CLUSTER OF EXCELLENCE
QUANTUM UNIVERSE

Out-of-equilibrium dark matter: production and cosmological signatures

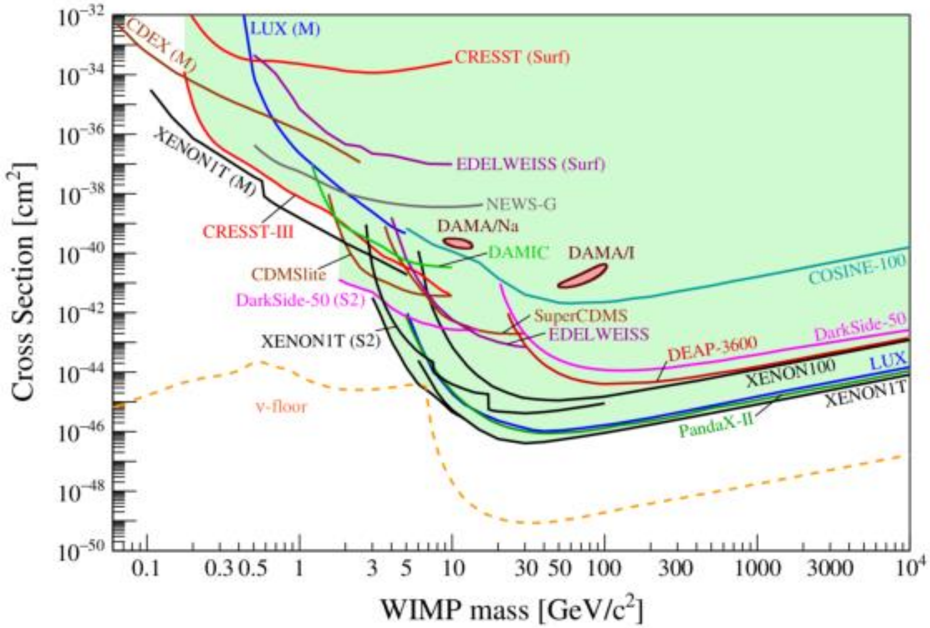
Mathias Pierre

Meeting "Théorie, Univers et Gravitation"

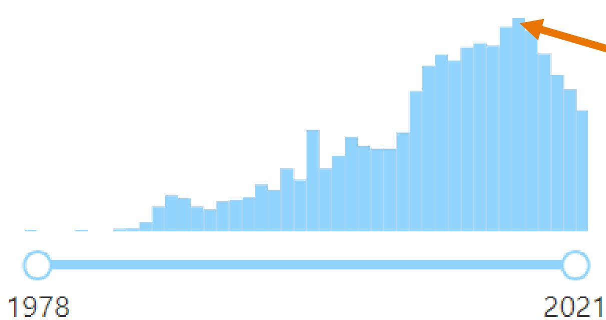
Institut Henri Poincaré

December 15th 2021

The waning of the WIMP?

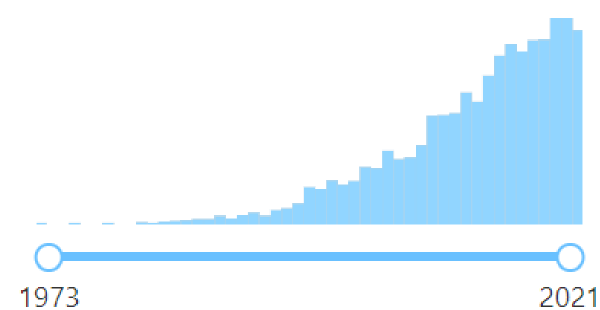


No WIMP actually detected!



2016

Inspire-HEP papers “WIMP”



Inspire-HEP papers “freeze-in”

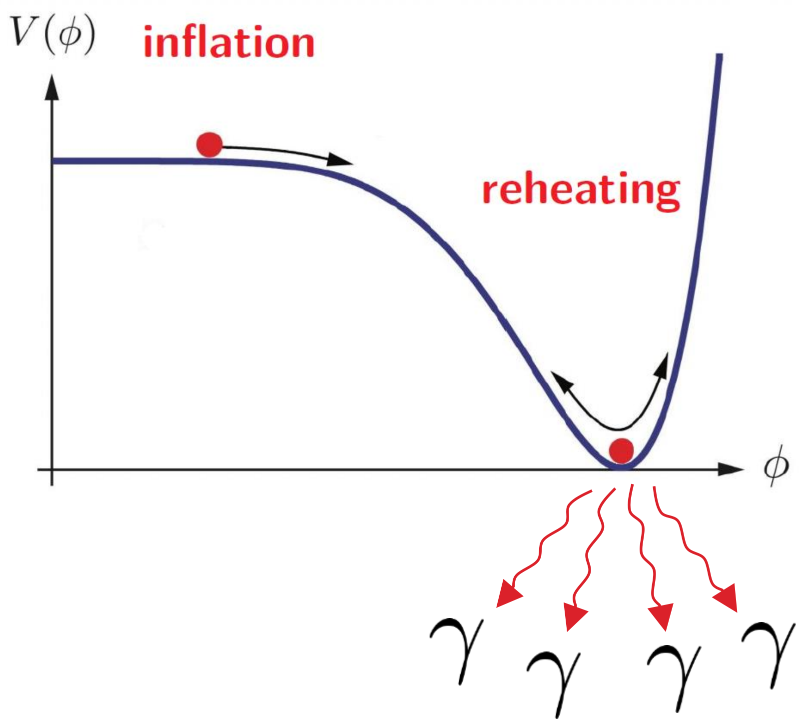
What are cosmological signatures of out-of-equilibrium DM?

Can we probe the DM production mechanism?

Based on [**arXiv:2011.13458**] – **JCAP 21**
with **G. Ballesteros** & **M. A. G. Garcia**

Production of out-of-equilibrium dark matter

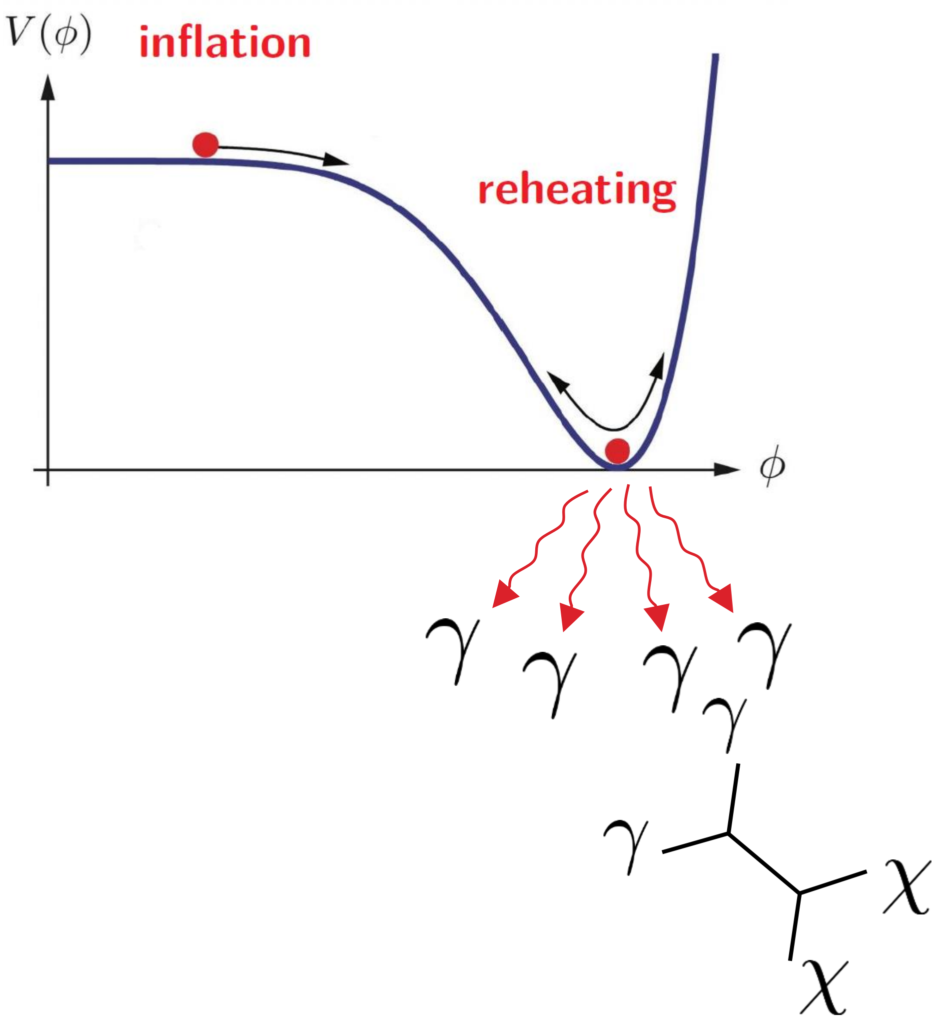
Out-of-equilibrium DM production



ϕ : inflaton

γ : generic SM particle

Out-of-equilibrium DM production



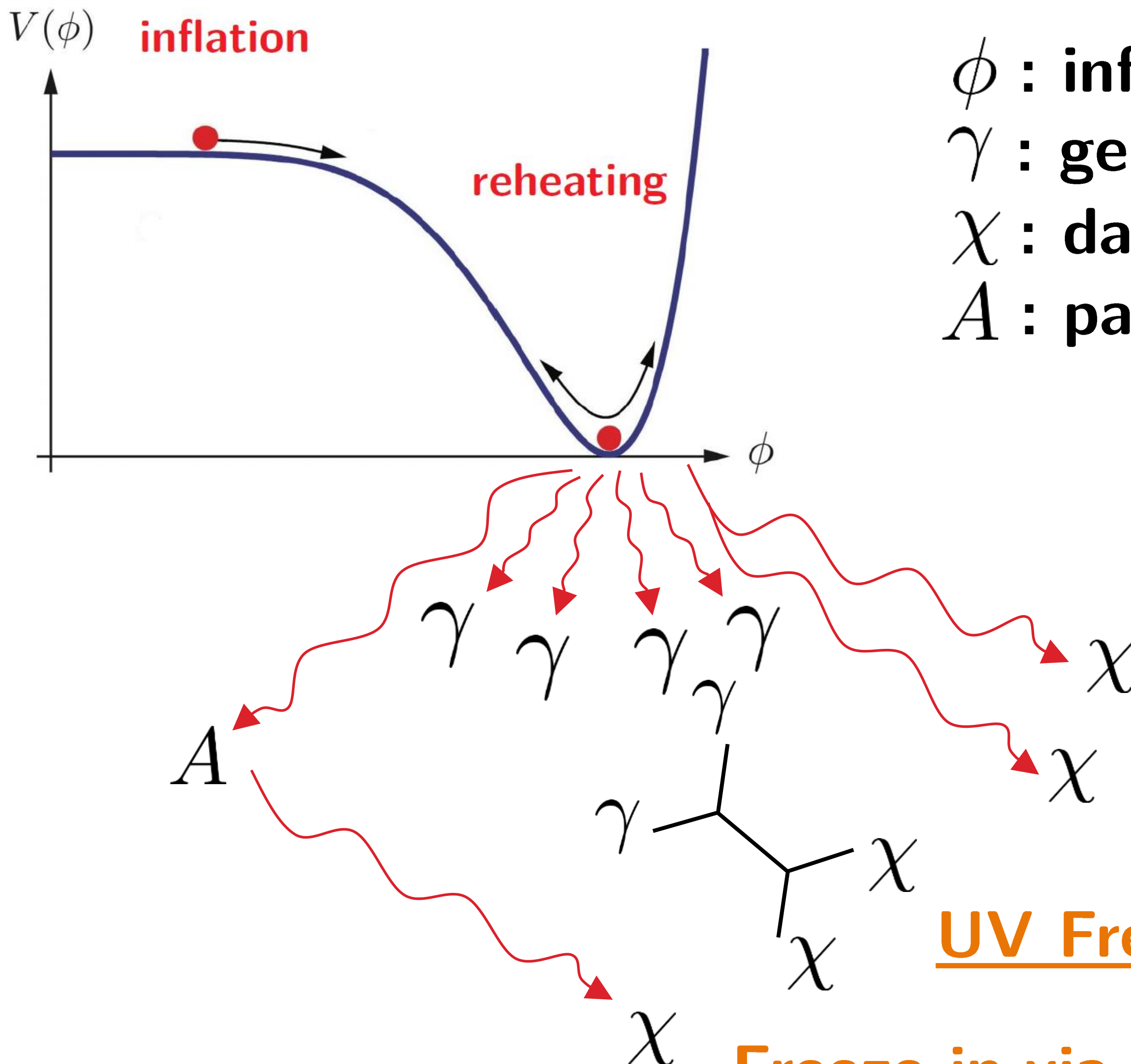
ϕ : **inflaton**

γ : **generic SM particle**

χ : **dark matter**

UV Freeze-in via scattering

Out-of-equilibrium DM production



ϕ : inflaton
γ : generic SM particle
χ : dark matter
A : particle (SM or not)

Decay of inflaton

UV Freeze-in via scattering

Freeze-in via decay

DM Phase space distribution

$$n_\chi(t) = \frac{g_\chi}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{p}\, f_\chi(p_0, t)$$

number-density

$$\rho_\chi(t) = \frac{g_\chi}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{p}\, p_0\, f_\chi(p_0, t)$$

energy-density

- Obtain **phase space distribution** by solving **Boltzmann** equation

$$\frac{\partial f_\chi}{\partial t} - H|\boldsymbol{p}|\frac{\partial f_\chi}{\partial|\boldsymbol{p}|} = \mathcal{C}[f_\chi(|\boldsymbol{p}|, t)]$$

- **Collision term** for processes $\chi + a + b + \cdots \longleftrightarrow i + j + \cdots$

$$\begin{aligned}\mathcal{C}\left[f_\chi\right] = -\frac{1}{2p_0}\int & \frac{g_a \mathrm{d}^3\boldsymbol{p}_a}{(2\pi)^3 2p_{a0}}\frac{g_b \mathrm{d}^3\boldsymbol{p}_b}{(2\pi)^3 2p_{b0}}\cdots\frac{g_i \mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3 2p_{i0}}\frac{g_j \mathrm{d}^3\boldsymbol{p}_j}{(2\pi)^3 2p_{j0}}\cdots \\ & \times (2\pi)^4\,\delta^{(4)}(p_\chi + p_a + p_b + \cdots - p_i - p_j - \cdots) \\ & \times \Big[|\mathcal{M}|^2_{\chi+a+b+\cdots\longrightarrow i+j+\cdots}\, f_a f_b \cdots f_\chi (1 \pm f_i)(1 \pm f_j)\cdots \\ & \quad - |\mathcal{M}|^2_{i+j+\cdots\longrightarrow \chi+a+b+\cdots}\, f_i f_j \cdots (1 \pm f_a)(1 \pm f_b)\cdots(1 \pm f_\chi)\Big]\end{aligned}$$

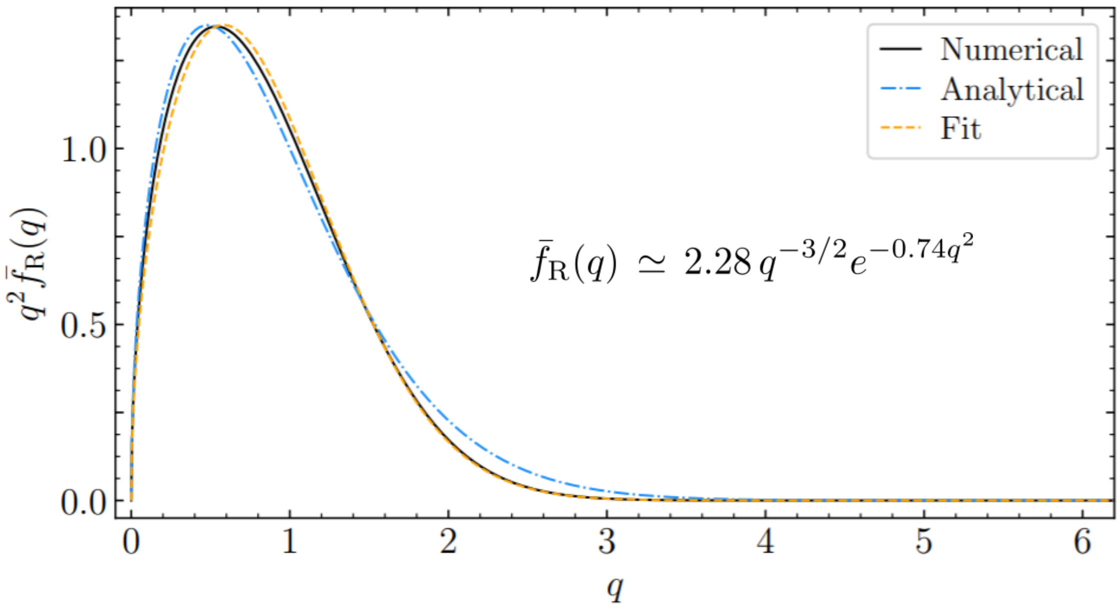
DM production from inflaton decay

- Consider **DM** produced from **perturbative** inflaton decay

$$\chi \longleftarrow \phi \longrightarrow \chi$$

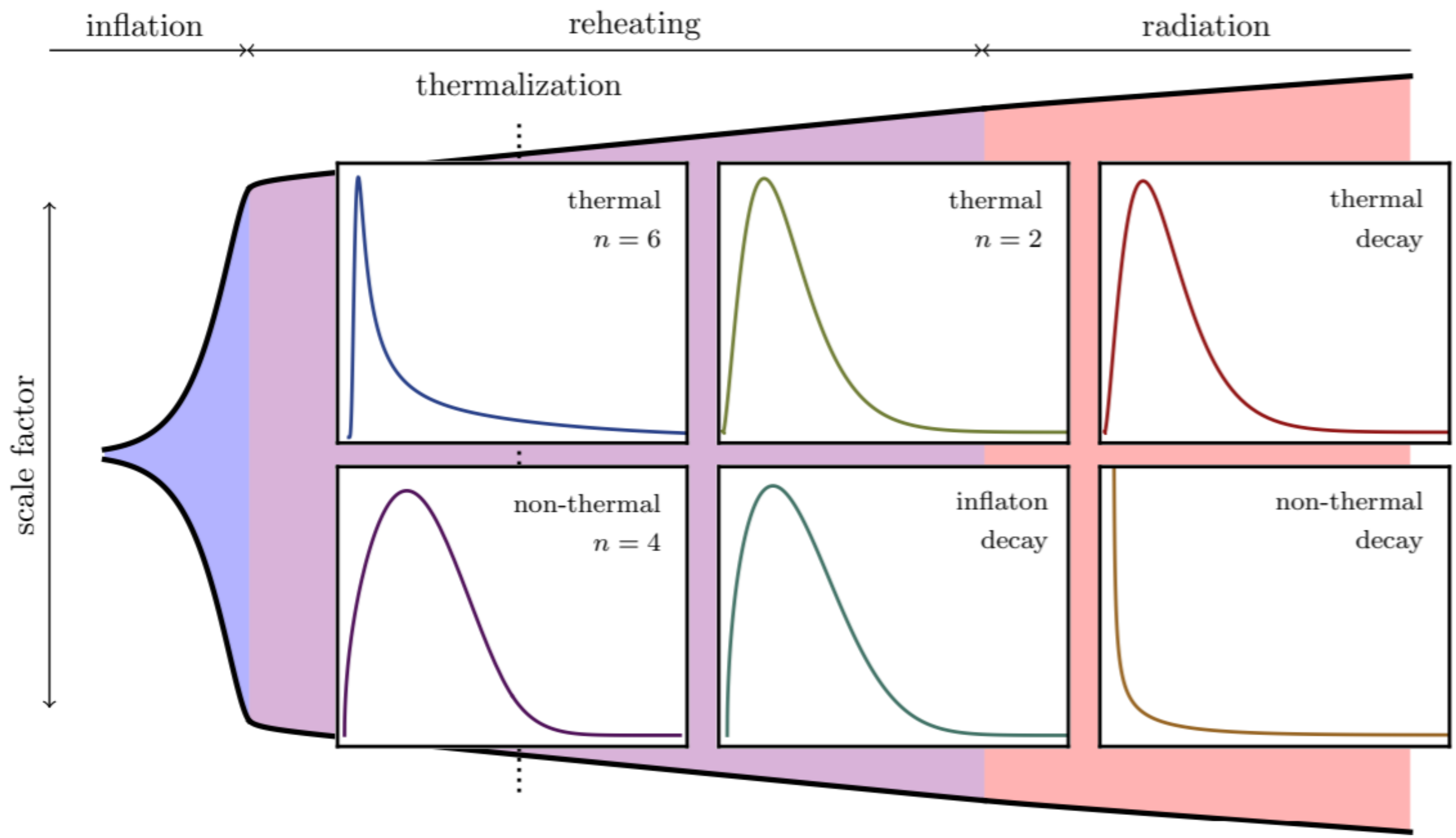
$$\mathcal{C}[f_\chi(p,t)] \;=\; \frac{8\pi^2}{g m_\phi^2}\,\Gamma_\phi\,\mathrm{Br}_\chi\,n_\phi(t)\,\delta(p - m_\phi/2)$$

$$f_\chi(p,t)\;\mathrm{d}^3\boldsymbol{p} \;=\; \frac{4\pi^4 \mathrm{Br}_\chi g_{*s}^{\mathrm{reh}}}{5 g_\chi}\left(\frac{T_{\mathrm{reh}}}{m_\phi}\right)^4\left(\frac{a_0}{a(t)}\right)^3 T_\star^3\,\bar{f}_{\mathrm{R}}(q)\;\mathrm{d}^3\boldsymbol{q} \qquad T_\star \;=\; \left(\frac{g_{*s}^0}{g_{*s}^{\mathrm{reh}}}\right)^{1/3}\frac{m_\phi}{2T_{\mathrm{reh}}}\,T_0$$

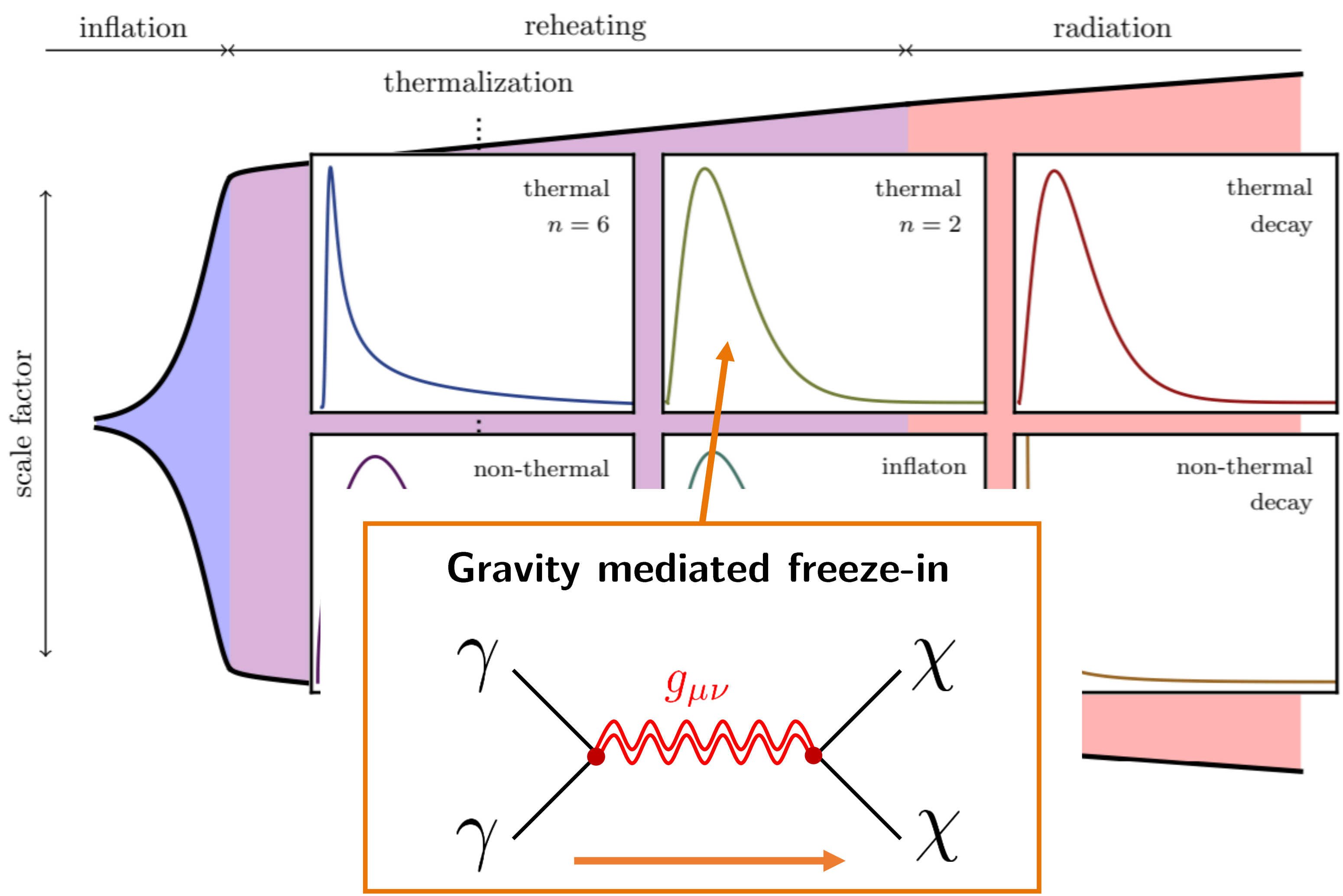


$$q \;\equiv\; \frac{p\,a(t)}{T_\star}$$

DM phase space distribution

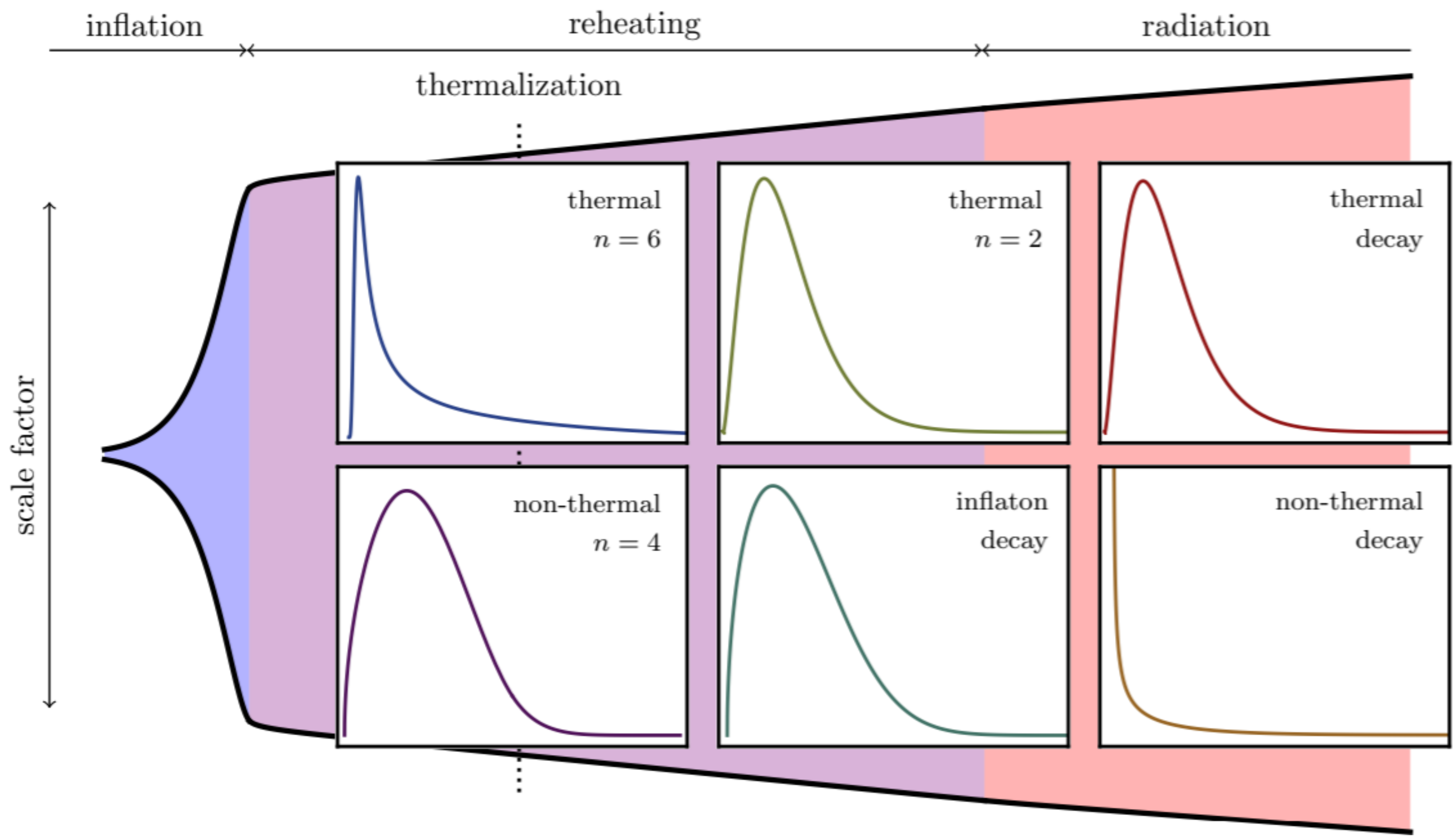


DM phase space distribution



[M. Garny, M. Sandora, M. S. Sloth - PRL 116 (2016) 10, 101302
N. Bernal, M. Dutra, Y. Mambrini, K. Olive, M. Peloso, **MP** - PRD 97 (2018) 11, 115020]

DM phase space distribution



- Most of the distributions fitted by $f(q) \propto q^{\alpha} \exp(-\beta\, q^{\gamma})$

What is the cosmological imprint of out-of-equilibrium dark matter?

Cosmological imprint

Non-Cold Dark Matter $w \neq 0$

- Expanding quantities around **homogenous background**

$$f(\boldsymbol{x}, \boldsymbol{p}, \tau) = \bar{f}(|\boldsymbol{p}|, \tau)[1 + \Psi(\boldsymbol{x}, \boldsymbol{p}, \tau)]$$

- In matter domination, matter **overdensities** δ follow

$$\ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2\left(1 - \frac{k^2}{k_{\rm FS}^2}\right)\delta = 0 \qquad w \ll 1$$

where $k_{\rm FS}^2(a) = \frac{9}{10}\frac{\mathcal{H}^2}{w(a)}$ is the **Free-Streaming wavenumber**

$\mathrm{d}\tau \equiv a\,\mathrm{d}t$: **Conformal time τ**

$\mathcal{H} \equiv a\,H$: **Conformal Hubble rate**

$w \equiv \bar{P}/\bar{\rho}$: **Equation-of-state parameter**

[C. Ma & E. Bertschinger. ApJ 455 (1995) 7-25]
[J. Lesgourgues & T. Tram, JCAP 09 (2011) 032]
[M. Kunz, S. Nesseris, I. Sawicki, PRD 94, 023510 (2016)]
[G. Ballesteros, M. A. G. Garcia & **MP**, 2011.13458]

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where $k_{\mathrm{FS}}^2(a) = \frac{9}{10}\frac{\mathcal{H}^2}{w(a)}$ is the **Free-Streaming wavenumber**

- If $w = 0$ all modes grow "**democratically**" : **CDM** limit
  $w \neq 0$ **cutoff in power spectrum** at $k_{\mathrm{H}}(a) \equiv \left[\int_0^a \frac{1}{k_{\mathrm{FS}}(\tilde{a})}\frac{\mathrm{d}\tilde{a}}{\tilde{a}}\right]^{-1}$

- **Only w controls the cutoff scale!**

$\mathrm{d}\tau \equiv a\,\mathrm{d}t$: **Conformal time τ**
$\mathcal{H} \equiv a\,H$: **Conformal Hubble rate**
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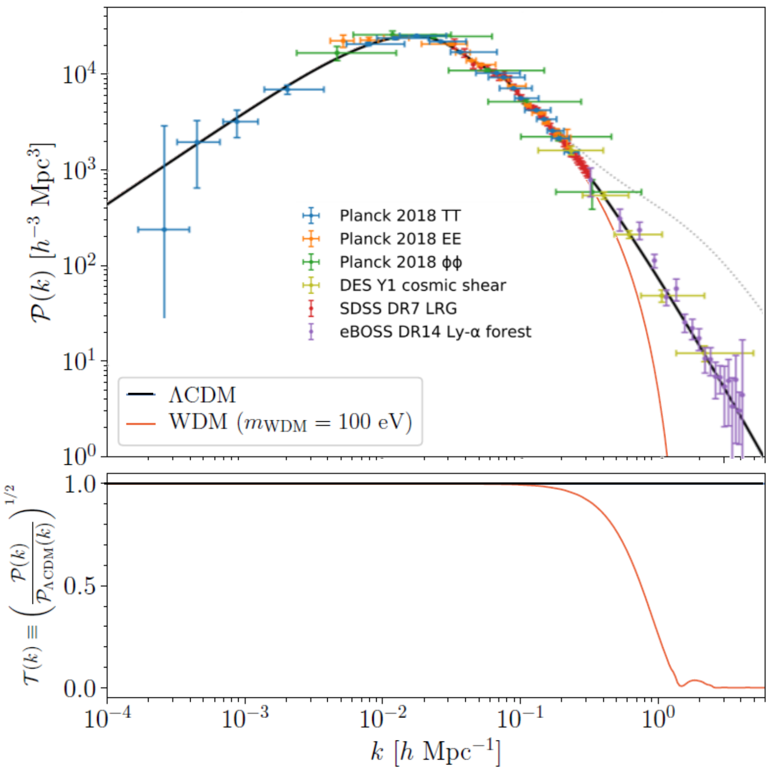
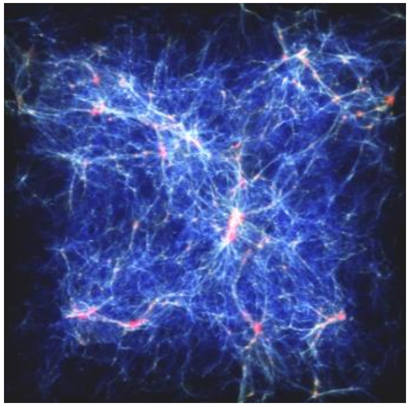
Non-Cold Dark Matter $w \neq 0$

- Lyman-alpha forest constraints Warm Dark Matter (**WDM**)

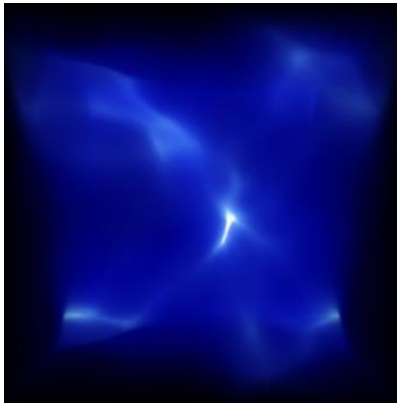
$$\bar{f}_{\mathrm{WDM}}(q) = \frac{1}{1 + e^{q/T_{\mathrm{WDM}}}} \Rightarrow \Omega_{\mathrm{WDM}} h^2 \simeq \left(\frac{m_{\mathrm{WDM}}}{94\ \mathrm{eV}}\right)\left(\frac{T_{\mathrm{WDM}}}{T_\nu}\right)^3 \simeq 0.12$$



ΛCDM



WDM



$m_{\mathrm{WDM}} = 100\,\mathrm{eV}$

[J. Baur et al. JCAP 08 (2016) 012]

$$m_{\mathrm{WDM}}^{\mathrm{Ly-}\alpha} = (1.9 - 5.3)\ \mathrm{keV\ at\ 95\%\ C.L.}$$

$$w_{\mathrm{WDM}}(m_{\mathrm{WDM}}^{\mathrm{Ly-}\alpha}) \sim 10^{-15} a^{-2}$$

[Braur et al. JCAP 08 (2016) 012 – Iršič et al. PRD 96 (2017) 2, 023522
Palanque Delabrouille et al. JCAP 04 (2020) 038 – Viel et al. PRD 88 (2013) 043502
Viel et al. PRD 71 (2005) 063534 – Narayanan et al. ApJ 543 (2000) L103-L106]

How warm is Non-Cold Dark Matter?

- **How to translate Lyman-alpha WDM bounds on any scenario ?**

$$w(m_{\rm DM}) = w_{\rm WDM}(m_{\rm WDM}^{\rm Ly-\alpha})$$

[S. Colombi, S. Dodelson, L. M. Widrow ApJ. 458 (1996) 1 - Kamada, N. Yoshida, K. Kohri, T. Takahashi JCAP 03 (2013) 008
K. J. Bae, R. Jinno, A. Kamada, K. Yanagi JCAP 03 (2020) 042 - A. Kamada & K. Yanagi JCAP 1911 (2019) 029]

$$w \simeq \frac{\delta P}{\delta \rho} = \frac{T_\star^2}{3m_{\rm DM}^2}\frac{\langle q^2\rangle}{a^2}$$

$w-$ matching

$$m_{\rm DM} = m_{\rm WDM}^{\rm Ly-\alpha}\left(\frac{T_\star}{T_{\rm WDM}}\right)\sqrt{\frac{\langle q^2\rangle}{\langle q^2\rangle_{\rm WDM}}}$$

- Compute **2nd moment of distribution** + **determine** $T_\star$

- If **distribution** can be fitted by $f(q) \propto q^\alpha \exp\left(-\beta\, q^\gamma\right)$

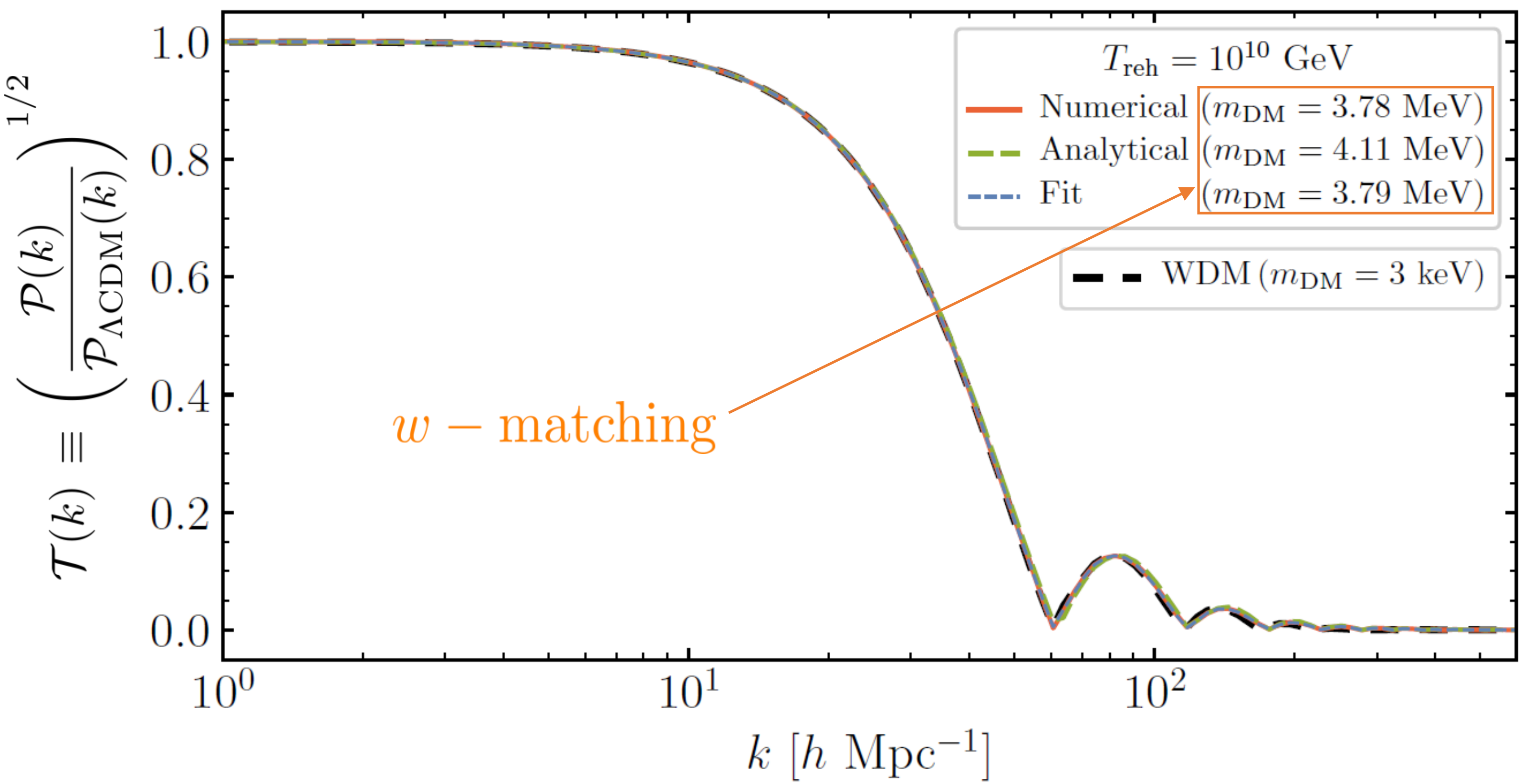
$w-$ matching →

$$m_{\rm DM} \simeq 7.56\ {\rm keV}\left(\frac{m_{\rm WDM}^{\rm Ly-\alpha}}{3\ {\rm keV}}\right)^{4/3}\left(\frac{\langle p\rangle_0}{T_0}\right)\sqrt{\frac{\Gamma\left(\frac{3+\alpha}{\gamma}\right)\Gamma\left(\frac{5+\alpha}{\gamma}\right)}{\Gamma^2\left(\frac{4+\alpha}{\gamma}\right)}}$$

How warm is Non-Cold Dark Matter?

- Example: **inflaton decay** case computed using **CLASS**

[D. Blas, J. Lesgourgues & T. Tram JCAP 07 (2011) 034 - J. Lesgourgues & T. Tram, JCAP 09 (2011) 032]



- **Excellent agreement** with $w - \mathrm{matching}$ for **all distributions**!

Inflaton decay: constraints

- **Lyman-alpha** bounds translate into

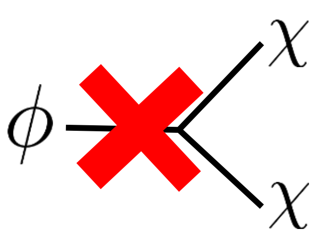
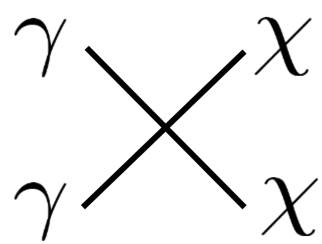
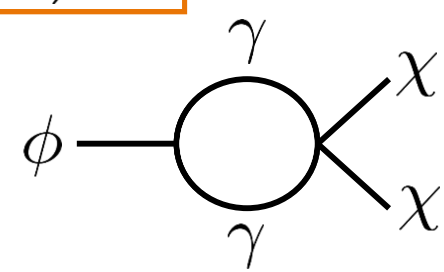
$$m_{\rm DM} \gtrsim \left(\frac{m_{\rm WDM}^{\rm Ly-\alpha}}{3\ \rm keV}\right)^{4/3} \left(\frac{106.75}{g_{*s}^{\rm reh}}\right)^{1/3} \left(\frac{m_\phi}{3\times 10^{13}\ \rm GeV}\right) \left(\frac{10^{10}\ \rm GeV}{T_{\rm reh}}\right) \begin{cases} 3.78\ \rm MeV\,, & \text{Numerical}\,,\\ 4.11\ \rm MeV\,, & \text{Analytical}\,,\\ 3.79\ \rm MeV\,, & \text{Fit}\,. \end{cases}$$

- For **low reheating temperature** $T_{\rm reh} \ll m_\phi$

$$m_{\rm DM} \gtrsim \rm EeV$$

- **Combining** with **relic density** condition

$${\rm Br}_\chi < 1.5\times 10^{-4} \left(\frac{g_{*s}^{\rm reh}}{106.5}\right)^{1/3} \left(\frac{3\,\rm keV}{m_{\rm WDM}^{\rm Ly-\alpha}}\right)^{4/3}$$

- Even if  , if  then 

[K. Kaneta, Y. Mambrini & Keith A. Olive Phys.Rev.D 99 (2019) 6, 063508]

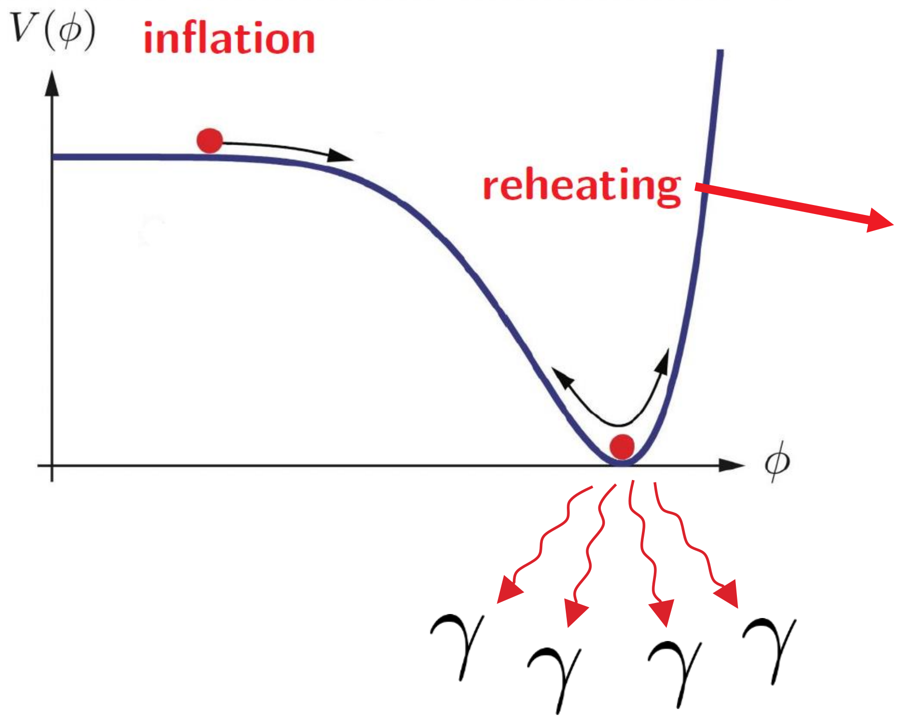
Summary

- **Out-of-equilibrium DM** can be produced **after inflation**
- **Cutoff** in power spectrum can be probed by Lyman-alpha
  Powerful tool to **probe out-of-equilibrium dark matter**
- Generalization via **Equation-of-State parameter** matching
  + much more... **in the paper**! [arXiv:2011.13458]

Thank you for your attention

Back-up Slides

Reheating in a nutshell



For a **quadratic** potential

$$\frac{d\rho_\phi}{dt} + 3H\rho_\phi = -\Gamma_\phi \rho_\phi$$

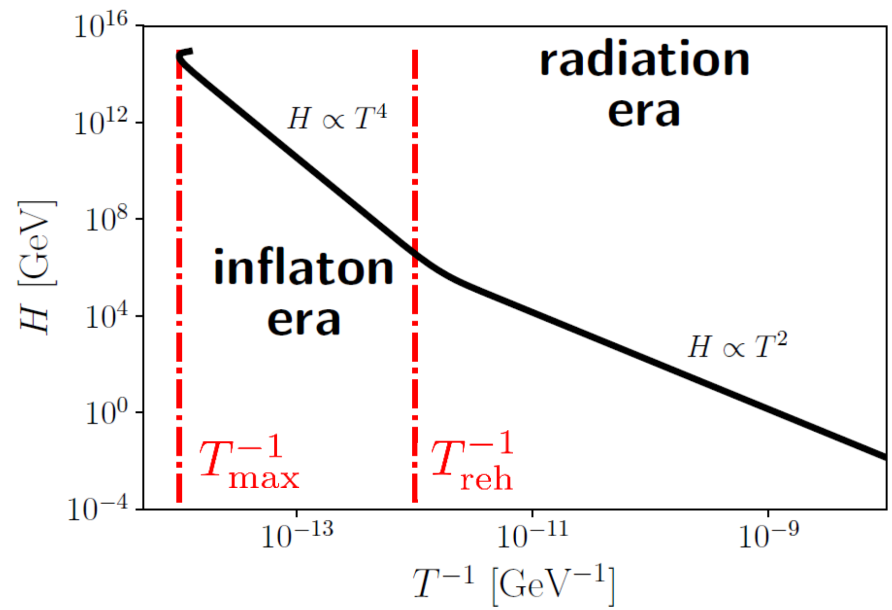
$$\frac{d\rho_\gamma}{dt} + 4H\rho_\gamma = \Gamma_\phi \rho_\phi$$

$$H^2 = \frac{1}{3M_{\rm Pl}}(\rho_\phi + \rho_\gamma)$$

$T_{\rm reh}$ **: Reheating temperature**

$$\rho_\phi(T_{\rm reh}) = \rho_\gamma(T_{\rm reh})$$

$$(\Leftrightarrow t_{\rm reh} \simeq \Gamma_\phi^{-1})$$



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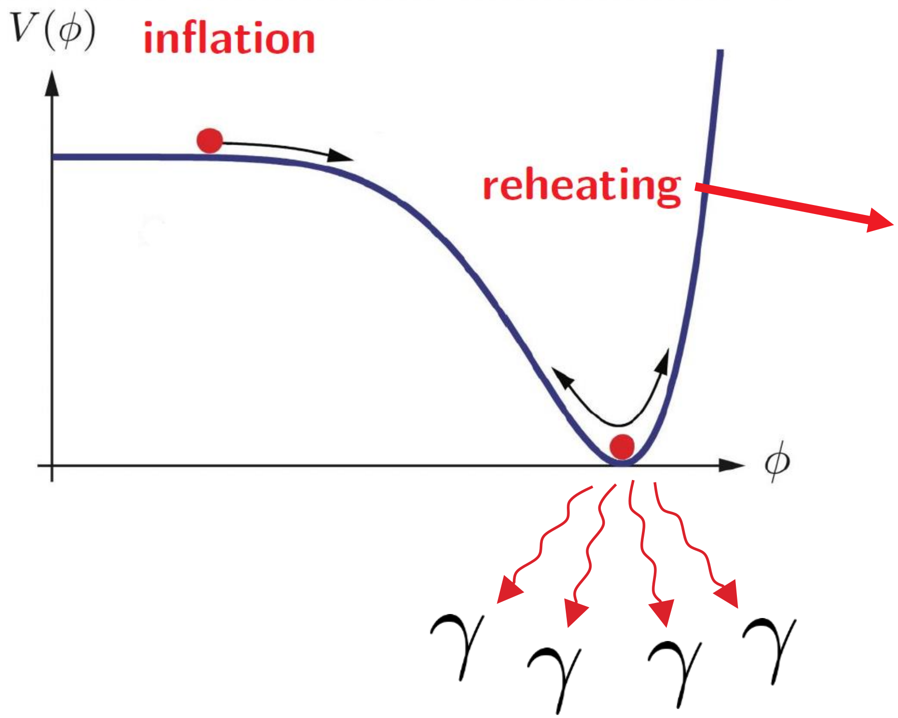

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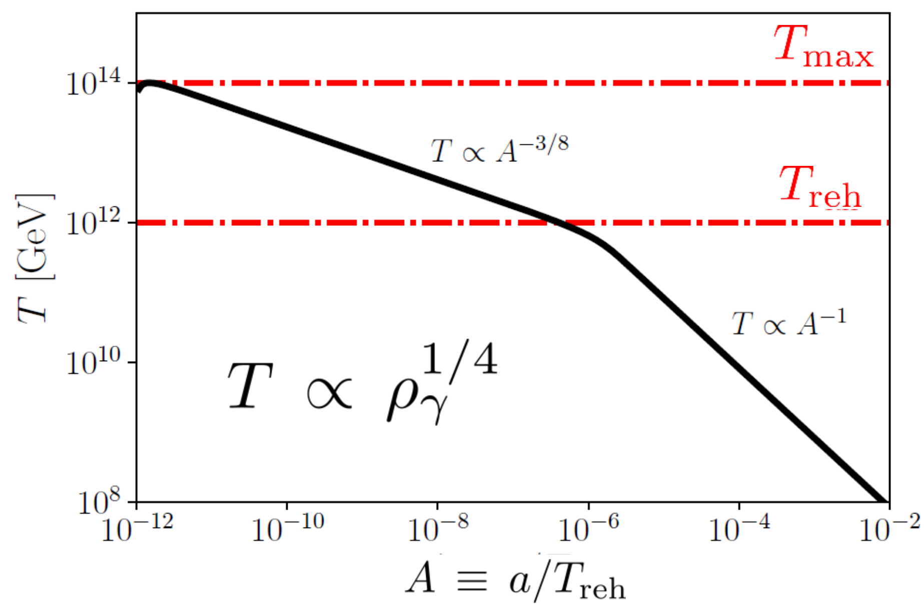


DM Phase space distribution

- **Solution** to the Boltzmann equation gives

$$f_\chi(p_0,t) = \int_{t_i}^{t} \mathcal{C}[f_\chi]\left(\frac{a(t)}{a(t')}|\boldsymbol{p}|, t'\right) \mathrm{d}t'$$

- For $t > t_{\rm dec}$ after **decoupling** (when production stops)

$$\frac{\partial f_\chi}{\partial t} - H|\boldsymbol{p}|\frac{\partial f_\chi}{\partial|\boldsymbol{p}|} = 0 \quad \Rightarrow \quad f_\chi(|\boldsymbol{p}|,t) = \bar{f}\left(|\boldsymbol{p}|\frac{a(t)}{a_{\rm dec}}, t_{\rm dec}\right)$$

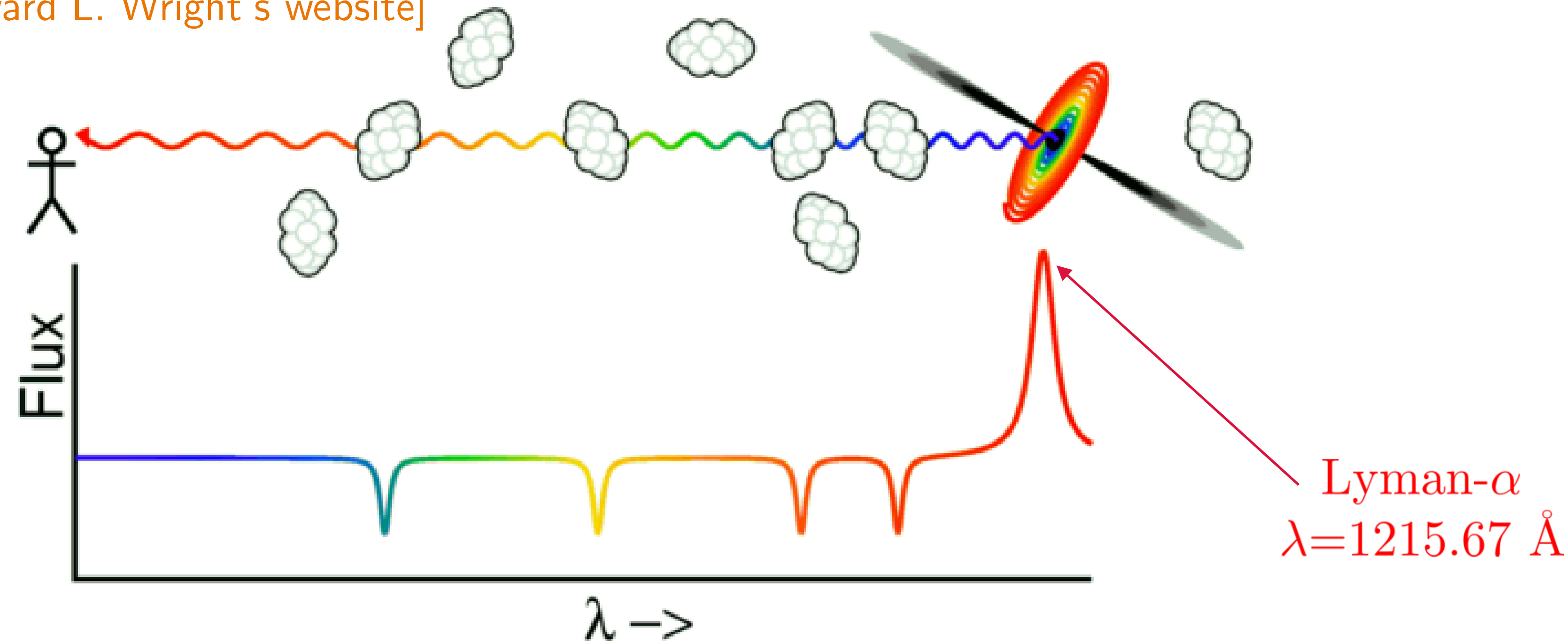
- After decoupling, distribution function only depends on the **comoving momentum**

$$q \equiv \frac{p\,a(t)}{T_\star} \qquad n_\chi(t) = \frac{g_\chi}{(2\pi)^3}\frac{T_\star^3}{a^3}\int \mathrm{d}^3\boldsymbol{q}\,\bar{f}_\chi(q)$$

$T_\star \equiv T_{\rm NCDM}$ in **CLASS** [J. Lesgourgues & T. Tram, JCAP 09 (2011) 032]

Lyman-alpha forest

[Edward L. Wright's website]

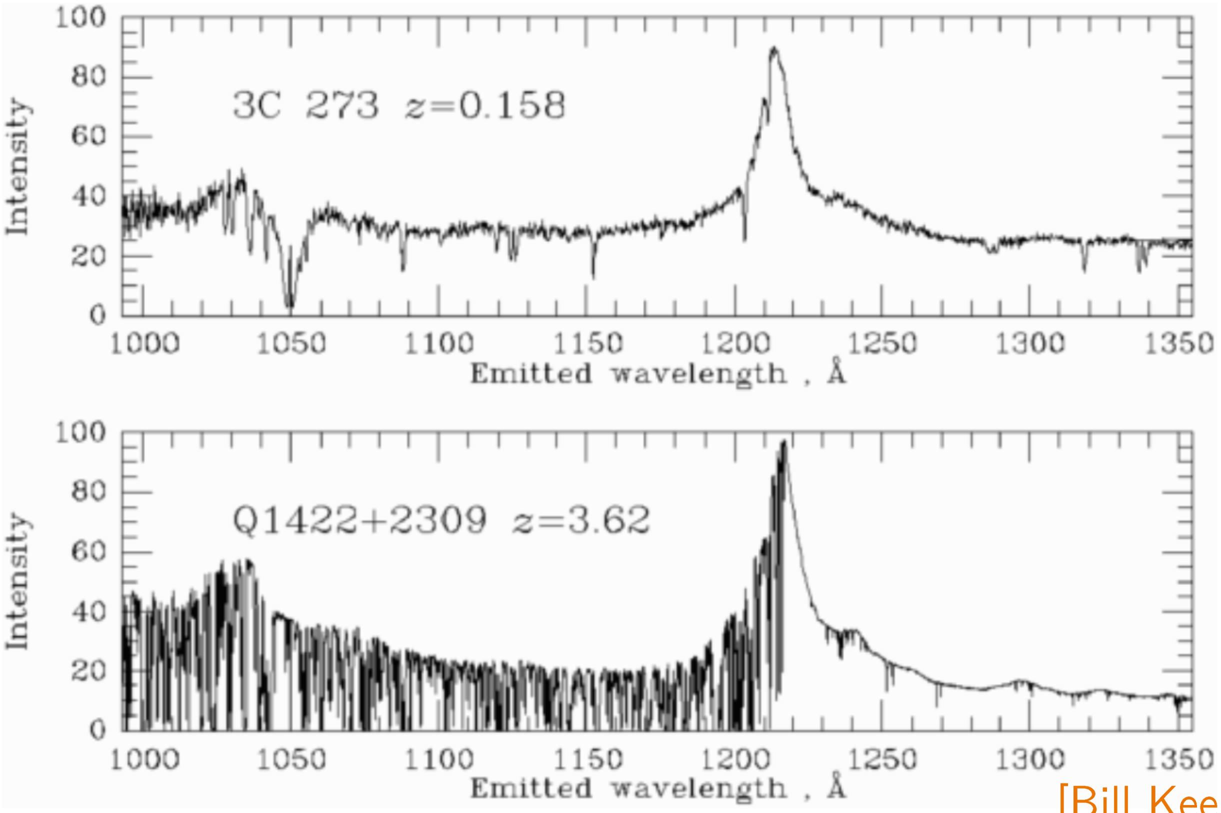


- Quasi-Stellar Objects (QSO) are luminous astrophysical objects powered by gas spiraling at high velocity into an massive black hole

- Light emitted by distant QSO is absorbed in foreground structures

- Allows for a 1D measure of overdensities along line of sight

Lyman-alpha forest



[Bill Keel's website]

- Comparison of QSO spectra at low and high redshift in QSO rest frame

NCDM Cosmology

- Expand around (homogenous) **background quantities**

$$f(\boldsymbol{x},\boldsymbol{p},\tau)=\bar{f}(|\boldsymbol{p}|,\tau)[1+\Psi(\boldsymbol{x},\boldsymbol{p},\tau)]$$

- Expand **fluctuations** in term of **Legendre polynomials**

$$\Psi(\boldsymbol{k},\hat{\boldsymbol{n}},q,\tau)=\sum_{\ell=0}^{\infty}(-i)^{\ell}(2\ell+1)\Psi_{\ell}(\boldsymbol{k},q,\tau)P_{\ell}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}})$$

- Express **fluctuations** in terms of **Legendre coefficients**

$$\delta\bar{\rho}=4\pi\left(\frac{T_\star}{a}\right)^4\int q^2\epsilon\bar{f}(q)\Psi_0\,\mathrm{d}q,\qquad \text{energy density fluctuation}$$

$$\delta\bar{P}=\frac{4\pi}{3}\left(\frac{T_\star}{a}\right)^4\int q^2\frac{q^2}{\epsilon}\bar{f}(q)\Psi_0\,\mathrm{d}q,\qquad \text{pressure (density) fluctuation}$$

$$(\bar{\rho}+\bar{P})\theta=4\pi k\left(\frac{T_\star}{a}\right)^4\int q^3\bar{f}(q)\Psi_1\,\mathrm{d}q,\qquad \text{velocity divergence}$$

$$(\bar{\rho}+\bar{P})\sigma=\frac{8\pi k}{3}\left(\frac{T_\star}{a}\right)^4\int q^2\frac{q^2}{\epsilon}\bar{f}(q)\Psi_2\,\mathrm{d}q,\qquad \text{anisotropic stress.}$$

NCDM Cosmology

- The phase space distribution satisfies **collisionless Boltzmann equation**

$$\frac{\partial f}{\partial \tau} + \frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\partial f}{\partial x^i} + \frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\partial f}{\partial q} + \frac{\mathrm{d}n_i}{\mathrm{d}\tau}\frac{\partial f}{\partial n_i} = 0\,,$$

- **Plugging** distribution expansion in **Legendre polynomials** give

$$\mathrm{d}s^2 = a(\tau)\big(-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\big)$$

in **synchronous** gauge $h \equiv h_{ii}$

$$\dot{\Psi}_0 = -\frac{qk}{\epsilon}\Psi_1 + \frac{1}{6}\dot{h}\frac{\mathrm{d}\ln\bar{f}}{\mathrm{d}\ln q}\,,$$

$$\dot{\Psi}_1 = \frac{qk}{3\epsilon}\Big(\Psi_0 - 2\Psi_2\Big)\,,$$

$$\dot{\Psi}_2 = \frac{qk}{5\epsilon}\Big(2\Psi_1 - 3\Psi_3\Big) - \Big(\frac{1}{15}\dot{h} + \frac{2}{5}\dot{\eta}\Big)\frac{\mathrm{d}\ln\bar{f}}{\mathrm{d}\ln q}\,,$$

$$\dot{\Psi}_\ell = \frac{qk}{(2\ell+1)\epsilon}\Big(\ell\Psi_{\ell-1} - (\ell+1)\Psi_{\ell+1}\Big)\,, \quad [\ell \geq 3]$$

For a non-relativistic species, higher multipoles are typically suppressed by (positive) powers of $q/\epsilon \sim p/m_{\mathrm{DM}}$, making any Ψ_ℓ with $\ell \geq 2$ much smaller than Ψ_0 and Ψ_1. In this case, the Boltzmann hierarchy can be truncated imposing $\Psi_\ell = 0$ for $\ell > 1$. In this (non-relativistic) case Ψ_0 depends only mildly on the variable q, and the integrals are dominated by the low $q \ll \epsilon$ regime so that we can identify $\delta P/\delta\rho \simeq \bar{P}/\bar{\rho} = w$.

NCDM Cosmology

- **Neglecting higher multipoles**, for very **non-relativistic** DM, integrating over momenta gives

$$\dot{\delta} = -(1+w)\left(\theta + \frac{\dot{h}}{2}\right) - 3\mathcal{H}\left(\hat{c}_s^2 - w\right)\delta + 9\mathcal{H}^2(1+w)\left(\hat{c}_s^2 - c_a^2\right)\frac{\theta}{k^2},$$

$$\dot{\theta} = -\mathcal{H}\left(1 - 3\hat{c}_s^2\right)\theta + \frac{\hat{c}_s^2}{1+w}k^2\delta,$$

- In **matter domination**, from **Einstein equations**, metric perturbation follow

$$\ddot{h} + \mathcal{H}\dot{h} + 3(1+3w)\mathcal{H}^2\delta = 0,$$

- Which can be translated to evolution of **matter density fluctuations**

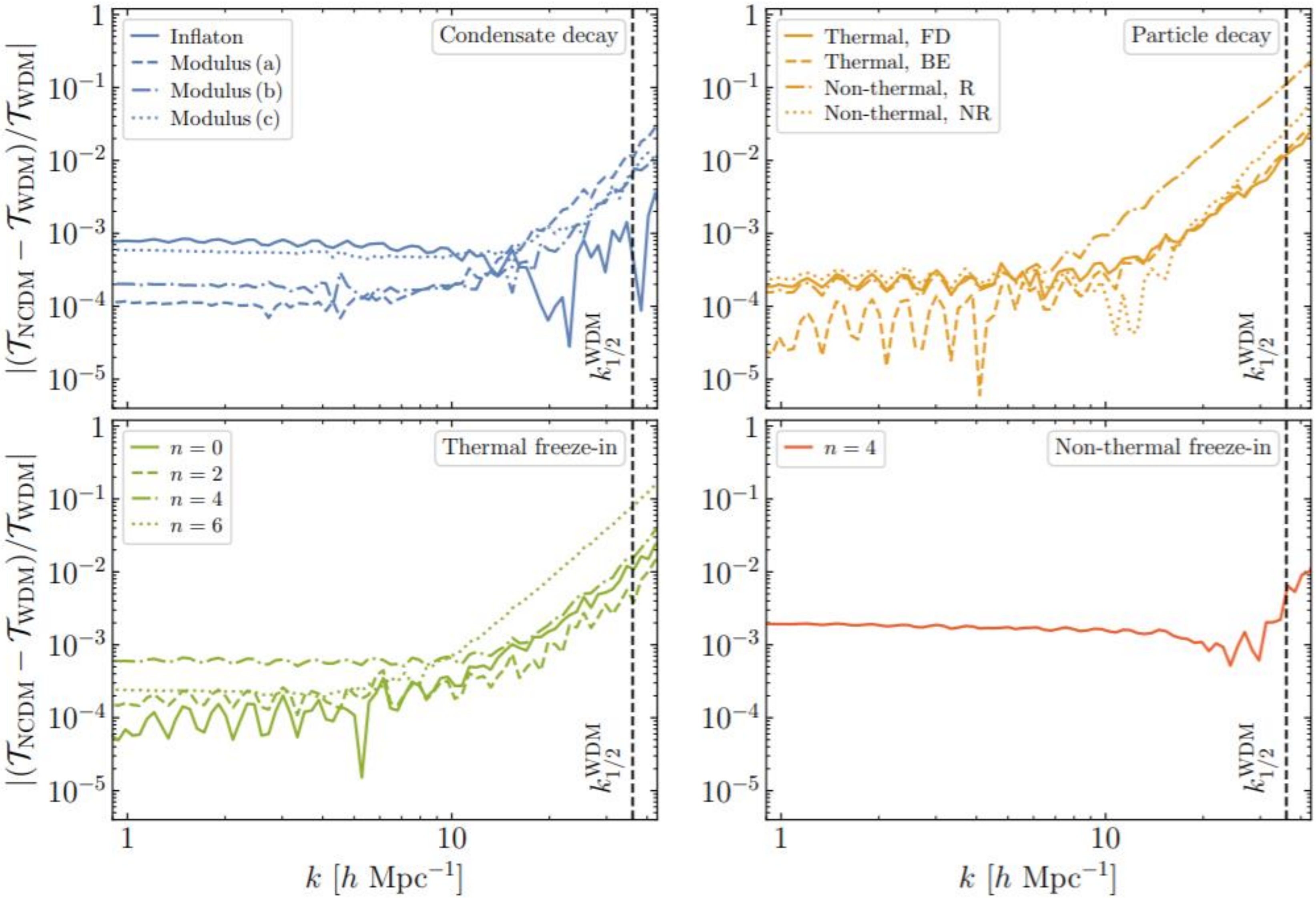
$$\ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2\left(1 - w\frac{10}{9}\frac{k^2}{\mathcal{H}^2}\right)\delta = 0.$$

General phase space distribution

$$f(q) \propto q^{\alpha} \exp\left(-\beta\, q^{\gamma}\right)$$

Scenario		α	β	γ
Inflaton decay		-3/2	0.74	1.00
Moduli decay	during reheating	-3/2	1.00	3/2
	after reheating	-1.00	1.00	2.00
Thermal decay		-1/2	1.00	1.00
Non-thermal decay	non-relativistic	-	-	-
	relativistic	-5/2	0.74	2.00
UV Freeze-in ($n = 0$)	BB	0.70	1.13	1.00
	FB	0.51	1.10	1.00
	FF	0.29	1.11	1.00
UV Freeze-in ($n = 2$)	BB	0.51	0.91	1.00
	FB	0.42	0.90	1.00
	FF	0.33	0.90	1.00
UV Freeze-in ($n = 4$)	BB	0.21	0.06	1.98
	FB	0.21	0.06	2.04
	FF	0.21	0.05	2.10
UV Freeze-in ($n = 6$)	BB	-	-	-
	FB	-	-	-
	FF	-	-	-
Non-thermal UV Freeze-in		-3/2	2.5	2.6

Precision on transfer functions



Contribution to N_{eff} ?

$$\Delta N_{\text{eff}} = \frac{8}{7}\left(\frac{T}{T_\nu}\right)^4 \frac{\rho_\chi - m_{\text{DM}} n_\chi}{\rho_\gamma}$$

$$= \frac{8\pi\Omega_\chi}{7\Omega_\gamma}\left(\frac{g_{*s}(T)}{g_{*s}^0}\right)^{4/3}\left(\frac{T}{T_\nu}\right)^4\left(\frac{T_\star}{m_{\text{DM}}}\right)$$

$$\times\left[\left\langle\sqrt{q^2 + \left(\frac{g_{*s}^0}{g_{*s}(T)}\right)^{2/3}\left(\frac{m_{\text{DM}}}{T_\star}\right)^2\left(\frac{T_0}{T}\right)^2}\right\rangle - \left(\frac{g_{*s}^0}{g_{*s}(T)}\right)^{1/3}\left(\frac{m_{\text{DM}}}{T_\star}\right)\left(\frac{T_0}{T}\right)\right].$$

- **Saturating** the Lyman-alpha bound gives

$$\Delta N_{\text{eff,max}} \simeq \frac{1.4\times 10^{-4}}{\sqrt{\langle q^2\rangle}}\left(\frac{g_{*s}(T)}{g_{*s}^0}\right)^{4/3}\left(\frac{\Omega_\chi h^2}{0.1}\right)\left(\frac{3\,\text{keV}}{m_{\text{WDM}}}\right)^{4/3}\left(\frac{T}{T_\nu}\right)^4$$
$$\times\left[\left\langle\sqrt{q^2+\mu_*(T)^2}\right\rangle - \mu_*(T)\right],$$

$$\mu_*(T) \equiv \sqrt{\langle q^2\rangle}\left(\frac{g_{*s}^0}{g_{*s}(T)}\right)^{1/3}\left(\frac{3\,\text{keV}}{m_{\text{WDM}}}\right)^{4/3}\left(\frac{7.56\,\text{keV}}{T}\right).$$

$$\boxed{\Delta N_{\text{eff}}(T_{\text{BBN}}) \lesssim 5.4\times 10^{-4}\left(\frac{\langle q\rangle}{\sqrt{\langle q^2\rangle}}\right)\left(\frac{\Omega_\chi h^2}{0.1}\right)\left(\frac{3\,\text{keV}}{m_{\text{WDM}}}\right)^{4/3},}$$