Thermal regularization of t-channel singularities in cosmology

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based on

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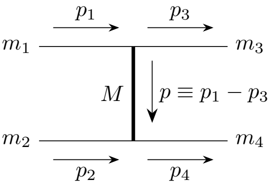
t-channel singularities in cosmology and particle physics

arXiv:2108.01757, currently under review

Planck 2022

Paris, 1 June 2022

- definition:



$$\mathcal{M} \sim \frac{1}{t - M^2}\,, \qquad t \equiv p^2$$

$t = M^2 \quad \Rightarrow \quad$ singular matrix element

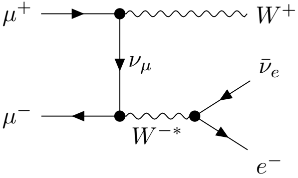
$\qquad\quad\;\; \Rightarrow \quad$ infinite cross section

⋆ IR regularization not applicable if $M > 0$

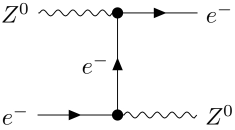
⋆ BW resummation not applicable if $\Gamma = 0$

- SM and BSM examples:

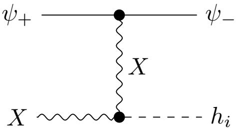
muon colliders



weak Compton scattering

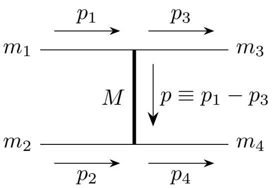


dark matter
in early Universe



(model: A. Ahmed et al., Eur.Phys.J.C 78 (2018) 11, 905)

$2 \leftrightarrow 2$ process: when does the t-channel singularity occur?



- Mandelstam variables:

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t \equiv p^2 = (p_1 - p_3)^2$$

- matrix element:

$$\mathcal{M} \sim \frac{1}{t - M^2}$$

- cross section

$$\sigma(s) \leftarrow \int_{t_{\min}(s)}^{t_{\max}(s)} \frac{dt}{(t - M^2)^2}$$

- thermally av. cross section

$$\langle \sigma v \rangle(T) \leftarrow \int \sigma(s)\, f(E_1, E_2, T)\, ds$$

- singularity condition:

$$t_{\min}(s) < M^2 < t_{\max}(s) \qquad \Rightarrow \qquad \text{singularity}$$

$$t_{\min} = m_1^2 + m_3^2 - 2E_1E_3 - 2|\vec{p}_1||\vec{p}_3| \qquad t_{\max} = m_1^2 + m_3^2 - 2E_1E_3 + 2|\vec{p}_1||\vec{p}_3|$$

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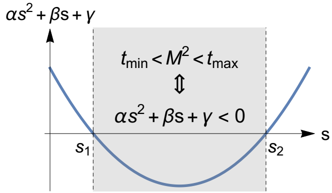
- in terms of the CMS energy ($\sqrt{s}$):

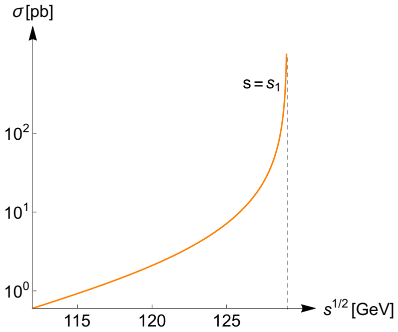
$$t_{\rm min}(s) < M^2 < t_{\rm max}(s)$$
$$\Leftrightarrow \quad s_1 < s < s_2$$

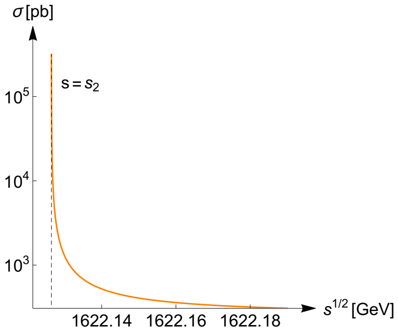


$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

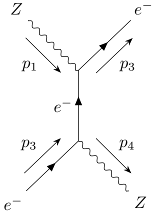
α, β, γ – functions of m_1, m_2, m_3, m_4 and M

example: weak Compton scattering





$$Ze^- \to Ze^- =$$



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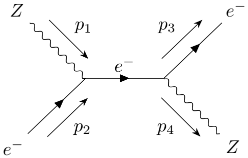

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$$t_{\min}(s) < M^2 < t_{\max}(s) \quad \Rightarrow \quad \text{singularity}$$

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- in terms of the CMS energy ($\sqrt{s}$):

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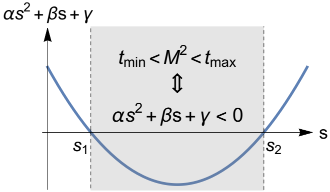


$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$

α, β, γ – functions of m_1, m_2, m_3, m_4 and M

- thermally averaged cross section $\leftarrow$ integration over $\sqrt{s} \in [s_{\min}, \infty)$
(weighted by thermal distribution functions)

- conclusion for the cosmological case:

if $s_2 > s_{\min} \equiv \max\{(m_1+m_2)^2, (m_3+m_4)^2\}$,
singularity in the allowed range $\quad \Rightarrow \quad \langle \sigma v \rangle = \infty$

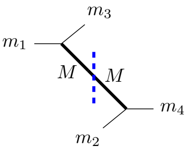
$$s_2 \equiv \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

α, β, γ – functions of m_1, m_2, m_3, m_4 and M

if $\underline{s_2 > s_{\min} \equiv \max\{(m_1+m_2)^2, (m_3+m_4)^2\}}$, singularity in the allowed range

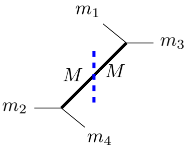
$\Updownarrow$

$m_1 > M + m_3$ and $m_4 > M + m_2$



or

$m_2 > M + m_4$ and $m_3 > M + m_1$



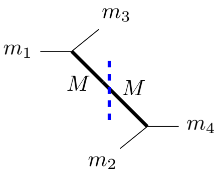
◇ Coleman-Norton theorem

S. Coleman & R. E. Norton, Nuovo Cim 38, 438–442 (1965)

"It is shown that a Feynman amplitude has singularities on the physical boundary if and only if the relevant Feynman diagram can be interpreted as a picture of an energy- and momentum-conserving process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time"

Known approaches to the problem
$\longrightarrow$ complex mass of unstable particles

I. Ginzburg, Nucl.Phys.B Proc.Suppl. 51 (1996) 85-89



idea: finite lifetime should affect the wavefunction

- at rest: $e^{im_1 t} \to e^{im_1 t} e^{-\Gamma_1 t}$

$$= e^{i\widetilde{m}_1 t}\,, \qquad \widetilde{m}_1 \equiv m_1 \left(1 + i\frac{\Gamma_1}{m_1}\right)$$

- after Lorentz boost: $p_1 \to \widetilde{p}_1 \equiv p_1 \left(1 + i\frac{\Gamma_1}{m_1}\right)$

$\to$ problem: $(\widetilde{p}_1 - \widetilde{p}_3)^2 \neq (\widetilde{p}_4 - \widetilde{p}_2)^2 \Rightarrow$ lack of symmetry

(momentum conservation...)

G. L. Kotkin et al., Yad. Fiz. 42 (1982) 692
G. L. Kotkin et al., Int. Journ. Mod. Phys. A 7 (1992) 4707
K. Melnikov & V. G. Serbo, Nucl.Phys. B483 (1997) 67
C. Dams & R. Kleiss, Eur.Phys.J.C29 (2003) 11
C. Dams & R. Kleiss, Eur.Phys.J. C36 (2004) 177

idea: at colliders, the beams have finite size

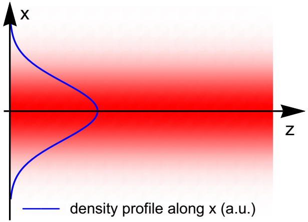
⇓

they should not be treated as plain waves

example:

Gaussian beam moving along z axis

$$n(x,y) \sim e^{-\frac{x^2+y^2}{2a^2}} \qquad a \text{ – beam width}$$



$$\int \frac{dt}{|t-M^2+i\epsilon|^2} \to \int \frac{a^3\, e^{-\frac{a^2\kappa^2}{2}}}{(2\pi)^{3/2}} \frac{d^3\kappa\, dt}{(t-M^2+i\epsilon-\vec{\kappa}\cdot\vec{q})(t-M^2-i\epsilon+\vec{\kappa}\cdot\vec{q})}$$

$$\sim \frac{\pi a}{|\vec{q}|}, \qquad \vec{q} \equiv \left[\frac{E_3}{E_1}\vec{p}_1 - \vec{p}_3\right]_{t=M^2}$$

→ problem: inapplicable in cosmological context

Idea

- early Universe = hot gas
- every particle interacts with a thermal medium
- the mean life time cannot be infinite $\Rightarrow$ effective width
- QFT in a thermal medium: Keldysh-Schwinger formalism

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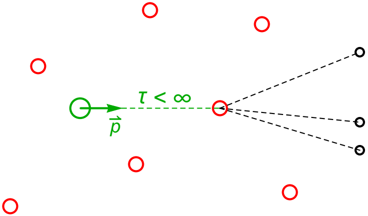
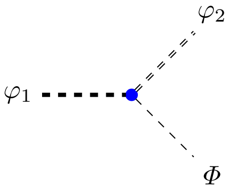


Toy model

- 3 real scalars: φ_1, φ_2, Φ
- Lagrangian:

$$\mathcal{L} = [\text{kinetic terms}] + \mu\, \varphi_1 \varphi_2 \Phi$$



- discrete symmetries:

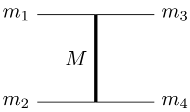
$$\mathbb{Z}_2 : \quad (\varphi_1\,,\ \varphi_2\,,\ \Phi) \longrightarrow (-\varphi_1\,,\ \varphi_2\,,\ -\Phi)\ ,$$
$$\mathbb{Z}_2' : \quad (\varphi_1\,,\ \varphi_2\,,\ \Phi) \longrightarrow (\varphi_1\,,\ -\varphi_2\,,\ -\Phi)\ ,$$

$\Rightarrow$ no power-3 terms except $\mu \varphi_1 \varphi_2 \Phi$

- power-4 terms (e.g. $\varphi_1^2\, \varphi_2^2$) dropped for simplicity

The singularity

- general case:



singular if

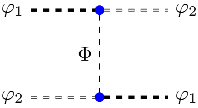
$$m_1 > m_3 + M \quad \text{and} \quad m_4 > m_2 + M$$

or

$$m_3 > m_1 + M \quad \text{and} \quad m_2 > m_4 + M$$

- toy model:

$$\mathcal{L} = [\text{kinetic terms}] + \mu\, \varphi_1 \varphi_2 \Phi$$



singular if

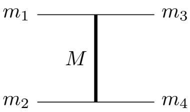
$$m_1 > m_2 + M$$

or

$$m_2 > m_1 + M$$

The singularity

- general case:



singular if

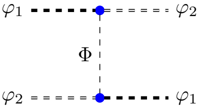
$m_1 > m_3 + M$ and $m_4 > m_2 + M$

or

$m_3 > m_1 + M$ and $m_2 > m_4 + M$

- toy model:

$$\mathcal{L} = [\text{kinetic terms}] + \mu\,\varphi_1\varphi_2\Phi$$



singular if

$\rightarrow \quad m_1 > m_2 + M$

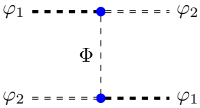
or

$m_2 > m_1 + M$

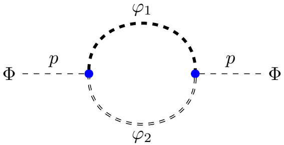
One-loop self-energy

$\mathcal{L} = [\text{kinetic terms}] + \mu\, \varphi_1 \varphi_2 \Phi \,,$ $\qquad m_1 > m_2 + M$

- singular process:



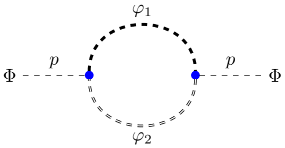
- one-loop contribution to mediator's self-energy:



$$i\Pi(x,y) = \mu^2 \; i\Delta_1(x,y)\; i\Delta_2(y,x),$$

- non-zero imaginary part of self-energy acquired as a result of thermal interactions with the medium of particles (Keldysh-Schwinger formalism)

Calculation of one-loop self-energy



$$\mathcal{L} = [\text{kinetic terms}] + \mu\,\varphi_1\varphi_2\Phi$$
$$m_1 > m_2 + M$$

- one-loop contribution to the self-energy:

$$i\Pi(x,y) = \mu^2\; i\Delta_1(x,y)\; i\Delta_2(y,x),$$

- non-zero imaginary part of self-energy acquired as a result of thermal interactions with the medium of particles (Keldysh-Schwinger formalism)

$$\Pi^+(p,T) = \frac{i}{2}\mu^2 \int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^+(k+p)\,\Delta_2^{\mathsf{sym}}(k,T) + \Delta_1^{\mathsf{sym}}(k,T)\,\Delta_2^-(k-p)\Big],$$

$$\Delta_l^{\mathsf{sym}}(k,T) \equiv -\frac{i\pi}{E_l}\Big(\delta(E_l-k_0)+\delta(E_l+k_0)\Big)\times\big[2f(E_l,T)+1\big]\,, \qquad l=1,2$$

$$\Delta_l^{\pm}(p) \equiv \frac{1}{p^2-m_l^2 \pm i\,\mathsf{sgn}(p_0)\,\varepsilon}\,, \quad E_l \equiv \sqrt{\vec{k}^{\,2}+m_l^2}\,, \quad f(E_l,T) = (e^{E_l/T}-1)^{-1}$$

- after tedious calculations:

$$\Sigma(|\vec{p}|,T) \equiv \Im\Pi^{+}(|\vec{p}|,T) = -\frac{\mu^2}{16\pi}\,\frac{1}{|\vec{p}|/T} \times \left[\ln\frac{e^{(A+C)/T}-1}{e^{A/T}-1} - \ln\frac{e^{(A+B+C)/T}-1}{e^{(A+B)/T}-1}\right]$$

$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - \sqrt{\lambda}\,|\vec{p}|}{2M^2}\,, \qquad B \equiv E_p\,, \qquad C \equiv \frac{\sqrt{\lambda}\,|\vec{p}|}{M^2}$$

$$E_p \equiv \sqrt{\vec{p}^{\,2} + M^2}$$

$$\lambda \equiv [m_1^2 - (m_2 + M)^2] \times [m_1^2 - (m_2 - M)^2]$$

effective width:

$$\Gamma_{\text{eff}}(|\vec{p}|,T) \equiv M^{-1}|\Sigma(|\vec{p}|,T)|$$

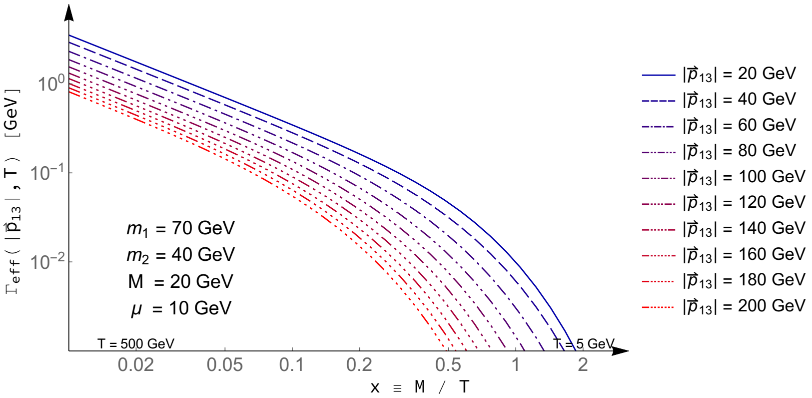
$$\Downarrow$$

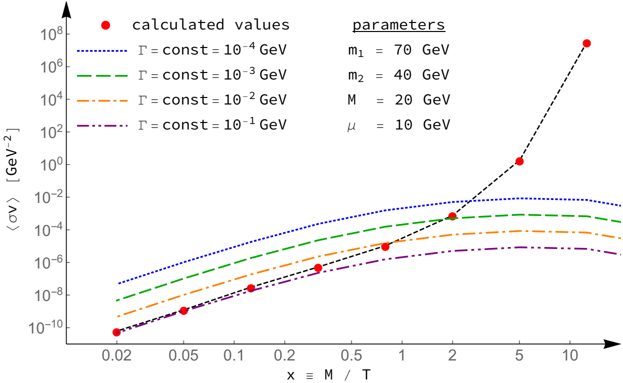
Breit-Wigner propagator:

$$\frac{1}{(t-M^2)^2} \to \frac{1}{(t-M^2)^2 + M^2\Gamma_{\text{eff}}(|\vec{p}|,T)^2}$$

Results: effective width

$$\Gamma_{\text{eff}}(|\vec{p}|, T) \approx \begin{cases} \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times \left[\ln\left(1 + \frac{C}{A}\right) - \ln\left(1 + \frac{C}{A+B}\right)\right] & \text{for high } T \\ \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times e^{-A/T}(1 - e^{-C/T})(1 - e^{-B/T}) & \text{for low } T \end{cases}$$



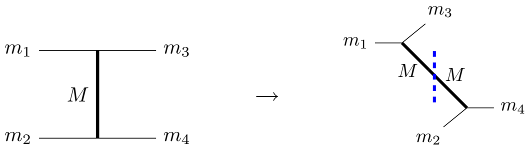


$$\langle\sigma v\rangle_{12\to 34}(T) = \mu^4 \int d\Pi_1\, d\Pi_2\, f(E_1, E_2, T) \int d\Pi_3\, d\Pi_4\, \frac{(2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)}{(t - M^2)^2 + M^2 \Gamma_{\text{eff}}(|\vec{p}|, T)^2}$$

$$d\Pi_k \equiv \frac{d^3 p_k}{(2\pi)^3\, 2E_k}$$ – phase-space element $(k = 1, 2, 3, 4)$

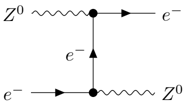
Summary

- t-channel singularity of $\langle \sigma v \rangle$ occurs if
  - the process can be seen as a sequence of decay and fusion



  - the mediator has no width (is stable)

- the singularity is present both in SM and BSM physics



- known approaches are either unsatisfactory or inapplicable
- interaction with the medium results in a non-zero effective width (obtained within the Keldysh-Schwinger formalism) that regulates the singularity
- the effective width depends on temperature and mediator's momentum (momentum transfer) and behaves in an expected, natural way

$$\Gamma_{\text{eff}} = \Gamma_{\text{eff}}(T, |\vec{p}|)$$

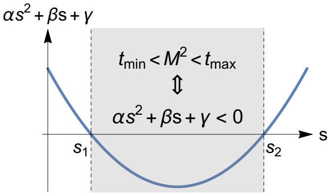
BACKUP SLIDES

Values of s_1, s_2 in terms of masses

- in terms of the CMS energy ($\sqrt{s}$):

$$t_{\min}(s) < M^2 < t_{\max}(s)$$
$$\Leftrightarrow \quad s_1 < s < s_2$$

$$s_{1,2} \equiv \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$



$$\alpha \equiv M^2$$
$$\beta \equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)$$
$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2)$$

Known approaches to the problem
→ Breit-Wigner propagator

$$\mathcal{M} \sim \frac{i}{p^2-M^2}$$

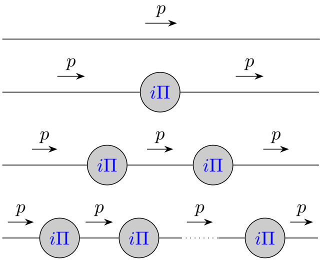
$$+ \frac{i}{p^2-M^2}\, i\Pi\, \frac{i}{p^2-M^2}$$

$$+ \frac{i}{p^2-M^2}\, i\Pi\, \frac{i}{p^2-M^2}\, i\Pi\, \frac{i}{p^2-M^2}$$

$$+ \dots$$

$$= \frac{i}{p^2-M^2} \sum_{k=0}^{\infty} \left(i\Pi \frac{i}{p^2-M^2} \right)^k$$

$$= \frac{i}{p^2-M^2+\Pi}$$



optical theorem $\Rightarrow \Im\Pi\big|_{p^2=M^2} = M\Gamma$

→ problem: $\Gamma = 0$ for a stable mediator

$$\Sigma(|\vec{p}|, T) \equiv \Im\Pi^{+}(|\vec{p}|, T)$$
$$= -\frac{\mu^2}{16\pi}\frac{1}{|\vec{p}|/T} \times \left[\ln\frac{e^{(A+C)/T}-1}{e^{A/T}-1} - \ln\frac{e^{(A+B+C)/T}-1}{e^{(A+B)/T}-1}\right]$$

$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - \sqrt{\lambda}\,|\vec{p}|}{2M^2}\,, \qquad B \equiv E_p\,, \qquad C \equiv \frac{\sqrt{\lambda}\,|\vec{p}|}{M^2}$$
$$E_p \equiv \sqrt{\vec{p}^2 + M^2}$$

$$\lambda \equiv [m_1^2 - (m_2 + M)^2][m_1^2 - (m_2 - M)^2]$$
$$= \left(m_1^2 - m_2^2 - M^2\right)^2 - 4m_2^2 M^2$$

Notice that

- $A > 0$, since $E_p > |\vec{p}|$ and $m_1^2 - m_2^2 - M^2 > \sqrt{\lambda}$,
- $B > 0$ and $C > 0$,
- A, B and C do not depend on T,
- $\Gamma_{\text{eff}} > 0$, since $\frac{xc-1}{x-1}$ is a decreasing function of x (for $x, c > 1$).

Limiting cases: no $\varphi_1 \longrightarrow \varphi_2\Phi$ decay

$$\Gamma_{\text{eff}}(|\vec{p}|, T) = \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times \left[\ln \frac{e^{(A+C)/T} - 1}{e^{A/T} - 1} - \ln \frac{e^{(A+B+C)/T} - 1}{e^{(A+B)/T} - 1} \right]$$

$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - \sqrt{\lambda}|\vec{p}|}{2M^2} , \qquad B \equiv E_p , \qquad C \equiv \frac{\sqrt{\lambda}|\vec{p}|}{M^2}$$

$$E_p \equiv \sqrt{\vec{p}^2 + M^2}$$

$$\lambda \equiv [m_1^2 - (m_2 + M)^2][m_1^2 - (m_2 - M)^2]$$

- for $m_1 = m_2 + M$:

$$\lambda = 0 \quad \Rightarrow \quad C = 0 \quad \Rightarrow \quad \text{both logarithms vanish} \quad \Rightarrow \quad \Sigma = 0$$

(cf. with singularity condition: $m_1 > m_2 + \Phi$)

Limiting cases: mediator at rest

$$\Gamma_{\text{eff}}(|\vec{p}|, T) = \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times \left[\ln \frac{e^{(A+C)/T} - 1}{e^{A/T} - 1} - \ln \frac{e^{(A+B+C)/T} - 1}{e^{(A+B)/T} - 1} \right]$$

$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - \sqrt{\lambda}|\vec{p}|}{2M^2}, \qquad B \equiv E_p, \qquad C \equiv \frac{\sqrt{\lambda}|\vec{p}|}{M^2}$$

$$E_p \equiv \sqrt{\vec{p}^2 + M^2}$$

$$\lambda \equiv [m_1^2 - (m_2 + M)^2][m_1^2 - (m_2 - M)^2]$$

- for $|\vec{p}| \to 0$: we expand around $|\vec{p}|/T \equiv \alpha = 0$ to get

$$\Gamma_{\text{eff}}(|\vec{p}|, T) = \frac{\mu^2}{16\pi M} \frac{1}{\alpha} \left(0 + \alpha \frac{\sqrt{\lambda}}{2M^2} \left[\frac{e^{\beta \frac{m_1^2 - m_2^2 - M^2}{2M}} + 1}{e^{\beta \frac{m_1^2 - m_2^2 - M^2}{2M}} - 1} - \frac{e^{\beta \frac{m_1^2 - m_2^2 + M^2}{2M}} + 1}{e^{\beta \frac{m_1^2 - m_2^2 + M^2}{2M}} - 1} \right] + \mathcal{O}(\alpha^2) \right)$$

$$\approx \frac{\mu^2}{32\pi} \frac{\sqrt{\lambda}}{M^3} \left[\frac{e^{\beta \frac{m_1^2 - m_2^2 - M^2}{2M}} + 1}{e^{\beta \frac{m_1^2 - m_2^2 - M^2}{2M}} - 1} - \frac{e^{\beta \frac{m_1^2 - m_2^2 + M^2}{2M}} + 1}{e^{\beta \frac{m_1^2 - m_2^2 + M^2}{2M}} - 1} \right].$$

$$\Gamma_{\text{eff}}(|\vec{p}|, T) = \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times \left[\ln \frac{e^{(A+C)/T} - 1}{e^{A/T} - 1} - \ln \frac{e^{(A+B+C)/T} - 1}{e^{(A+B)/T} - 1} \right]$$

- for $T \to \infty$ we expand the exponentials:

$$\Gamma_{\text{eff}}(|\vec{p}|, T) \approx \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times \left[\ln \frac{A+C}{A} - \ln \frac{A+B+C}{A+B} \right]$$

$$= \frac{\mu^2}{16\pi M} \frac{1}{|\vec{p}|/T} \times \left[\underbrace{\ln \left(1 + \frac{C}{A}\right) - \ln \left(1 + \frac{C}{A+B}\right)}_{>0} \right] \quad \to \infty$$

Limiting cases: low temperature

$$\Gamma_{\text{eff}}(|\vec{p}|,T) = \frac{\mu^2}{16\pi M}\,\frac{1}{|\vec{p}|/T} \times \left[\ln\frac{e^{(A+C)/T}-1}{e^{A/T}-1} - \ln\frac{e^{(A+B+C)/T}-1}{e^{(A+B)/T}-1}\right]$$

- for $T \to 0$ we rewrite the result

$$\Gamma_{\text{eff}}(|\vec{p}|,T) = \frac{\mu^2}{16\pi M}\,\frac{1}{|\vec{p}|/T} \times \left[\ln\frac{e^{C/T}-e^{-A/T}}{1-e^{-A/T}} - \ln\frac{e^{(B+C)/T}-e^{-A/T}}{e^{B/T}-e^{-A/T}}\right]$$

and introduce $\xi \equiv e^{-A/T} \ll 1$

$$\Gamma_{\text{eff}}(|\vec{p}|,T) = \frac{\mu^2}{16\pi M}\,\frac{1}{|\vec{p}|/T} \times \left[\ln\frac{e^{C/T}-\xi}{1-\xi} - \ln\frac{e^{(B+C)/T}-\xi}{e^{B/T}-\xi}\right]$$

- Expansion around $\xi = 0$:

$$\Gamma_{\text{eff}}(|\vec{p}|,T) = \frac{\mu^2}{16\pi M}\,\frac{1}{|\vec{p}|/T} \times \left[0 + \xi\,(1-e^{-C/T})(1-e^{-B/T}) + \mathcal{O}(\xi^2)\right]$$

$$\approx \frac{\mu^2}{16\pi M}\,\frac{1}{|\vec{p}|/T} \times e^{-A/T}(1-e^{-C/T})(1-e^{-B/T}) \quad \to 0$$