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Hamiltonian formalism for cosmological perturbations: the separate-universe approach

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Introduction

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- It is only formulated for slow-roll inflation or ultra-slow roll [Pattison et al. (2019)]. Generalizing it beyond slow roll may be easier with a Hamiltonian formulation.
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However, the stochastic formalism happens to be very restrictive:

- It is only formulated for slow-roll inflation or ultra-slow roll [Pattison et al. (2019)]. Generalizing it beyond slow roll may be easier with a Hamiltonian formulation.
- It relies on the so-called separate-universe approach which assumes that isotropic and anisotropic d.o.f. evolve independently.
- It requires a gauge transformation from the spatially-flat gauge to the uniform-expansion gauge (a.k.a. uniform-$\mathcal{N}$ gauge) where the noise is computed.

Introduction

Three questions:

- How to formulate the separate-universe approach in a Hamiltonian setting?
- What is the domain of validity of the separate-universe approach (and of the stochastic formalism)?
- How to fix a gauge in separate universe? [work in progress]

Hamiltonian formalism for general relativity

Total (Hamiltonian) constraint of general relativity [Langlois (1994)]:

$$C\left[N, N^i\right] = \int \mathrm{d}^3\vec{x} \left[N\left(\mathcal{S}^{(\mathrm{G})} + \mathcal{S}^{(\varphi)}\right) + N^i\left(\mathcal{D}_i^{(\mathrm{G})} + \mathcal{D}_i^{(\varphi)}\right)\right] \tag{1}$$

where N is the lapse function and N^i is the shift vector.

Invariance of the theory under time reparametrisation is ensured by the scalar constraint:

$$\mathcal{S}^{(\mathrm{G})} + \mathcal{S}^{(\varphi)} = 0\,, \tag{2}$$

Invariance under space reparametrisation is ensured by the diffeomorphism constraint:

$$\mathcal{D}_i^{(\mathrm{G})} + \mathcal{D}_i^{(\varphi)} = 0\,. \tag{3}$$

Cosmological perturbation theory (CPT)

They depend on fields:

$$\begin{cases} \varphi(\tau,\vec{x}) = \delta\varphi(\tau,\vec{x}) + \varphi(\tau), \\ \pi_\varphi(\tau,\vec{x}) = \delta\pi_\varphi(\tau,\vec{x}) + \pi_\varphi(\tau). \end{cases} \tag{4}$$

$$\begin{cases} \gamma_{ij}(\tau,\vec{x}) = \delta\gamma_{ij}(\tau,\vec{x}) + \gamma_{ij}(\tau), \\ \pi^{ij}(\tau,\vec{x}) = \delta\pi^{ij}(\tau,\vec{x}) + \pi^{ij}(\tau), \end{cases} \tag{5}$$

$$\begin{cases} N(\tau,\vec{x}) = \delta N(\tau,\vec{x}) + N(\tau), \\ N^i(\tau,\vec{x}) = \delta N^i(\tau,\vec{x})\,. \end{cases} \tag{6}$$

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We will consider the scalar sector of perturbations:

$$\delta N^i(\tau,\vec{k}) = i\frac{k^i}{k}\delta N_1(\tau,\vec{k})\,, \qquad \mathcal{D}_i(\tau,\vec{k}) = ik_i\mathcal{D}(\tau,\vec{k})\,. \tag{7}$$

Define an orthonormal basis $\left(M^1_{ij}, M^2_{ij}\right)$ such that:

$$\delta\gamma_{ij}(\tau,\vec{k}) = \delta\gamma_1(\tau,\vec{k})M^1_{ij} + \delta\gamma_2(\tau,\vec{k})M^2_{ij}(\vec{k})\,, \tag{8}$$

$$\delta\pi^{ij}(\tau,\vec{k}) = \delta\pi_1(\tau,\vec{k})M^{ij}_1 + \delta\pi_2(\tau,\vec{k})M^{ij}_2(\vec{k})\,. \tag{9}$$

We choose this basis such that $(\delta\gamma_1, \delta\pi_1)$ represents the purely isotropic part of gravitational perturbations and $(\delta\gamma_2, \delta\pi_2)$ the anisotropic one.

Cosmological perturbation theory (CPT) & Separate Universe (SU)

$$C \simeq C^{(0)} + C^{(1)} + C^{(2)} \,, \tag{10}$$

$C^{(0)}$ generates the dynamics of the FLRW background.
$C^{(1)}$ vanishes identically.
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Consider large scales $\rightarrow C^{(0)} + \overline{C^{(1)}} + \overline{C^{(2)}}$ isotropic and anisotropic degrees of freedom decouple.
Separate-universe approach (or "quasi-isotropic approach"). See *e.g.* [Wands et al. (2000)]

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Equivalent: Perturb the FLRW constraint $C^{(0)} \rightarrow C^{(0)} + \overline{C^{(1)}} + \overline{C^{(2)}}$.

CPT	separate-universe approach
δN	$\overline{\delta N}$
δN^i	0
$\delta\gamma_1$	$\overline{\delta\gamma_1}$
$\delta\pi_1$	$\overline{\delta\pi_1}$
$\delta\gamma_2$	0
$\delta\pi_2$	0
$\delta\varphi$	$\overline{\delta\varphi}$
$\delta\pi_\varphi$	$\overline{\delta\pi_\varphi}$

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- What is the domain of validity of the separate-universe approach (and of the stochastic formalism)?
- How to fix a gauge in separate-universe? [work in progress]

Consistency of the SU

$v \equiv$ covolume a^3

Let us compare the CPT constraints with the SU ones. Both linear scalar constraints match if:

$$\frac{k^2}{v^{2/3}} \ll \frac{1}{M_{\mathrm{Pl}}^2} \left| \frac{\pi_\varphi^2}{v^2} - V(\varphi) \right| . \tag{11}$$

Both quadratic scalar constraints match if:

$$\frac{k^2}{v^{2/3}} \ll \left| V_{,\varphi,\varphi} \right| . \tag{12}$$

Finally, the interactions between isotropic and anisotropic (gravitational) degrees of freedom can be neglected if one further imposes:

$$\frac{k^2}{v^{2/3}} \ll \frac{1}{M_{\mathrm{Pl}}^2} \left| \frac{\pi_\varphi^2}{v^2} + \frac{V(\varphi)}{2} \right| . \tag{13}$$

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Remark: those conditions are gauge dependent! For example if $\delta\gamma_1 = \overline{\delta\gamma_1} = 0$, equations (11) and (13) are not required.

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Fixing a gauge

- In CPT, we have a system with eight variables $\delta N, \delta N_1, \delta\varphi, \delta\pi_\varphi, \delta\gamma_1, \delta\pi_1, \delta\gamma_2, \delta\pi_2$.
- Two constraints $\mathcal{S}^{(1)} = 0$ and $\mathcal{D}^{(1)} = 0$.
- Two gauge transformations freeze two variables (for instance $\delta\varphi = \delta\pi_\varphi = 0$) .
- This imposes two additional conditions (for instance $\delta\dot{\varphi} = \delta\dot{\pi}_\varphi = 0$) .

We end with two variables only, *i.e.* a single physical degree of freedom. It can be parametrised in a gauge-invariant way, *e.g.* :

$$Q_{MS} := \delta\varphi + \frac{M_{\mathrm{Pl}}^2 \pi_\varphi}{\sqrt{6}\theta v^{5/3}} \left(\sqrt{2}\delta\gamma_1 - \delta\gamma_2\right) . \qquad (14)$$

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- In SU, we have five variables $\overline{\delta N}, \overline{\delta\varphi}, \overline{\delta\pi_\varphi}, \overline{\delta\gamma_1}, \overline{\delta\pi_1}$.
- One constraint $\overline{\mathcal{S}^{(1)}} = 0$.
- One gauge transformation freezes one variable (for instance $\overline{\delta\varphi} = 0$) .
- One additional condition (for instance $\dot{\overline{\delta\varphi}} = 0$) .

Here the physical degree of freedom can be parametrised as:

$$\overline{Q}_{MS} := \overline{\delta\varphi} + \frac{M_{\rm Pl}^2 \pi_\varphi}{\sqrt{3}\theta v^{5/3}} \overline{\delta\gamma_1} . \tag{15}$$

Fixing a gauge

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- Two constraints $\mathcal{S}^{(1)} = 0$ and $\mathcal{D}^{(1)} = 0$.
- Two gauge transformations freeze two variables (for instance $\delta N = 0$).
- No additional condition.

We end with two variables only, *i.e.* a single physical degree of freedom. It can be parametrised in a gauge-invariant way, *e.g.* :

$$Q_{MS} := ? \tag{14}$$

- In SU, we have five variables $\overline{\delta N}, \overline{\delta\varphi}, \overline{\delta\pi_\varphi}, \overline{\delta\gamma_1}, \overline{\delta\pi_1}$.
- One constraint $\overline{\mathcal{S}^{(1)}} = 0$.
- One gauge transformation freezes one variable (for instance $\overline{\delta N} = 0$).
- No additional condition.

Here the physical degree of freedom can be parametrised as:

$$\overline{Q}_{MS} := ? \tag{15}$$

The spatially-flat gauge

One sets $\delta\gamma_1 = \delta\gamma_2 = 0$. This means $\overline{\delta\gamma_1} = 0$ in SU.
But the expressions obtained for the perturbed lapse in CPT and in SU do not match:

$$\delta N = -N \frac{\pi_\varphi}{v\theta} \delta\varphi \,, \tag{16}$$

$$\overline{\delta N} = -\frac{2M_{\rm Pl}^2}{3} \frac{N}{\theta^2} \left(V_{,\varphi} \overline{\delta\varphi} + \frac{\pi_\varphi}{v^2} \overline{\delta\pi_\varphi} \right) . \tag{17}$$

It is therefore not consistent.
This mismatch is explained by the fact that $k\delta N_1$ is not k-suppressed in this gauge [see *e.g.* [Pattison et al. (2019)]].

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Maybe define the spatially-flat gauge in SU as:

$$\overline{\delta N} = -N \frac{\pi_\varphi}{v\theta} \overline{\delta\varphi} \,. \tag{18}$$

A same gauge may need to be written with two different prescriptions in CPT and in SU.

Conclusion

- We formulated the cosmological-perturbation theory (CPT) and the separate-universe (SU) approach in a Hamiltonian framework.
- At large-scales, the isotropic and anisotropic degrees of freedom decouple. The SU can be understood as a perturbed FLRW universe.
- By comparing CPT with SU, we determined the minimal scale to consider for the SU to be valid.
- However, this scale depends on the chosen gauge. One needs to find a systematic way to link gauges in CPT with gauges in SU. [work in progress]

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