Slicing maps: a sharp road to the Cori-Vauquelin-Schaeffer bijection and some of its probabilistic consequences

G. Miermont (ENS de Lyon), based on work with

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Journée Cartes 40 ans de la Bijection CV

October 11th, 2021, LaBRI, Bordeaux
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1. Generating series of maps and slices

Many problems in map enumeration boil down to computing generating series of the form

\[ Z_\Phi = Z_\Phi(t, t_1, t_2, \ldots) = \sum_{m \in \Phi} t^\# \text{Vertices}(m) t_1^\# \#_0^m t_2^\# \#_1^m t_3^\# \#_2^m \ldots \]

where \( \Phi \) is a given class of maps, typically of a given topology, or with various degree restrictions.

When \( \Phi \) is the class of rooted (and pointed) plane maps, an implicit expression is given by the CVS bijection, and its generalization by Bouttier-Di Francesco-Guitter.
Many map enumeration formulas have found a bijective explanation since these works, usually based on classes of decorated trees:

- blossoming trees
- labeled trees (mobiles).

Since ~2010, Bouttier and Guiller have advocated the use of somehow more primitive objects called slices.
Definition of a slice:

A planar map with one "exterior" face, with three incident corners A, B and C appearing in the exterior face contour order, and such that:

1. The boundary segment AB is geodesic
2. The boundary segment AC is uniquely geodesic
3. These two meet only at A.

The width is the length of the base BC.

Elementary slices (width 1) include:
- \(A \leq C\) (trivial)
- \(A = C\) (empty)
Slicing slices (Bouttier, Guitter)

A (non-trivial, non-empty) elementary slice may be further cut into slices by looking at \((-1)\times d\) distances to\( A\) of the vertices incident to the face opposite to the first edge on the left boundary.

This yields a lattice path.
Slicing slices (Bouttier, Guittet) (bipartite case)

A (non-trivial, non-empty) elementary slice may be further cut into slices by looking at \((-1)^{x\text{distances}}\) to \(A\) of the vertices incident to the face \(f\) opposite to the first edge on the left boundary.

This yields a lattice path

We shoot a leftmost geodesic to \(A\) from every corner incident to \(f\).

This decomposes the slice into \(\frac{\deg_f}{2}\) slices.
Slicing slices (Bouttier, Guittet) (bipartite case)

A (non-trivial, non-empty) elementary slice may be further cut into slices by looking at \((-1)\times\)distances to \(A\) of the vertices incident to the face \(f\) opposite to the first edge on the left boundary.

This decomposes the slice into \(\frac{\text{deg} f}{2}\) slices.

(one for every up-step)
A (non-trivial) non-empty elementary slice

A skip-free bridge
\[0 = l_0, l_1, \ldots, l_{2k} = l_0\]
\[l_{i+1} - l_i \geq -1\]
\[0 \leq i < k\]

Associate a lattice path, glue slices to the up steps
Proposition: This construction yields a bijection between slices and skip-free bridges marked with slices.

Iterating this correspondence yields the Bouttier-Di Francesco-Guitter bijection labeled mobiles.
2. Some remarkable formulas

a. Bipartite, rooted, pointed plane maps

Let

\[ R = R(t, t_2, t_3, t_4, \ldots) \quad (t_{2i+1} = 0 \forall i \geq 0) \]

\[ = Z_\Phi(t, t_2, t_4, t_6, \ldots) \]

where \( \Phi = \{ \text{planar rooted pointed positive maps} \} \)

\[ d_m(e^-, v) = d_m(e^+, v) + 1. \]

Then

\[ R = t + \sum_{k \geq 1} \binom{2k-1}{k} t^{2k} R^k \]
2. Some remarkable formulas

a. Bipartite, rooted, pointed plane maps

Let \( R = R(t, t_2, t_4, t_6, \ldots) \) \((t_{2i+1} = 0 \forall i \geq 0)\).

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\[
R = t + \sum_{k \geq 1} \binom{2k-1}{k} t_{2k} R^k
\]

We consider the leftmost geodesic starting from \( e \), ending at \( v \).
2. Some remarkable formulas

a. Bipartite, rooted, pointed plane maps

Let \( R = R(t, t_2, t_4, t_6, \ldots) \) \((t_{2i+1} = 0 \forall i \geq 0)\).

\[
R = \mathcal{Z}_\Phi(t, t_2, t_4, t_6, \ldots)
\]

where \( \Phi = \{\text{planar rooted pointed positive maps}\} \)

\[
d_m(e^-, v) = d_m(e^+, v) + 1.
\]

Then

\[
R = t + \sum_{k \geq 1} \binom{2k-1}{k} t_{2k} R^k
\]

We consider the leftmost geodesic starting from \( e \), ending at \( v \). Then we cut open along this path.
Fact: Unless we started from a conventional trivial (vertex) map, with weight $E$, corresponding to the empty slice, this yields an elementary non-empty slice, and this is one-to-one.
b. The disk formula

Fix $L > 0$ and $\Phi = \{\text{rooted, pointed plane bipartite maps with exterior face of deg } 2L\}$

"disks of perimeter $2L$"

Set $D = \mathbb{Z}_{\Phi}(t, t_2, t_4, t_6, \ldots)$

(the root face is not included as a $t_{\deg f}$ factor).

Then $D = \left( \frac{2L}{L} \right) R_L^L$

Bouttier - Guiller
Proof by a slice decomposition

Turning around the root face, the distance variations to \( v \) from a path of length \( 2L \) with steps \( \pm 1 \)

From every (up) step we start a leftmost geodesic from the corresponding boundary corner to \( v \).

This results (bijectively) in a collection of \( L \) non-trivial elementary slices glued on the up-steps of a lattice path of length \( 2L \) from 0 to 0.
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This results (bijectively) in a collection of $2$ non-trivial elementary slices glued on the up-steps of a lattice path of length $2L$ from $0$ to $0$.

Gluing the left and right boundaries together yields the wanted disk.
c. The cylinder formula

Let $L_1, L_2 > 0$. Let $\Phi = \{\text{Planar maps with two distinguished rooted faces of degrees } 2L_1, 2L_2\}$. Set $\alpha(L) = \frac{2L!}{L!(L-1)!}$. Then if $C = \mathcal{Z}_\phi(t_1, t_2, t_3, \ldots)$ we have

$$C = \alpha(L_1) \alpha(L_2) \frac{R^{L_1+L_2}}{L_1+L_2}$$

Eynard, Collet-Fussy
c. The cylinder formula

Let \( L_1, L_2 > 0 \). Let \( \Phi = \{ \text{Plane-maps with two distinguished rooted faces of degrees } 2L_1, 2L_2 \} \). Set \( \alpha(L) = \frac{2L!}{L!(L-1)!} \).

Then if \( C = \mathcal{Z}_\Phi (t_1, t_2, t_4, \ldots) \) we have

\[
C = \alpha(L_1) \alpha(L_2) \frac{R^{L_1+L_2}}{L_1+L_2}
\]

Eynard, Collet-Fuss

One can understand this formula by using two decompositions into "slices set on paths" similar to the case of disks (Bouttier - Gutttor)
This time, the two paths do not end up at their starting values.

\[ R^{L_1 + l} \]

\[ R^{L_2 - l} \]

Number of paths:

\[ \frac{2L_1}{L_1 + l} \]

\[ \frac{2L_2}{L_2 - l} \]
This time, the two paths do not end up at their starting values.
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\[ (2L \text{ choices of gluings}) \]

\[ \sum_{\phi} 2^{L_1} \left( \frac{2L_1}{L_1 + l} \right) \left( \frac{2L_2}{L_2 - l} \right) = \frac{\alpha(L_1) \alpha(L_2)}{L_1 + L_2} \]
d. The pairs of pants formula

Let $\Phi = \{\text{Rooted maps with } 3 \text{ distinguished faces, and degrees } 2L_1, 2L_2, 2L_3\}$

Then $P = \mathcal{Z}_{\Phi}(t_1, t_2, t_3, \ldots)$ satisfies:

$$P = \alpha(L_1) \alpha(L_2) \alpha(L_3) R^{L_1+L_2+L_3} \frac{d \ln R}{dt}$$

We give a geometrical/combinatorial proof of this identity, and a related one. By a decomposition analogous to cylinders, we may focus on the case of "tight pants", where each boundary is of minimal length in its free homotopy class.

One gets:

$$TP = R^{l_1+l_2+l_3} \frac{d \ln R}{dt} \quad \text{if } l_1, l_2, l_3 > 0$$

2L_1, 2L_2, 2L_3 degrees of tight boundaries.
An important novelty (essentially coming from the term \( \frac{d \ln R}{dt} \)) consists in introducing new elementary pieces:

- **geodesic triangles**
  - \((X_e, Y, f.g.)\)

**Red** = unique geodesic

**Red dots** = mandatory passage points for geodesics.

\(X_e\) = unique geodesic

\(\text{not a geodesic}\)
We build a tight pair of pants with perimeters $\ell_1, \ell_2, \ell_3$ with five pieces

\[ \begin{array}{c}
3 \text{ triangles of excesses } e_1, e_2, e_3 \\
2 \text{ triangles}
\end{array} \]

\[ \begin{aligned}
e_1 + e_2 &= 2\ell_1 \\
e_2 + e_3 &= 2\ell_2 \\
e_3 + e_1 &= 2\ell_3
\end{aligned} \]
We build a tight pair of pants with perimeters $2l_1, 2l_2, 2l_3$ with five pieces:

- 3 triangles of excess $e_1, e_2, e_3$
- 2 triangles

\[
\begin{align*}
2l_1 + e_1 + e_2 &= 2l_1 \\
2l_2 + e_2 + e_3 &= 2l_2 \\
2l_3 + e_3 + e_1 &= 2l_3
\end{align*}
\]

We conclude with:

**Fact 1:**

\[ X_e = R^e X_0 \]

**Fact 2:**

\[
\frac{d \ln R}{dt} = \frac{X_0^3 Y^2}{t^6}
\]
3. Some probabilistic consequences

**Goal**: We want to discuss limits of plane maps (with boundaries)

Let \( q = (q_1, q_2, \ldots) \) be an admissible sequence, so one can define

\[
P_q^\ast(m) = \prod_{g \in \text{Faces}(m)} \frac{q \deg(g)^{\frac{1}{2}}}{W_q}, \quad q - \text{Boltzmann law on bipartite, plane maps.}
\]

and

\[
P_q^{(p)}(m) = \prod_{g \in \text{Faces}(m) \setminus \text{root}} \frac{q \deg(g)^{\frac{1}{2}}}{W_q^{(p)}}, \quad q - \text{Boltzmann law on bipartite, plane maps with root face degree } 2p
\]

(similarly, \( P_q^{(p_1p_2 \cdots p_t)} \)).

What can we say of \( P_q^{(p)} \) as \( p \to \infty \)?
(Scaling) limits of metric spaces:

The Gromov–Hausdorff–Prokhorov distance between \((X,d,\mu),(X',d',\mu')\) compact, metric, measured spaces, is

\[
\inf_{\phi,\phi'} \left( \delta_{\text{Haus}}(\phi(X),\phi'(X')) \lor \delta_{\text{Pro}}(\phi_*\mu,\phi'_*\mu') \right)
\]

where \(\phi,\phi'\) are isometric embeddings of \((X,d)\), \((X',d')\) in a common metric space \((Z,\delta)\)
a. Scaling limits for random plane maps and slices

**Thm (M. for \( q = a \delta_2 \), Le Gall):** Let \( q \) be finitely supported.

Let \( M_n \) have law \( P_q ( \cdot \mid \# V(m) = n ) \).

Then for an appropriate constant \( c = c_q \)

\[
(V(M_n), \frac{c}{n^{4/3}}d_{M_n}, \frac{1}{n} \sum_{v \in V(M_n)} \delta_{m(v)}) \xrightarrow{d} (S, D_S, d)
\]

in distribution for the GHP topology, where the limit is a random measured compact metric space (the Brownian map, or Brownian sphere), independent of \( q \).

**Remark:** One can improve a lot the assumptions on \( q \) (Marzouk).
Under the same context as above, let $S_n$ be a random slice obtained by cutting along the leftmost geodesic of the map $M_n$ as above.

**Theorem (Le Gall):** One has the joint convergence

$$
\lim_{n \to \infty} (V(M_n), cn^{-\frac{1}{n}} d_{M_n}, e^-, \nu) \xrightarrow{(d)} (S, D_S, x, y) $$

$$
\lim_{n \to \infty} (V(S_n), cn^{-\frac{1}{n}} d_{S_n}, e^-, \nu) \xrightarrow{(d)} (S', D_S', x, y)
$$

The limit $(S', D_S')$ is the Brownian slice with unit area. The Brownian slice with area $A$ has the law of $(S', A^{-\frac{1}{n}} D_S)$.
b. Scaling limits of disks

Thm (Bettinelli-M.): Assume \( q \) is a finitely supported weight sequence. Then if \( M_p \) has law \( P_{\frac{p}{q}} \), one has the criticality condition...

\[
\left( V(M_p), \left( \frac{2p}{3} \right)^{\frac{1}{2}} d_{M_p}, \frac{1}{p^2} \sum_{x \in V(M_p)} \delta_{x} \right)_{p \to \infty} \overset{(d)}{\to} \left( \text{FBD}_1, d, \mu \right)
\]

in distribution for the GHP topology.

The limit is called the **free Brownian disk** with perimeter 1. Its area \( \mu(\text{FBD}_1) \) has law \( \frac{f_{\frac{3}{2}}(x)}{x} \, dx \), where \( f_{\frac{3}{2}} \) is a stable \( \left( \frac{3}{2} \right) \) probability density function.

We let \( \text{FBD}_L^{(d)} = (\text{FBD}_1, L^{\frac{1}{2}}d, L^{2} \mu) \).

A curiosity: \( q \) does not appear in the rescaling (why ??)
Thm (Bettinelli): A.s. $\text{FBD}_L$ is homeomorphic to $\{z \in \mathbb{C}: |z| < 1\}$

\[
\text{with } \dim_{\text{Haus}}(\text{FBD}_L) = 4, \quad \dim_{\text{Haus}}(\partial \text{FBD}_L) = 2
\]

Brownian disks are known to appear naturally in the study of random maps and the Brownian map:

- Metric structure of $(S,D)$
  (Miller-Sheffield, Le Gall)
- SAW on random maps (Gwynne-Miller)
- Percolation exploration / SLE$_6$
  on LQG$_{\sqrt{8/3}}$ surfaces disconnect pieces with law $\text{FBD}_L$ (Gwynne-Miller)
- Also relevant in Holden-Sun's Cardy embedding papers.
Slicing maps with boundaries

Let \( Z^* \) be the set of pointed quadrangulations with a boundary perimeter \( 2p \) and \( n \) non-root faces.

Then there is a bijection

\[
Z^* \leftrightarrow \left\{ (\mathbf{l}, s) : \mathbf{l} = (l_0, ..., l_p) \text{ a skip-free walk of length } p \right\}
\]

from \( 0 \) to \( 0 \), \( s = (s_1, ..., s_p) \) slices consisting in taking left-most geodesics from boundary to the distinguished point.

and conversely:

**Take** \(-\mathbf{l}\)

**Glue**
Slicing maps with boundaries

Let \( \mathcal{Z}_p \) be the set of pointed quadrangulations with a boundary from \( \mathbb{D}_p \) to \( \mathbb{D}_p \) of perimeter \( 2p \).

Then there is a bijection

\[
\mathcal{Z}_p \leftrightarrow \left\{ (f, s) : f = (l_0, \ldots, l_p) \text{ a skip-free walk of length } p \right\}
\]

consisting in taking leftmost geodesics from boundary to the distinguished point and conversely:

take leftmost geodesics to \( \nu \) from the boundary

\[
\text{glue}
\]
We now want to choose $M_p$ with law $\mathbb{P}_{\frac{1}{12} \delta_2}$, whose image under the slice decomposition above is the law of $(L_p, (I_{(1)}, \ldots, I_{(n)}))$ where:

- $L_p$ is a uniform skip-free bridge from $0$ to $0$ of length $p$.
- $I_{(1)}, I_{(2)}, \ldots, I_{(n)}$ are iid slices of $\mathbb{P}_{\frac{1}{12} \delta_2}$-distributed quadrangulations.

Recall: $\# \{\text{quadrangulations with } n \text{ faces} \} \sim C 12^n n^{-5/2}$ as $n \to \infty$.

So $\# \{\text{Slices with } n \text{ faces} \} \sim \frac{C}{2} 12^n n^{-3/2}$.

Which implies that $\mathbb{P}_{\frac{1}{12} \delta_2}(\# F(I_1) > n) \sim \frac{C}{n^{1/2}}$.

That is, $\frac{\# F(I_{(1)}) + \ldots + \# F(I_{(n)})}{p^2} \xrightarrow{d} A$ where $A$ has a Stable $(\frac{1}{2})$ law $\int_{\frac{1}{2}} dx$ (this explains why the area of $\text{FBD}_d$ has density $\int_{\frac{1}{2}} dx$; one has to unbiased the marked vertex).
Then: \( \left( L_p \left( \frac{l}{\sqrt{2\pi}} \right) \right) \xrightarrow{(d)} \left( \sqrt{3} B_{t \in [0,1]}^{\text{bridge}} \right) \)

a standard Brownian bridge.

and \( \sum_{i=1}^{p} \delta \left( \frac{i}{p} \right) \mathcal{J}(\frac{i}{p}) \rightarrow \text{Poisson RM} \ (dt \otimes \lambda \ d\mathbb{W}) \)

where \( d\mathbb{W} = \int_{0}^{\infty} \frac{dA}{2 \sqrt{2\pi} A^{3/2}} \mathbb{S}_{1/2} \)

\( \mathbb{S}_{1/2} \) law of a slice of area \( A \)

\( \lambda \) constant > 0.

Gluing along geodesics is well-behaved w.r.t. GH distance, but gluing infinitely many spaces is not!
Then: \( \left( \frac{L_p(L_p)}{\sqrt{2\pi/3}} \right) \stackrel{(d)}{\longrightarrow} (\sqrt{3} B_{t_{\text{eff}}}^{\text{bridge}}) \)

a standard Brownian bridge

and \( \sum_{i=1}^{p} \delta \left( \frac{i}{p}, \frac{j_0}{\sqrt{p}} \right) \rightarrow \text{Poisson RM} \)  
\( (dt \otimes \lambda \, d\Theta) \)

where \( d\Theta = \int_{0}^{\infty} \frac{dA}{2\sqrt{2\pi A^{3/2}}} \) Slice_{\text{law of a slice of area } A} \)

Law of a slice of area \( A \)

Key lemma: Geodesics to \( s \) do not intersect the boundary (except maybe at the starting point).

\( \Rightarrow \) Typical geodesics intersect finitely many slices

and the metric gluing is well-behaved.

\( \Rightarrow \) this gluing
Key lemma: Geodesics to $s_*$ do not intersect the boundary (except maybe at the starting point).

Proved via Shepp’s covering theorem for Poisson random measures [Bettinelli].

Can the observer Sitting on the dotted line see the Green bridge through the slices? No!

The neighborhood of a Brownian bridge looks like $2$-sided Brownian motion $\times \sqrt{3}$.
Key lemma: Geodesics to $s_*$ do not intersect the boundary (except maybe at the starting point).

Proved via Shepp's covering theorem for Poisson random measures [Bettinelli].

Can the observer sitting on the dotted line see the Blue bridge through the yellow bars?

No!
Can the observer sitting on the dotted line see the Blue bridge through the yellow bars?

No!

In this form, using excursion theory, the problem becomes a Shepp covering problem of $(0, \infty)$ by intervals $(t_i, t_i + \epsilon_i)$, where $\sum_{i \geq 1} \delta(t_i, x_i)$ is a Poisson random measure with intensity $dt \otimes \nu(dx/\epsilon^2)$, where $\nu$ happens to be 1 (critical case!)

Can the observer
Sitting on the dotted line see the
Blue bridge through the slices?

No! But almost!

In this form, using excursion theory, the problem becomes
a Shepp covering problem of \((0, \infty)\) by intervals \((t_i, t_i+\xi_i)\),
where \(\sum_{i \geq 1} \delta(t_i, \xi_i)\) is a Poisson random measure with intensity
\(dt \otimes \frac{vdz}{z^2}\), where \(v\) happens to be 1 (critical case!)
A word on non-compact limits: slices and Brownian half-plane.

The local picture of the Brownian disk near a boundary point (the root say) is described by the Brownian half-plane (Bau-M. Ray, also Gwynne-Miller):

\[(FBD_L, \frac{1}{\varepsilon} D, \text{root}) \xrightarrow{\varepsilon \downarrow 0} \text{BHP}, \quad L > 0 \text{ fixed}.\]

For the local Gromov-Hausdorff topology.

The description of BHP in terms of slices is nicer than for FBD.

Gluing all slices between times $L$ and $L'$ gives a “thick slice” of width $L' - L$, and those have a (homogeneous) semigroup property w.r.t gluing.
c. Minimal separating curves in cylinders and pairs of pants.

Let $Q_n$ be a uniformly chosen quadrangulation with two boundaries of perimeters $2L_1^{(n)}$, $2L_2^{(n)}$, where $L_i^{(n)} \sim \sqrt{2n} \xi_i$. Let $2\ell^{(n)}$ be the length of the shortest path separating the boundaries, and $A_1^{(n)}$, $A_2^{(n)}$ the number of quadrangles on each side.

**Thm (Bouttier-Guitter-M.)**: One has

\[
\frac{(A_1^{(n)})^{\frac{1}{n}}}{(\frac{2}{n})^{\frac{1}{n}}} \sim \frac{\ell^{(n)}}{n^{\frac{1}{2}}} \xrightarrow{d} (A_1, R)\]

where $A_1$ and $\ell$ are independent, $A_1$ has a conditioned stable ($\frac{1}{2}$) distribution, and $R \sim \text{Rayleigh} \left( \frac{1}{L_1} + \frac{1}{L_2} \right)^{-1}$. 

Let $Q_n$ be a uniformly chosen quadrangulation with three boundaries of perimeters $2L_1^{(n)}, 2L_2^{(n)}, 2L_3^{(n)}$ and $L_i^{(n)} \sim \sqrt{2n} \xi_i$. Let $2l_i^{(n)}$ be the length of the shortest path homotopic to the $i$-th boundary. And $A_0^{(n)}, A_1^{(n)}, A_2^{(n)}, A_3^{(n)}$ the "interior" and "exterior" areas.

**Thm (Bouttier-Guitter-M.):** One has

$$\left((A_i^{(n)}/n), \left(\frac{2}{n}\right)^{1/4} l_i^{(n)}\right)_{i \in \{1, 4\}, \text{kick}} \xrightarrow{n \to \infty} (A_i), (R_i)$$

where $A, R$ are independent, and $R_i \sim \text{Rayleigh}(L_i)$. 

**Note:** $\min(R_1, R_2, R_3) \sim \text{Rayleigh}\left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)$. 

Thanks for listening

Joyeux anniversaire!
Thanks for listening

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