Scattering of Test Fields in the Interior of Black Holes

Mokdad (Rémi) Mokdad
IMB – Univeristé de Bourgogne

Collaborators:
D. Häfner (IF – UGA), J.-P. Nicolas (LMBA – UBO), R. Nasser (ETH – Zürich)

Théorie, Univers et Gravitation
Institut Henri Poincaré – Paris
Dec. 13, 2021

- Einstein-Maxwell coupled system:
  - Coupled equations:

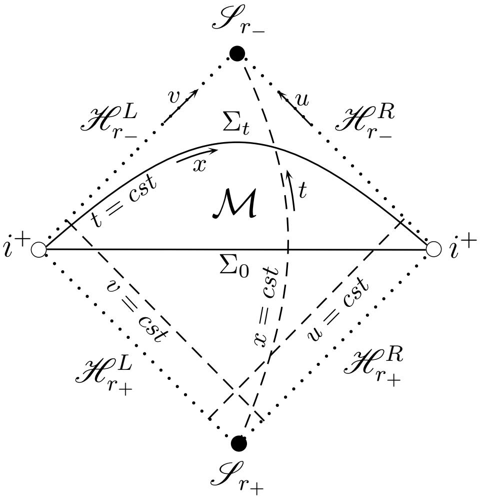
$$\begin{cases} \mathbf{G_{ab}} + \Lambda g_{ab} = 8\pi \left(\frac{1}{4} g_{ab} F^{cd} F_{cd} - F_{ac} F_b{}^c\right); \\ \nabla^a F_{ab} = 0 \quad ; \quad \nabla_{[a} F_{bc]} = 0 \end{cases}$$

  - A family of spherically symmetric solutions: RN(A)dS black hole spacetime, $\mathcal{M} = \mathbb{R}_t \times]0, +\infty[_r \times \mathcal{S}^2_{\theta,\varphi}$

$$g = f(r)\mathrm{d}t^2 - \frac{1}{f(r)}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2 \,,$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \Lambda r^2 \,;.$$

The General model

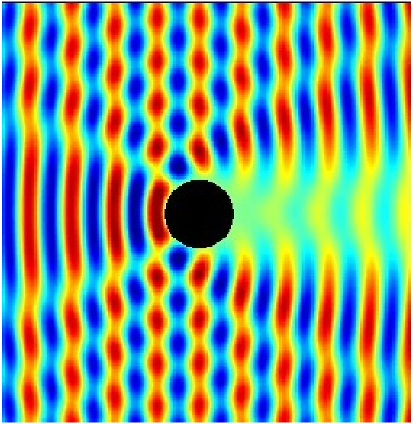


The interior between the Cauchy and the event horizons.

Scattering

General Idea:

$$\text{Past Profile} \xleftrightarrow{\text{Scattering Operator}} \text{Future Profile}$$



- 1^{st} Approach – Stationary: Via the transmission and reflection coefficients that gives the scattering matrix.

$$\text{Dynamic in time} \xrightarrow{\text{Fourier}} \text{Stationary: fixed frequency}$$

Approaches for Scattering

- 1^{st} Approach – Stationary: Via the transmission and reflection coefficients that gives the scattering matrix.

$$\text{Dynamic in time} \xrightarrow{\text{Fourier}} \text{Stationary: fixed frequency}$$

- 2^{nd} Approach – Dynamic: Via the wave operators.

$$W^{\pm} = s-\lim_{t\to\pm\infty} U(0,t)U_0(t,0) \quad ; \quad \Omega^{\pm} = s-\lim_{t\to\pm\infty} U_0(0,t)U(t,0)\,.$$

Scattering Operator $S = \Omega^+ W^-$.

- 1^{st} Approach – Stationary: Via the transmission and reflection coefficients that gives the scattering matrix.

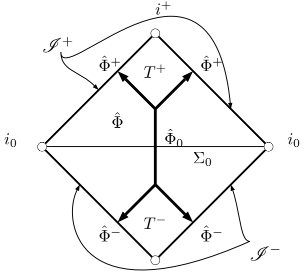
$$\text{Dynamic in time} \xrightarrow{\text{Fourier}} \text{Stationary: fixed frequency}$$

- 2^{nd} Approach – Dynamic: Via the wave operators.

$$W^{\pm} = s-\lim_{t\to\pm\infty} U(0,t)U_0(t,0) \quad ; \quad \Omega^{\pm} = s-\lim_{t\to\pm\infty} U_0(0,t)U(t,0)\,.$$

Scattering Operator $S = \Omega^+ W^-$.

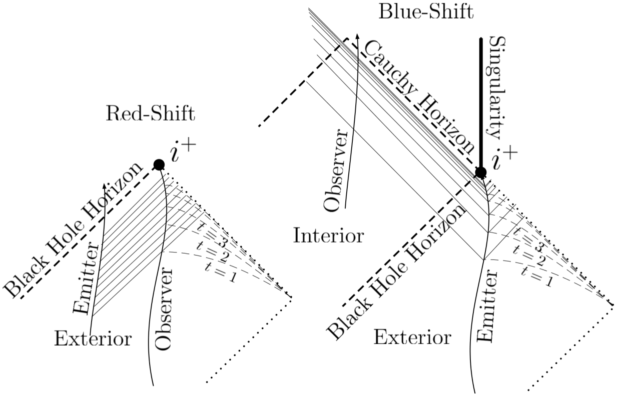
- 3^{rd} Approach – Geometric: Via the trace operators. Rescale and compactify, then take "traces".
Scattering operator $S = T^+(T^-)^{-1}$



Historical Context of Scattering Inside BHs

and the motivation by the cosmic censorship conjecture

- R. Penrose and M. Simpson (1973): numerical observation of blue shift inside RN black hole at the Cauchy horizon.



Historical Context of Scattering Inside BHs

and the motivation by the cosmic censorship conjecture

- R. Penrose and M. Simpson (1973): numerical observation of blue shift inside RN black hole at the Cauchy horizon.
- S. Chandrasekhar and J.B. Hartle (1982): investigated a blow-up for perturbations of linearized gravity in RN using "stationary" scattering.

- R. Penrose and M. Simpson (1973): numerical observation of blue shift inside RN black hole at the Cauchy horizon.
- S. Chandrasekhar and J.B. Hartle (1982): investigated a blow-up for perturbations of linearized gravity in RN using "stationary" scattering.
- C. Kehle and Y. Shlapentokh-Rothman (2019): scattering for Linear waves inside a RN *directly* between the two horizons with "stationary" scattering.

Historical Context of Scattering Inside BHs

and the motivation by the cosmic censorship conjecture

- R. Penrose and M. Simpson (1973): numerical observation of blue shift inside RN black hole at the Cauchy horizon.
- S. Chandrasekhar and J.B. Hartle (1982): investigated a blow-up for perturbations of linearized gravity in RN using "stationary" scattering.
- C. Kehle and Y. Shlapentokh-Rothman (2019): scattering for Linear waves inside a RN *directly* between the two horizons with "stationary" scattering.
- D. Häfner, J.P. Nicolas, and M.M. (2020): scattering for charged and massive Dirac fields inside RN(A)dS.

- R. Penrose and M. Simpson (1973): numerical observation of blue shift inside RN black hole at the Cauchy horizon.
- S. Chandrasekhar and J.B. Hartle (1982): investigated a blow-up for perturbations of linearized gravity in RN using “stationary” scattering.
- C. Kehle and Y. Shlapentokh-Rothman (2019): scattering for Linear waves inside a RN *directly* between the two horizons with “stationary” scattering.
- D. Häfner, J.P. Nicolas, and M.M. (2020): scattering for charged and massive Dirac fields inside RN(A)dS.
- M.M. (2021): conformal approach for Dirac inside RN(A)dS using the waves re-interpretation method.

Historical Context of Scattering Inside BHs

and the motivation by the cosmic censorship conjecture

- R. Penrose and M. Simpson (1973): numerical observation of blue shift inside RN black hole at the Cauchy horizon.
- S. Chandrasekhar and J.B. Hartle (1982): investigated a blow-up for perturbations of linearized gravity in RN using "stationary" scattering.
- C. Kehle and Y. Shlapentokh-Rothman (2019): scattering for Linear waves inside a RN *directly* between the two horizons with "stationary" scattering.
- D. Häfner, J.P. Nicolas, and M.M. (2020): scattering for charged and massive Dirac fields inside RN(A)dS.
- M.M. (2021): conformal approach for Dirac inside RN(A)dS using the waves re-interpretation method.
- M.M. and R. Nasser (2021): scattering breakdown for linear waves between the horizons and an intermediate hypersurface inside RN(A)dS.

Dirac Equation

Charged and massive Dirac equations:

$$\begin{cases} \left(\nabla^{AA'} - iqA^{AA'}\right)\phi_A & = \frac{m}{\sqrt{2}}\chi^{A'} \, , \\ \left(\nabla_{AA'} - iqA_{AA'}\right)\chi^{A'} & = -\frac{m}{\sqrt{2}}\phi_A \, , \end{cases}$$

Describes the behaviour of:

- Massive (mass m)
- Charged (charge q)
- Spin-$\frac{1}{2}$ particle (a bi-spinor (ϕ, χ))
- In an ambient electrostatic field with potential A.

Dirac Equation

Charged and massive Dirac equations:

$$\begin{cases} (\nabla^{AA'} - iqA^{AA'})\phi_A & = \frac{m}{\sqrt{2}}\chi^{A'} , \\ (\nabla_{AA'} - iqA_{AA'})\chi^{A'} & = -\frac{m}{\sqrt{2}}\phi_A , \end{cases}$$

Describes the behaviour of:

- Massive (mass m)
- Charged (charge q)
- Spin-$\frac{1}{2}$ particle (a bi-spinor (ϕ, χ))
- In an ambient electrostatic field with potential A.

Dirac Current:

- Current: Causal vector field $J^a = \phi^A \bar{\phi}^{A'} + \bar{\chi}^A \chi^{A'}$.

Dirac Equation

Charged and massive Dirac equations:

$$\begin{cases} \left(\nabla^{AA'} - iqA^{AA'}\right)\phi_A & = \frac{m}{\sqrt{2}}\chi^{A'} \,, \\ \left(\nabla_{AA'} - iqA_{AA'}\right)\chi^{A'} & = -\frac{m}{\sqrt{2}}\phi_A \,, \end{cases}$$

Describes the behaviour of:

- Massive (mass m)
- Charged (charge q)
- Spin-$\frac{1}{2}$ particle (a bi-spinor (ϕ, χ))
- In an ambient electrostatic field with potential A.

Dirac Current:

- Current: Causal vector field $J^a = \phi^A \bar{\phi}^{A'} + \bar{\chi}^A \chi^{A'}$.
- J is divergence-free: Dirac equations $\implies \nabla^a J_a = 0$.

Dirac Equation

Charged and massive Dirac equations:

$$\begin{cases} (\nabla^{AA'} - iqA^{AA'})\phi_A &= \frac{m}{\sqrt{2}}\chi^{A'} \, , \\ (\nabla_{AA'} - iqA_{AA'})\chi^{A'} &= -\frac{m}{\sqrt{2}}\phi_A \, , \end{cases}$$

Describes the behaviour of:

- Massive (mass m)
- Charged (charge q)
- Spin-$\frac{1}{2}$ particle (a bi-spinor (ϕ, χ))
- In an ambient electrostatic field with potential A.

Dirac Current:

- Current: Causal vector field $J^a = \phi^A \bar{\phi}^{A'} + \bar{\chi}^A \chi^{A'}$.
- J is divergence-free: Dirac equations $\implies \nabla^a J_a = 0$.
- A Hilbert space of initial data $\mathcal{H}$ with a conserved L^2-norm.

Dirac Equation

Charged and massive Dirac equations:

$$\begin{cases} \left(\nabla^{AA'} - iqA^{AA'}\right)\phi_A & = \frac{m}{\sqrt{2}}\chi^{A'} \,, \\ \left(\nabla_{AA'} - iqA_{AA'}\right)\chi^{A'} & = -\frac{m}{\sqrt{2}}\phi_A \,, \end{cases}$$

Describes the behaviour of:

- Massive (mass m)
- Charged (charge q)
- Spin-$\frac{1}{2}$ particle (a bi-spinor (ϕ, χ))
- In an ambient electrostatic field with potential A.

Dirac Current:

- Current: Causal vector field $J^a = \phi^A \bar{\phi}^{A'} + \bar{\chi}^A \chi^{A'}$.
- J is divergence-free: Dirac equations $\implies \nabla^a J_a = 0$.
- A Hilbert space of initial data $\mathcal{H}$ with a conserved L^2-norm.
- Unitary evolution.

Scattering Theory for Dirac Fields

Theorem (D.Häfner , J.-P. Nicolas , M.M. – 2020)

The future and past direct wave operators $W^{\pm}$ and inverse wave operators $\Omega^{\pm}$ are well-defined on $\mathcal{H}$ as the strong limits:

$$W^{\pm} = s - \lim_{t \to \pm\infty} \mathcal{U}(0,t) e^{itH_0^{\pm}},$$

$$\Omega^{\pm} = s - \lim_{t \to \pm\infty} e^{-itH_0^{\pm}} \mathcal{U}(t,0).$$

They are unitary operators on $\mathcal{H}$. Moreover,

$$W^{\pm}\Omega^{\pm} = \Omega^{\pm}W^{\pm} = \mathrm{Id}_{\mathcal{H}}.$$

Scattering Theory for Dirac Fields

Theorem (D.Häfner , J.-P. Nicolas , M.M. – 2020)

The future and past direct wave operators $W^{\pm}$ and inverse wave operators $\Omega^{\pm}$ are well-defined on $\mathcal{H}$ as the strong limits:

$$W^{\pm} = s - \lim_{t\to\pm\infty} \mathcal{U}(0,t) e^{itH_0^{\pm}},$$

$$\Omega^{\pm} = s - \lim_{t\to\pm\infty} e^{-itH_0^{\pm}} \mathcal{U}(t,0).$$

They are unitary operators on $\mathcal{H}$. Moreover,

$$W^{\pm}\Omega^{\pm} = \Omega^{\pm}W^{\pm} = \mathrm{Id}_{\mathcal{H}}.$$

The scattering operator $S = \Omega^{+}W^{-}$ is a unitary operator on $\mathcal{H}$.

Scattering Theory for Dirac Fields

Theorem (D.Häfner , J.-P. Nicolas , M.M. – 2020)

The future and past direct wave operators $W^\pm$ *and inverse wave operators* $\Omega^\pm$ *are well-defined on* $\mathcal{H}$ *as the strong limits:*

$$W^\pm = s - \lim_{t \to \pm\infty} \mathcal{U}(0,t) e^{itH_0^\pm},$$

$$\Omega^\pm = s - \lim_{t \to \pm\infty} e^{-itH_0^\pm} \mathcal{U}(t,0).$$

They are unitary operators on $\mathcal{H}$. *Moreover,*

$$W^\pm \Omega^\pm = \Omega^\pm W^\pm = \mathrm{Id}_{\mathcal{H}}.$$

The scattering operator $S = \Omega^+ W^-$ is a unitary operator on $\mathcal{H}$.

Theorem (M.M. – 2021)

The equivalent geometric (conformal) Scattering theory via trace operators using the waves re-interpretation method.

Wave Equation

The geometric wave equation:

$$\Box_g \phi = 0.$$

In (t, x, θ, φ) coordinates:

$$\Box_g = \nabla^a \nabla_a = \frac{1}{f}(\partial_x^2 - \partial_t^2) - \frac{2}{r}\partial_t - \frac{1}{r^2}\Delta_{\mathcal{S}^2}$$

The energy-momentum tensor

$$\mathbf{T}_{ab} := \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \, .$$

It satisfies:

- Divergence-free: $\nabla^a \mathbf{T}_{ab} = \nabla_b \phi \Box \phi$ so, $\Box_g \phi \implies \nabla^a \mathbf{T}_{ab} = 0.$
- Dominant Energy Cond. : X and Y causal $\implies \mathbf{T}_{ab} X^a Y^b \geq 0.$

For X a vector field (observer) and S a hypersurface, we can define the "Energy" of a wave ϕ:
Let $J^a = \mathbf{T}^{ab} X_b$,

$$\mathcal{E}_X[\phi](S) := \int_S i_J \mathrm{dVol}_{\mathbf{g}},$$

For X a vector field (observer) and S a hypersurface, we can define the "Energy" of a wave ϕ:
Let $J^a = \mathbf{T}^{ab} X_b$,

$$\mathcal{E}_X[\phi](S) := \int_S i_J \mathrm{dVol}_{\mathbf{g}},$$

- If X is timelike and S is spacelike, $\mathcal{E}$ is definite positive (by D.O.E.), and can be used as a norm on ϕ.
- If X is Killing, $\mathcal{E}$ is conserved (by Stokes' theorem).

For X a vector field (observer) and S a hypersurface, we can define the "Energy" of a wave ϕ:
Let $J^a = \mathbf{T}^{ab} X_b$,

$$\mathcal{E}_X[\phi](S) := \int_S i_J \mathrm{dVol}_{\mathbf{g}},$$

- If X is timelike and S is spacelike, $\mathcal{E}$ is definite positive (by D.O.E.), and can be used as a norm on ϕ.
- If X is Killing, $\mathcal{E}$ is conserved (by Stokes' theorem).

However, no timelike Killing vector field inside the black hole!
Therefore, no energy norm is conserved...

Choose X to be $T := \partial_t$.

$$\begin{aligned}\mathcal{E}[\phi](t) : &= \mathcal{E}_T[\phi](\Sigma_t) = \int_{\Sigma_t} \mathbf{T}_{00} r^2 \mathrm{d}x \wedge \mathrm{d}\omega^2 \\ &= \frac{1}{2} \int_{\mathbb{R}_x \times \{t\} \times \mathcal{S}^2_\omega} \left((\partial_t \phi)^2 + (\partial_x \phi)^2 - \frac{f}{r^2} |\nabla_{\mathcal{S}^2} \phi|^2 \right) r^2 \mathrm{d}x \mathrm{d}^2\omega\end{aligned}$$

T extends smoothly and becomes normal to the horizons:

$$\mathcal{E}_T[\phi](\mathscr{H}^R_{r_-}) = \int_{\mathscr{H}^R_{r_-}} \mathbf{T}_{00} r^2 \mathrm{d}u \wedge \mathrm{d}\omega^2 = \int_{\mathbb{R}_u \times \{r_-\} \times \mathcal{S}^2} (\partial_u \phi)^2 r_-^2 \mathrm{d}u \mathrm{d}^2\omega,$$

$$\mathcal{E}_T[\phi](\mathscr{H}^L_{r_-}) = \int_{\mathscr{H}^L_{r_-}} \mathbf{T}_{00} r^2 \mathrm{d}v \wedge \mathrm{d}\omega^2 = \int_{\mathbb{R}_v \times \{r_-\} \times \mathcal{S}^2} (\partial_v \phi)^2 r_-^2 \mathrm{d}v \mathrm{d}^2\omega,$$

Scattering of waves directly between the two horizons

Theorem (C. Kehle, Y. Shlapentokh-Rothman – 2019)

For data Φ_- on $\mathscr{H}_{r_+}$, let be ϕ the corresponding solution to the wave equation. The scattering map $S : \mathcal{H}^- \to \mathcal{H}^+$ given by $S(\Phi_-) = \phi|_{\mathscr{H}_{r_-}}$ is a Hilbert space isomorphism.
In more traditional language, the Theorem yields existence, uniqueness, and asymptotic completeness of scattering states.

Theorem (M.M. , R. Nasser – 2021)

For data Φ_0 on Σ_0, let be ϕ the corresponding solution to the wave equation. The trace maps $T^{\pm} : \mathcal{H}(0) \to \mathcal{H}^{\pm}$ given by $T^{\pm}(\Phi_0) = \phi|_{\mathscr{H}_{r_{\pm}}}$ are linear bounded operators ***but have unbounded inverses*** *since: there exists a sequence $(\phi_n^{\pm})_n$ of solutions to the wave equation such that $\phi_n^{\pm}|_{\Sigma_0} \in \mathcal{C}_c^{\infty}(\Sigma_0)$ and $\mathcal{E}[\phi_n^{\pm}](0) = 1$ for all n, and*

$$\lim_{n\to\infty} \lim_{t\to\pm\infty} \mathcal{E}[\phi_n^{\pm}](t) = 0.$$

Theorem (M.M. , R. Nasser – 2021)

For data Φ_0 on Σ_0, let be ϕ the corresponding solution to the wave equation. The trace maps $T^\pm : \mathcal{H}(0) \to \mathcal{H}^\pm$ given by $T^\pm(\Phi_0) = \phi|_{\mathscr{H}_{r_\pm}}$ are linear bounded operators ***but have unbounded inverses*** *since: there exists a sequence $(\phi_n^\pm)_n$ of solutions to the wave equation such that $\phi_n^\pm|_{\Sigma_0} \in \mathcal{C}_c^\infty(\Sigma_0)$ and $\mathcal{E}[\phi_n^\pm](0) = 1$ for all n, and*

$$\lim_{n\to\infty} \lim_{t\to\pm\infty} \mathcal{E}[\phi_n^\pm](t) = 0.$$

The breakdown happens only at high angular momentum and zero *spatial* frequency!

Questions for physicists!

- How come there is scattering for scalar waves between the horizons but not between the horizons and an intermediate hypersurface?!

Questions for physicists!

- How come there is scattering for scalar waves between the horizons but not between the horizons and an intermediate hypersurface?!
- What could be the physical mechanism causing such a breakdown?

Questions for physicists!

- How come there is scattering for scalar waves between the horizons but not between the horizons and an intermediate hypersurface?!
- What could be the physical mechanism causing such a breakdown?
- Can this breakdown be related to blue-shift effects even though it is on both horizons?

Questions for physicists!

- How come there is scattering for scalar waves between the horizons but not between the horizons and an intermediate hypersurface?!
- What could be the physical mechanism causing such a breakdown?
- Can this breakdown be related to blue-shift effects even though it is on both horizons?
- Is the zero spatial frequency a “resonance” ? And how to define resonances inside black holes?

Thank you