New exact cylindrically symmetric solutions of GR

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Outline

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Introduction

Cylindrical analytic solutions: the pioneers

Cylindrically symmetric vacuum (exterior)

- 1919 Levi-Civita: vacuum static cylindrical spacetimes
- 1932 Lewis: vacuum exterior gravitationally sourced by an infinite cylinder of stationary matter rotating around its symmetry axis
- 1937 Einstein-Rosen: time-dependent vacuum exterior

Cylindrically symmetric source (interior)

- 1924 Lanczos (1937 van Stockum): stationary rigidly rotating infinite dust cylinder with (Lanczos) and without (Lanczos and van Stockum) a cosmological constant
- and more: static, stationary, (non) rigidly rotating, dust, isotropic perfect fluid, thin shells, etc. But no known exact solution for anisotropic fluid

Stationary sources: a new approach

Continuation of a couple of previous works:

- Preliminary study of rigidly rotating stationary cylindrical anisotropic fluids Debbasch, Herrera, Pereira, Santos (2006) GRG **38**, 1825
- Generalization to non-rigid rotation and deeper analysis of rigid rotation with an emphasis on gravito-electromagnetic properties MNC, Santos (2020) PRD **102**, 044026

New results: Exact interior solutions for a rigidly rotating fluid with anisotropic pressure bounded by a cylindrical hypersurface and matched to an exterior Lewis-Weyl vacuum:

- Purely axial pressure MNC (2021) PRD **104**, 064040
- Purely azimuthal pressure MNC (2021) arXiv 2111.13938

General framework

Cylindrical spacetime inside the source

The general stress-energy tensor

$$T_{\alpha\beta} = (\rho + P_r)V_\alpha V_\beta + P_r g_{\alpha\beta} + (P_\phi - P_r)K_\alpha K_\beta + (P_z - P_r)S_\alpha S_\beta$$

$$V^\alpha V_\alpha = -1, \quad K^\alpha K_\alpha = S^\alpha S_\alpha = 1, \quad V^\alpha K_\alpha = V^\alpha S_\alpha = K^\alpha S_\alpha = 0$$

The stationary cylindrically symmetric line element

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + 2k\mathrm{d}t\mathrm{d}\phi + \mathrm{e}^\mu(\mathrm{d}r^2 + \mathrm{d}z^2) + l\mathrm{d}\phi^2,$$

$$-\infty \leq t \leq +\infty, \quad 0 \leq r, \quad -\infty \leq z \leq +\infty, \quad 0 \leq \phi \leq 2\pi$$

Stationarity + cylindrical symmetry = metric coefficients functions of the radial coordinate only

Purely axial pressure solution

Equation of state and 4-velocity of the fluid

Equation of state

$$P_r = P_\phi = 0 \qquad \frac{P_z}{\rho} = h(r)$$

Corotating frame

$$V^\alpha = v\delta^\alpha_0 \qquad fv^2 = 1$$

$$S^\alpha = \mathrm{e}^{-\mu/2}\delta^\alpha_2$$

Intermediate function: $D^2 = fl + k^2$

Field equations

$$-f\mu'' - 2f\frac{D''}{D} + f'' - f'\frac{D'}{D} + \frac{3f(f'l' + k'^2)}{2D^2} = 2\kappa\rho f \mathrm{e}^{\mu}$$

$$k\mu'' + 2k\frac{D''}{D} - k'' + k'\frac{D'}{D} - \frac{3k(f'l' + k'^2)}{2D^2} = -2\kappa\rho k \mathrm{e}^{\mu}$$

$$\frac{\mu' D'}{2D} + \frac{f'l' + k'^2}{4D^2} = 0$$

$$\frac{D''}{D} - \frac{\mu' D'}{2D} - \frac{f'l' + k'^2}{4D^2} = \kappa P_z \mathrm{e}^{\mu}$$

$$l\mu'' + 2l\frac{D''}{D} - l'' + l'\frac{D'}{D} - \frac{3l(f'l' + k'^2)}{2D^2} = 2\kappa\rho\frac{k^2 \mathrm{e}^{\mu}}{f}$$

Bianchi identity

$$T^{\beta}_{1;\beta} = \frac{1}{2}\rho\frac{f'}{f} - \frac{1}{2}P_z\mu' = 0$$

$$\frac{f'}{f} - h\mu' = 0$$

Solving the field equations

Particular choice of coordinates issuing from the degree of gauge freedom

$$\mu' = \frac{2h'}{1-h} + \frac{2h'}{h} \qquad\qquad \mathrm{e}^{\mu} = c_\mu \frac{h^2}{(1-h)^2}$$

Inserted into Bianchi

$$\frac{f'}{f} = \frac{2h'}{1-h} \qquad\qquad f = \frac{c_f}{(1-h)^2}$$

G_{00} combined with G_{03} $\qquad kf' - fk' = 2cD,$

$$k = \frac{c_f}{(1-h)^2}\left[c_0 - \frac{2c}{c_f^2}\int_{r_0}^{r}(1-h(v))^4\, D(v)\mathrm{d}v\right]$$

Interim results

$$D^2 = c_f^2 c_k^2 \frac{(-2\ln h + 4h - h^2 + c_\beta)}{(1-h)(1+h)},$$

$$k = \frac{c_f}{(1-h)^2}\left[c_0 + c_k(-2\ln h + 4h - h^2 + c_\beta)\right],$$

$$l = c_f c_k^2 \frac{(1-h)}{(1+h)}(-2\ln h + 4h - h^2 + c_\beta) - \frac{c_f}{(1-h)^2}\left[c_0 + c_k(-2\ln h + 4h - h^2 + c_\beta)\right]^2.$$

Axisymmetry and regularity conditions

From axisymmetry and to ensure elementary flatness in the vicinity of the axis of rotation

$$l \overset{0}{=} 0, \qquad f \overset{0}{=} \mathrm{e}^{\mu} \overset{0}{=} 1, \qquad k \overset{0}{=} 0, \qquad D \overset{0}{=} 0, \qquad l' \overset{0}{=} 0.$$

$$f' \overset{0}{=} k' \overset{0}{=} k'' - k' \frac{D'}{D} \overset{0}{=} 0$$

Implying

$$c_f \overset{0}{=} (1-h)^2, \qquad c_\mu \overset{0}{=} \frac{c_f}{h^2}, \qquad c_0 = 0, \qquad c_\beta \overset{0}{=} 2\ln h - 4h + h^2.$$

Junction conditions through Σ

Stationary system = the Weyl class of the Lewis metric used to represent the exterior vacuum spacetime

$$\mathrm{d}s^2 = -F\mathrm{d}t^2 + 2K\mathrm{d}t\mathrm{d}\phi + \mathrm{e}^M(\mathrm{d}R^2 + \mathrm{d}z^2) + L\mathrm{d}\phi^2$$

$$F = aR^{1-n} - a\delta^2 R^{1+n},$$

$$K = -(1 - ab\delta)\delta R^{1+n} - abR^{1-n},$$

$$\mathrm{e}^M = R^{(n^2-1)/2},$$

$$L = \frac{(1-ab\delta)^2}{a} R^{1+n} - ab^2 R^{1-n},$$

$$\delta = \frac{c}{an},$$

Darmois'junction conditions imply

$$P_r \stackrel{\Sigma}{=} 0$$

Final form of the solution

$$f = \left(\frac{1-h_0}{1-h}\right)^2, \qquad \mathrm{e}^{\mu} = \left(\frac{1-h_0}{h_0}\right)^2 \left(\frac{h}{1-h}\right)^2,$$

$$k = (1-h_0)^2 \frac{\left[2\ln\frac{h_0}{h} + 4(h-h_0) - (h^2-h_0^2)\right]}{(1-h)^2}$$

$$l = (1-h_0)^2 \left[2\ln\frac{h_0}{h} + 4(h-h_0) - (h^2-h_0^2)\right] \left\{\frac{1-h}{1+h} - \frac{\left[2\ln\frac{h_0}{h} + 4(h-h_0) - (h^2-h_0^2)\right]}{(1-h)^2}\right\}$$

$$r = \frac{(1-h_0)^2}{c} \int_{h_0}^{h} \left\{\frac{1+u}{u^2(1-u)^3\left[2\ln\frac{h_0}{u} + 4(u-h_0) - (u^2-h_0^2)\right]}\right\}^{\frac{1}{2}} \mathrm{d}u$$

$$\rho = \frac{2c^2h_0^2}{\kappa(1-h_0)^4} \frac{(1-h)^4}{h^2(1+h)^2} \left\{\frac{h\left[2\ln\frac{h_0}{h} + 4(h-h_0) - (h^2-h_0^2)\right]}{(1+h)} + 2(1-h)^2\right\}$$

Hydrodynamical scalars

Acceleration vector modulus

$$\dot{V}^\alpha \dot{V}_\alpha = \frac{c^2 h_0^2}{(1-h_0)^6} \frac{(1-h)^3}{(1+h)} \left[2 \ln \frac{h_0}{h} + 4(h-h_0) - (h^2 - h_0^2) \right]$$

Rotation scalar

$\omega^2 = \dfrac{c^2}{f^2 \mathrm{e}^\mu}$ with $f \overset{0}{=} \mathrm{e}^\mu \overset{0}{=} 1 \Rightarrow \omega^2 \overset{0}{=} c^2 \Rightarrow c =$ the amplitude of the vorticity on the axis

Shear and expansion

The shear and expansion vanish owing to rigid and stationary rotation

Singularities

First singularity $h = +1$

The whole metric set diverges. The density ρ vanishes and does not change sign = coordinate singularity.

Second singularity $h = -1$

The density ρ diverges and changes sign. Curvature singularity.

Third singularity $h = 0$

The density diverges and is no more defined for sign $h \neq$ sign h_0 . Curvature singularity.

Constraints from mathematical structure

Coming from: the signature of the metric

the definition domain of the logarithm function $\Rightarrow \frac{h_0}{h} > 0$

Imply two classes of solutions:

- $0 < h_0 < 1 \quad \Rightarrow \quad 0 < h_1 < h < h_2 < 1$
- $-1 < h_0 < 0 \quad \Rightarrow \quad -1 < h < h_3 < 0$

with h_1, h_2, h_3 depending on h_0 according to the relations:

$$(1-h_1)^3 - (1+h_1)\left[2\ln\frac{h_0}{h_1} + 4(h_1 - h_0) - (h_1^2 - h_0^2)\right] = 0$$

$$2\ln\frac{h_0}{h_2} + 4(h_2 - h_0) - (h_2^2 - h_0^2) = 0$$

$$(1-h_3)^3 - (1+h_3)\left[2\ln\frac{h_0}{h_3} + 4(h_3 - h_0) - (h_3^2 - h_0^2)\right] = 0$$

Parameters and constraints, a summary

The solution exhibits two independent constant parameters: h_0 and c

- h_0 is the value of $h(r) \equiv P_z(r)/\rho(r)$ on the axis of symmetry

$h_0 = 1$ implying the vanishing of all the metric functions is forbidden

- c measures the amplitude of the rotation scalar on the axis

Admissible ranges and singularities

The three singularities which are the limits of the ranges for h and h_0

$0 < h_0 < 1 \quad \Rightarrow \quad 0 < h_1 < h < h_2 < 1$ and $-1 < h_0 < 0 \quad \Rightarrow \quad -1 < h < h_3 < 0$

can thus be ignored.

Purely azimuthal pressure solution

Equation of state and 4-velocity of the fluid

Equation of state

$$P_r = P_z = 0 \qquad\qquad \frac{P_\phi}{\rho} = h(r)$$

Corotating frame

$$V^\alpha = v\delta^\alpha_0 \qquad\qquad fv^2 = 1$$

$$K^\alpha = -\frac{kv}{D}\delta^\alpha_0 - \frac{fv}{D}\delta^\alpha_3$$

Intermediate function

$$D^2 = fl + k^2$$

Field equations

$$-f\mu'' - 2f\frac{D''}{D} + f'' - f'\frac{D'}{D} + \frac{3f(f'l' + k'^2)}{2D^2} = 2\kappa\rho f \mathrm{e}^{\mu}$$

$$k\mu'' + 2k\frac{D''}{D} - k'' + k'\frac{D'}{D} - \frac{3k(f'l' + k'^2)}{2D^2} = -2\kappa\rho k \mathrm{e}^{\mu}$$

$$\frac{\mu' D'}{2D} + \frac{f'l' + k'^2}{4D^2} = 0$$

$$\frac{D''}{D} - \frac{\mu' D'}{2D} - \frac{f'l' + k'^2}{4D^2} = 0$$

$$l\mu'' + 2l\frac{D''}{D} - l'' + l'\frac{D'}{D} - \frac{3l(f'l' + k'^2)}{2D^2} = 2\frac{\kappa}{f}\left(\rho k^2 + P_\phi D^2\right)\mathrm{e}^{\mu}$$

Bianchi identity

$$T^{\beta}_{1;\beta} = \frac{1}{2}\left(\rho + P_\phi\right)\frac{f'}{f} - P_\phi\frac{D'}{D} = 0$$

$$\frac{1}{2}(1+h)\frac{f'}{f} - h\frac{D'}{D} = 0$$

Junction conditions

$$P_r \stackrel{\Sigma}{=} 0$$

Solving the field equations

$G_{11} + G_{22} \Rightarrow D'' = 0$ $\qquad$ $D = c_1 r + c_2 = r + c_2$

Particular choice of coordinates similar to that for the axial case

$$\frac{f'}{f} = \frac{2h'}{1-h} \Leftrightarrow f = \frac{c_f}{(1-h)^2}$$

Inserted into Bianchi

$$\frac{1+h}{h(1-h)}h' = \frac{D'}{D} \Leftrightarrow D = \frac{h}{c_5(1-h)^2}$$

G_{00} combined with G_{03} $\qquad$ $kf' - fk' = 2cD$

$$k = f\left(c_4 - 2c\int_0^r \frac{D(v)}{f(v)^2}\mathrm{d}v\right) - \frac{c_4}{f_0}$$

Interim results

$$k = \frac{c_f}{(1-h)^2}\left\{c_4 - \frac{2c}{c_f^2}\left[\frac{h_0^2 - h^2}{2} + 2(h_0 - h) + 2\ln\left(\frac{1-h_0}{1-h}\right)\right]\right\} - \frac{c_4}{f_0}$$

$$\begin{aligned} l \;=\; & \frac{h^2}{c_f(1-h)^2} - \frac{(1-h)^2}{c_f}\left\{\frac{c_f}{(1-h)^2}\left[c_4\right.\right. \\ & - \left.\left.\frac{2c}{c_f^2}\left(\frac{h_0^2 - h^2}{2} + 2(h_0 - h) + 2\ln\left(\frac{1-h_0}{1-h}\right)\right)\right] - \frac{c_4}{f_0}\right\}^2 \end{aligned}$$

$$\mathrm{e}^{\mu} = c_\mu \frac{(1-h)^{1+\frac{4c^2}{c_f^2}}}{(1+h)} \exp\left[\frac{c^2}{c_f^2} h(4+h)\right]$$

$$\frac{h}{(1-h)^2} = r + c_2 c_5$$

Regularity conditions

To ensure

- axisymmetry $l \stackrel{0}{=} 0$

- elementary flatness in the vicinity of the axis of rotation, add

$$f \stackrel{0}{=} \mathrm{e}^{\mu} \stackrel{0}{=} 1, \quad k \stackrel{0}{=} 0, \quad l' \stackrel{0}{=} 0.$$

Implying

$$c_4 = h_0 = 0 \Leftrightarrow P_\phi \stackrel{0}{=} 0, \qquad c_f = c_\mu = 1, \quad c_2 = 0.$$

The energy density

After rescaling the r coordinate from a factor $c_1 c_5$

$$r = D = \frac{h}{(1-h)^2} \qquad h = 1 + \frac{1}{2r} + \epsilon\sqrt{\frac{1}{r} + \frac{1}{4r^2}}$$ with $\epsilon = \pm 1$ and $r \neq 0$

Using them, together with the other intermediate results, into the field equations gives the energy density

$$\rho = \frac{2}{\kappa}(1-h)^{3-4c^2}\left[2c^2 + \frac{(1-h)}{h(1+h)^3}\right]\exp\left[-c^2 h(4+h)\right]$$

Behaviour of h(r)

First derivative of h wrt r

$$h' = -\frac{1}{2r^2}\left[1 + \epsilon\frac{1+2r}{\sqrt{1+4r}}\right]$$

$\epsilon > 0 \Rightarrow h < 0 \Rightarrow P_\phi < 0$ ruled out

$\epsilon < 0$ valid and implies $h > 0 \Rightarrow P_\phi > 0$

and h monotonically increasing from the axis to the boundary Σ

Final form of the solution

$$f = \frac{1}{(1-h)^2} \qquad e^{\mu} = \frac{(1-h)^{1+4c^2}}{(1+h)} \exp\left[c^2 h(4+h)\right]$$

$$k = \frac{c}{(1-h)^2}\left[h^2 + 4h + 4\ln(1-h)\right]$$

$$l = \frac{1}{(1-h)^2}\left\{h^2 - c^2\left[h^2 + 4h + 4\ln(1-h)\right]^2\right\}$$

$$\rho = \frac{2}{\kappa}(1-h)^{3-4c^2}\left[2c^2 + \frac{(1-h)}{h(1+h)^3}\right]\exp\left[-c^2 h(4+h)\right]$$

$$D = \frac{h}{(1-h)^2} = r \qquad h = 1 + \frac{1}{2r} - \sqrt{\frac{1}{r} + \frac{1}{4r^2}} \quad r \neq 0$$

Hydrodynamical scalars of the fluid

- Acceleration vector modulus

$$\dot{V}^{\alpha}\dot{V}_{\alpha} = \frac{(1-h)^{3-4c^2}}{1+h} \exp\left(-c^2 h(4+h)\right)$$

- Rotation scalar

$$\omega^2 = c^2 (1-h)^{3-4c^2} (1+h) \exp\left(-c^2 h(4+h)\right)$$

- $h = h_0 = 0 \Rightarrow \omega^2 \overset{0}{=} c^2 \Rightarrow c =$ the amplitude of the vorticity on the axis
- Shear and expansion

The shear and expansion vanish owing to rigid and stationary rotation

Sign constraints and metric signature

These constraints arise from:

- the presence of $\ln(1-h)$ in the metric functions
- the weak energy condition $\rho > 0$
- the signature of the metric $(-+++)$

Table 1. Sign constraints			
h	0	$h > 0$	$+1$
c	$c \to \infty$	$c > 0$ $c^2 < \frac{h^2}{\left[h^2+4h+4\ln(1-h)\right]^2}$	$c \to 0$

Singularities

Two possible singular loci:

- $h = +1$ where r diverges $\Rightarrow$ not reached inside the bounded cylinder

- $h = 0$ mere coordinate singularity

Comparison of both cases, axial and azimuthal

- Three over five field equations are strictly the same.
- Same gauge choice = same f metric function.
- However, we end up with very different other metric functions.
- P_ϕ vanishes on the axis, not P_z.
- The modulus of the acceleration vector vanishes on the axis for axial pressure, it is unity for azimuthal one.
- The amplitude of the rotation scalar is the absolute value of c.
- The pressure can be negative for axial, only positive for azimuthal.
- Three possible singular loci for axial, only two for azimuthal. But different nature and relevance.
- No dust limit in any case since h=const., hence h=0, is excluded.

Applications to astrophysical systems

Purely axial pressure alone:

- In rigidly rotating stationary cylindrical dust, confinement of test particles occurs in the radial direction, while motion in the axial direction is free. Proposal: relevance to extragalactic jet formation Opher, Santos, Wang (1996) in the axis direction.

Anisotropic pressure in any direction – axial or azimuthal:

- Rotation can halt cylindrical relativistic gravitational collapse Apostolatos, Thorne (1992) Pressure with vanishing radial component = negligible influence on radial collapse. Conjecture: such spacetimes = final stage of collapsing cylindrical fluids.
- The finding of these new solutions is a first step towards exploring fluid anisotropy in cylindrical symmetry, e.g., starting point for perturbative or numerical approaches.

Conclusions

Gravitational sources: Stationary rigidly rotating cylindrical anisotropic fluids with pressure directed along the axis of symmetry or azimuthally

New exact solutions of the GR field equations: Interior source matched to an exterior stationary Lewis vacuum of the Weyl (real) class

Analysed physical properties:

- Hydrodynamical tensors, vectors, scalars
- Singularities and definition intervals
- Interpretation of the constant parameters and discussion of the solutions as a whole
- Applications to astrophysical systems

Future works: find other exact solutions for, e.g., purely radiative pressure and/or non rigidly rotating fluids