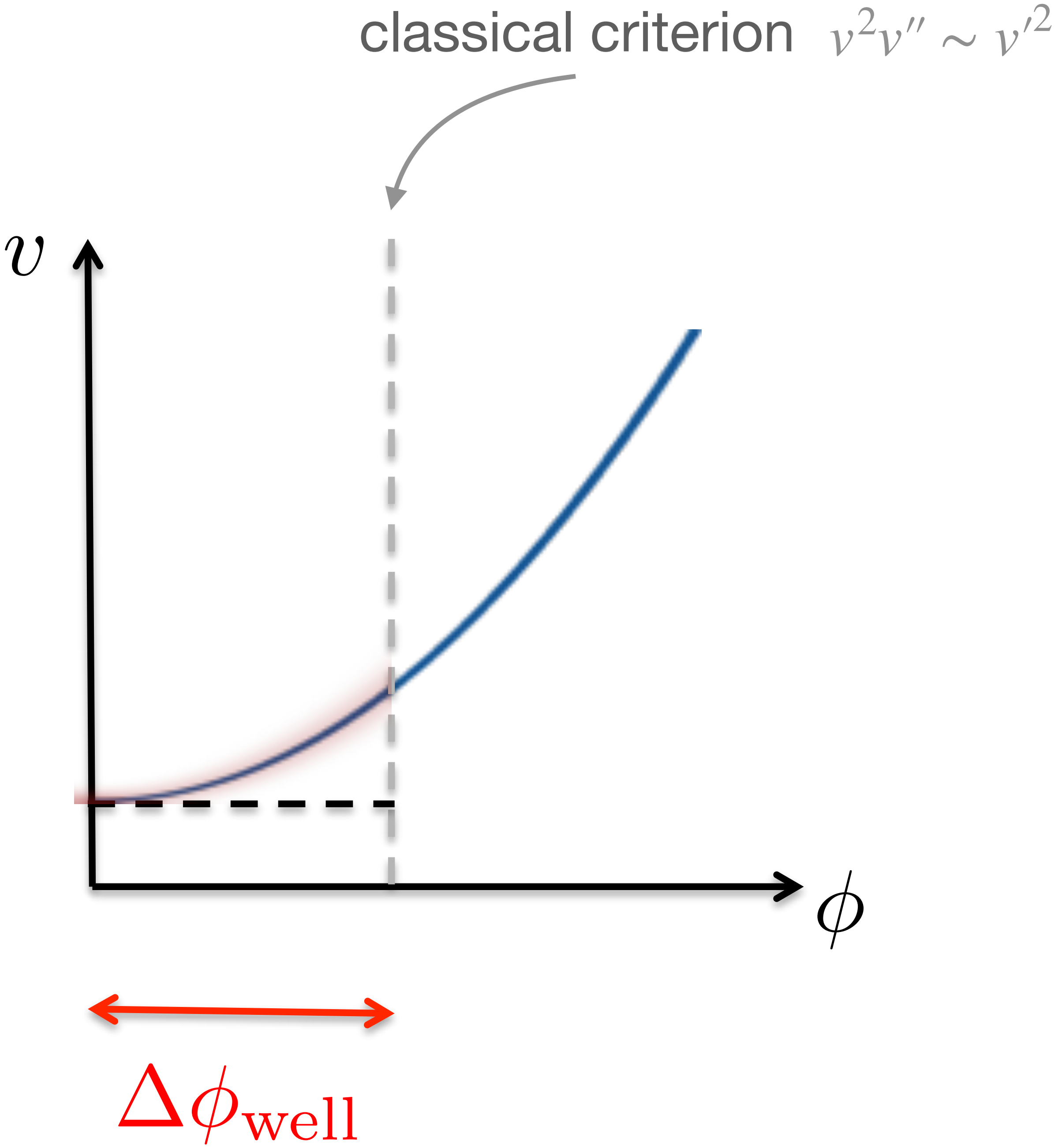
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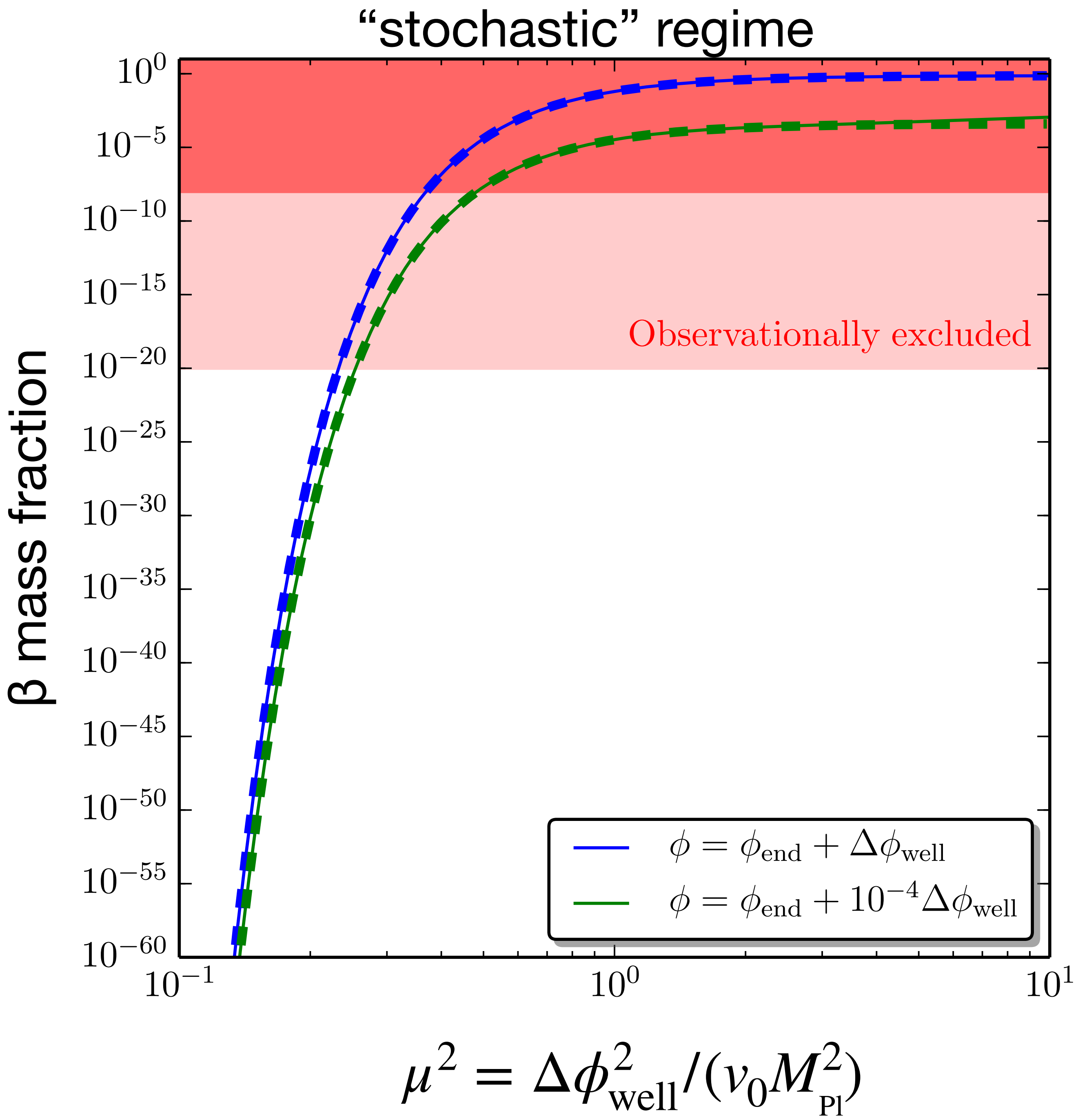
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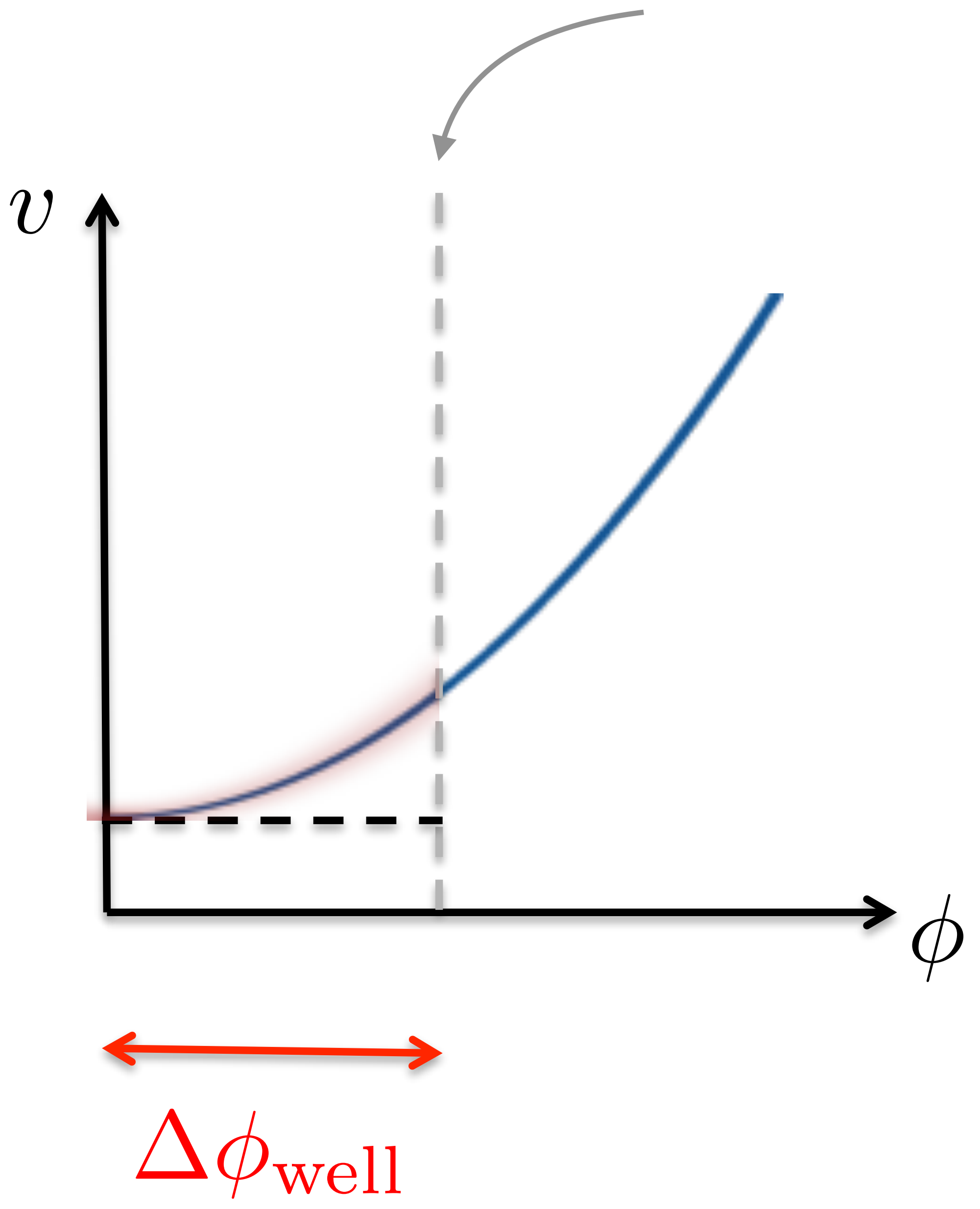
Remark: $\langle \mathcal{N} \rangle = \mu^2 \frac{\phi}{\Delta\phi}\left(1 - \frac{\phi}{2\Delta\phi}\right)$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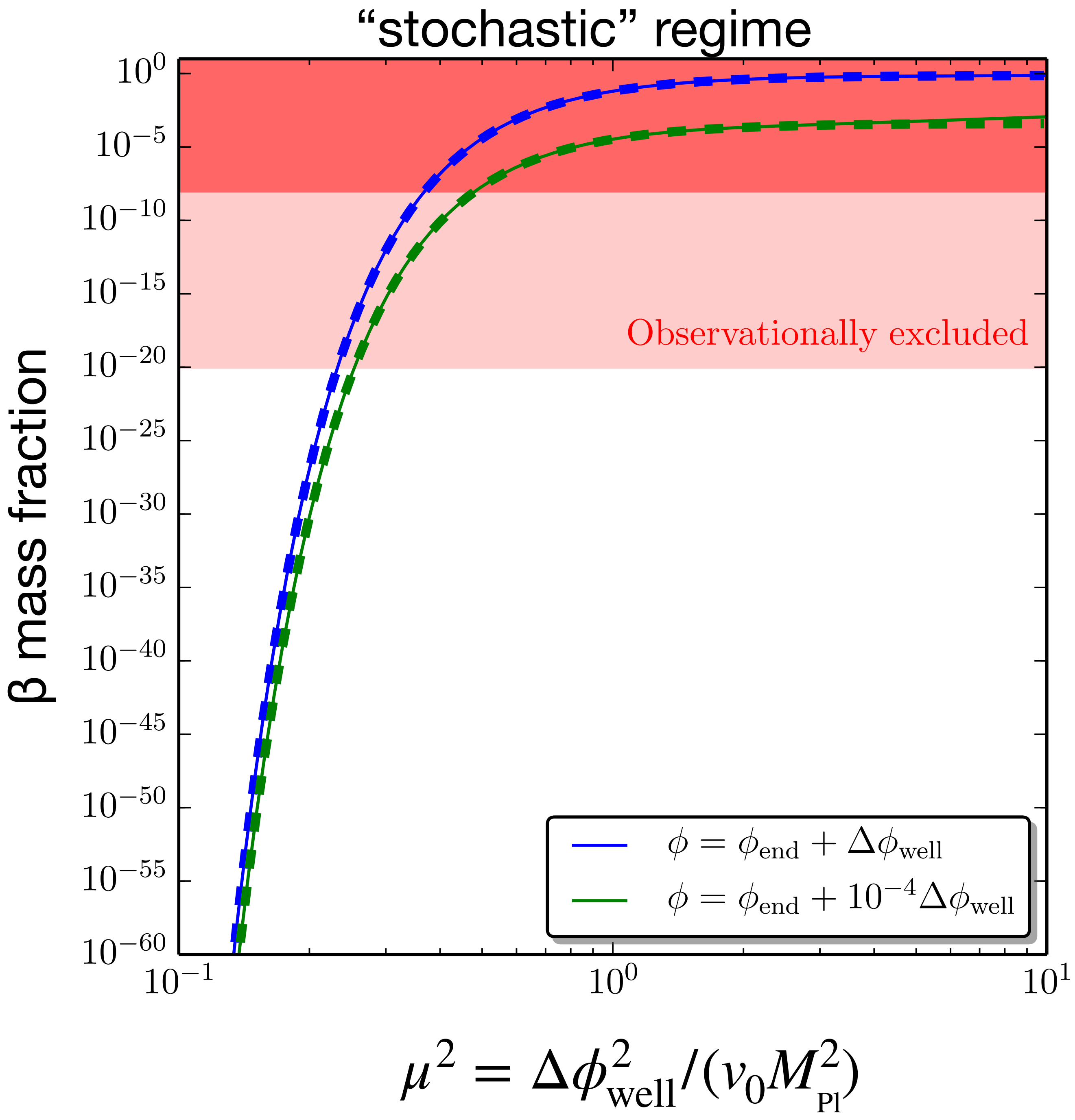
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