Slicing maps: a sharp road to the Cori-Vauquelin-Schaeffer bijection and some of its probabilistic consequences

G. Miermont (ENS de Lyon), based on work with

Jérémie Bettinelli (CNRS & LIX, Ecole polytechnique)

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Journée Cartes 40 ans de la Bijection CV

October 11th, 2021, LaBRI, Bordeaux

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1. Generating series of maps and slices

Many problems in map enumeration boil down to computing generating series of the form

$$\boxed{\begin{aligned} Z_\Phi &= Z_\Phi(t, t_1, t_2, \dots) \\ &= \sum_{\underline{m} \in \Phi} t^{\#\text{Vertices}(\underline{m})}\, t_1^{\#\square_{\underline{m}}}\, t_2^{\#\bigcirc_{\underline{m}}}\, t_3^{\#\triangle_{\underline{m}}} \cdots \end{aligned}}$$

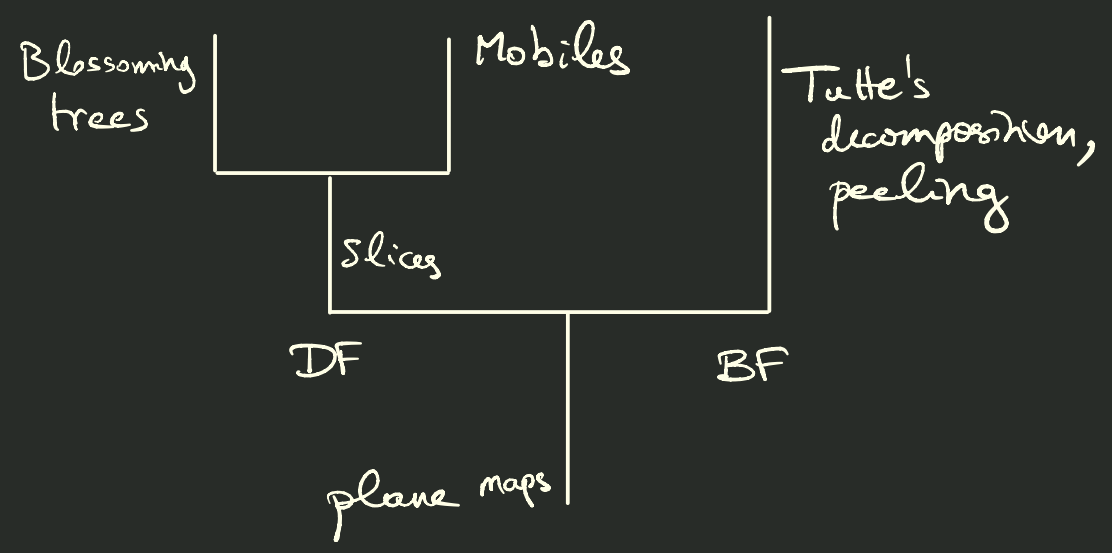
where Φ is a given class of maps, typically of a given topology, or with various degree restrictions.

When Φ is the class of rooted (and pointed) plane maps, an implicit expression is given by the CVS bijection, and its generalization by Bouttier-Di Francesco-Guitter.

Many map enumeration formulas have found a bijective explanation since these works, usually based on classes of decorated trees:

- blossoming trees
- labeled trees (mobiles).

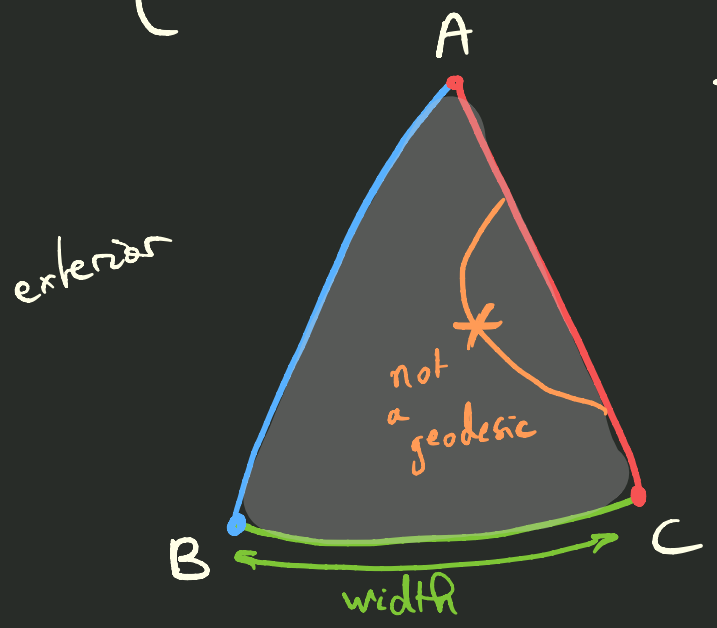
Since ~ 2010, Bouttier and Guitter have advocated the use of somehow more primitive objects called slices.



Definition of a slice :

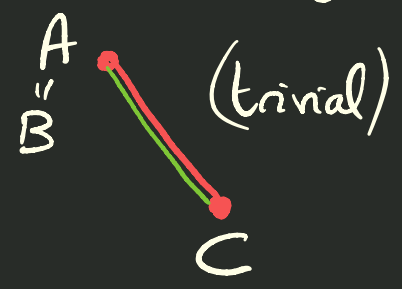
A planar map with one "exterior" face, with three incident corners A, B and C appearing in the exterior face contour order, and such that :

- The boundary segment AB is geodesic
- The boundary segment AC is uniquely geodesic
- These two meet only at A.

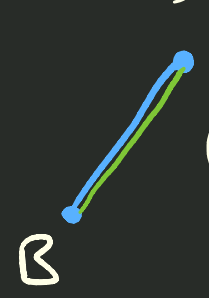


The width is the length of the base BC

Elementary slices (width 1) include



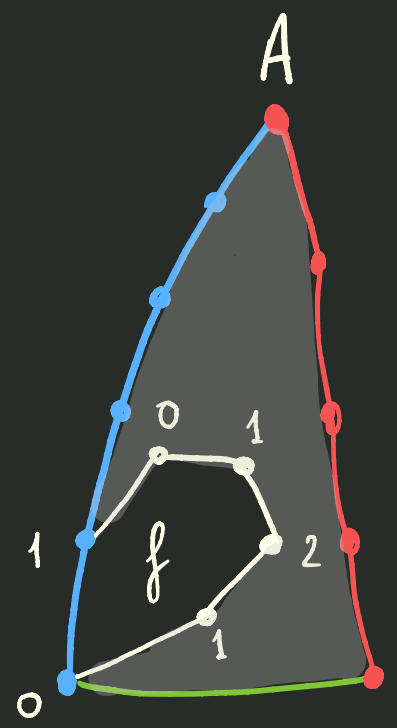
(trivial)



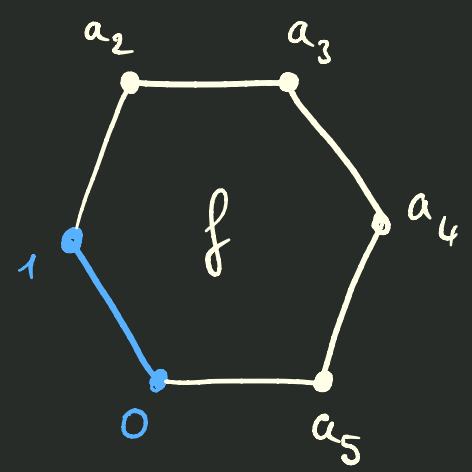
(empty).

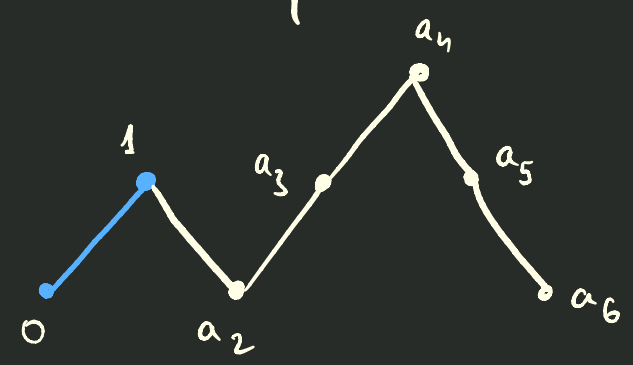
Slicing slices (Bouttier, Guitter)

A (non-trivial, non-empty) elementary slice may be further cut into slices by looking at $(-1)\times$ distances to A of the vertices incident to the face opposite to the first edge on the left boundary



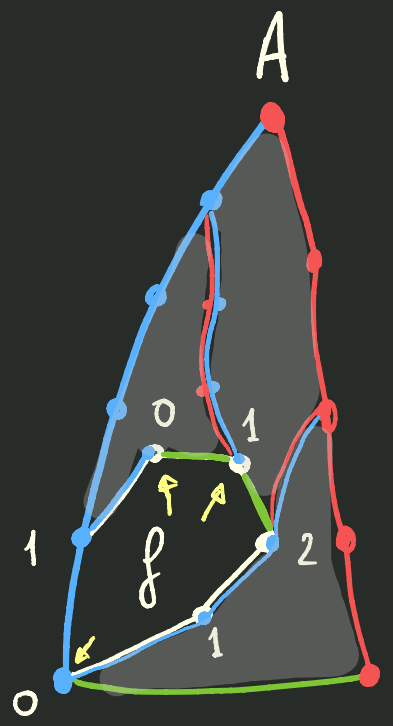
This yields a lattice path





Slicing slices (Bouttier, Guitter) (bipartite case)

A (non-trivial, non-empty) elementary slice may be further cut into slices by looking at $(-1)\times$ distances to A of the vertices incident to the face f opposite to the first edge on the left boundary



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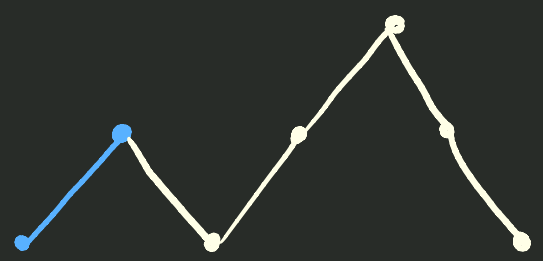

We shoot a leftmost geodesic to A from every corner incident to f

This decomposes the slice into $\frac{\deg f}{2}$ slices.

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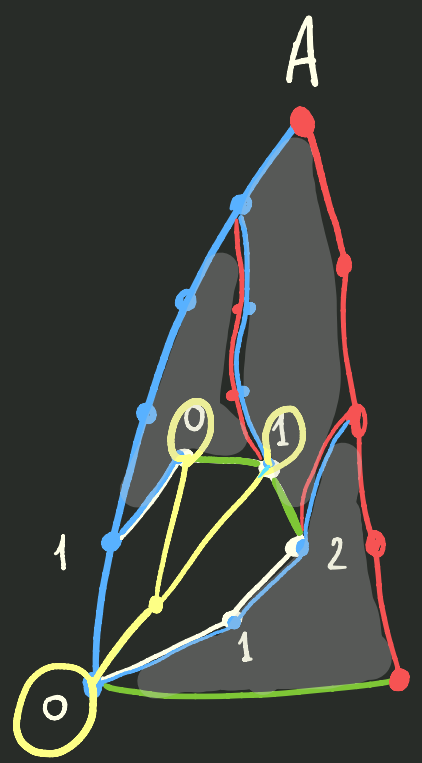
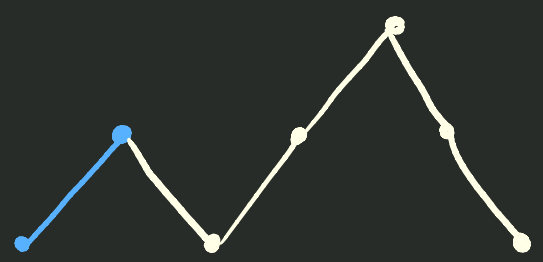
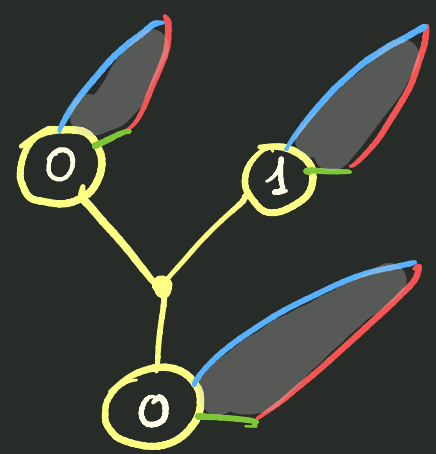
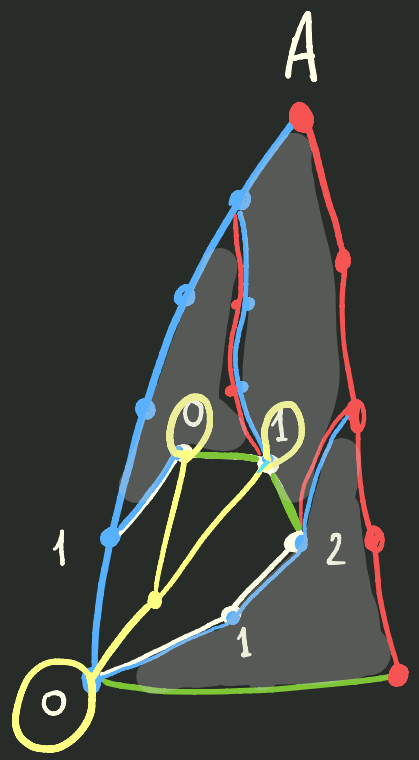


This decomposes the slice into $\frac{\deg f}{2}$ slices.

(one for every up-step)

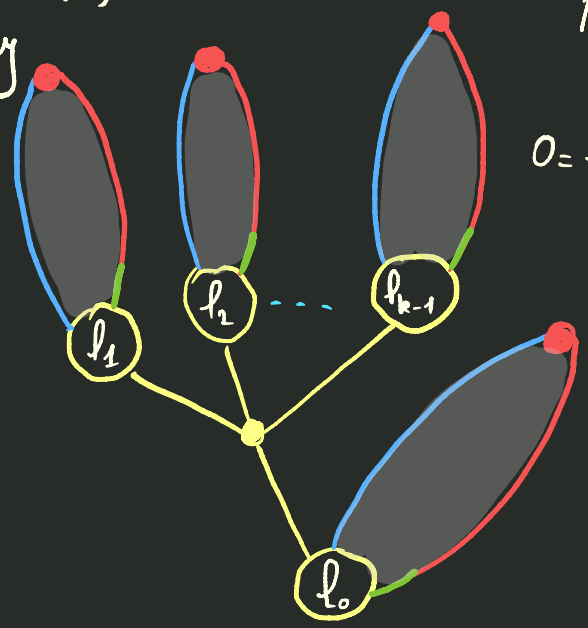






A (non-trivial) non-empty

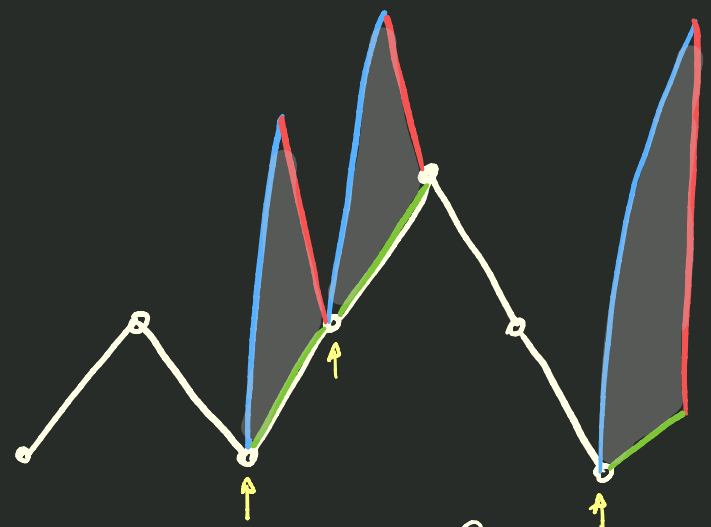
$\longmapsto$ elementary slice



A skip-free bridge

$0 = \ell_0, \ell_1, \dots, \ell_{2k} = \ell_0$

$$\boxed{\begin{array}{c} \ell_{i+1} - \ell_i \geq -1 \\ 0 \leq i < k \end{array}}$$



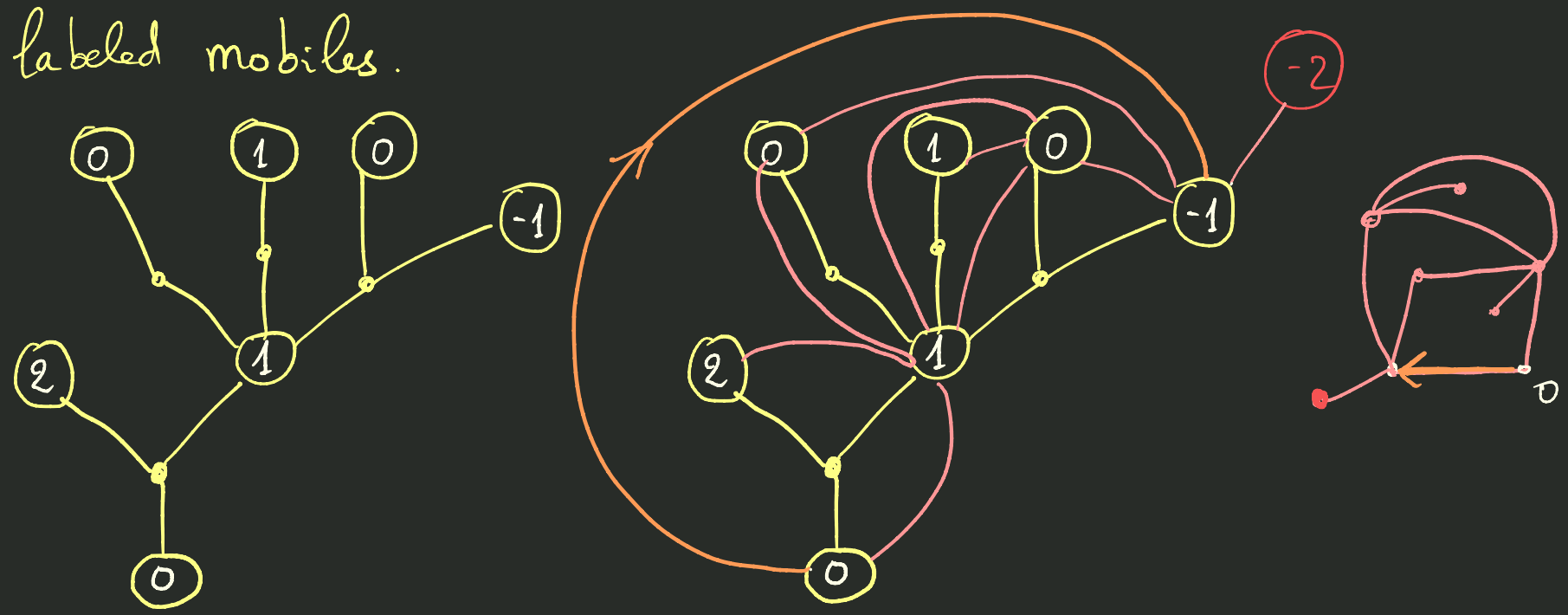
Associate a lattice path, glue slices to the up steps

glue $\longleftarrow$

Proposition: This construction yields a bijection between slices and skip-free bridges marked with slices.

Iterating this correspondence yields the Bouttier-Di Francesco-Guitter bijection

labeled mobiles.



2. Some remarkable formulas

a. Bipartite, rooted, pointed plane maps

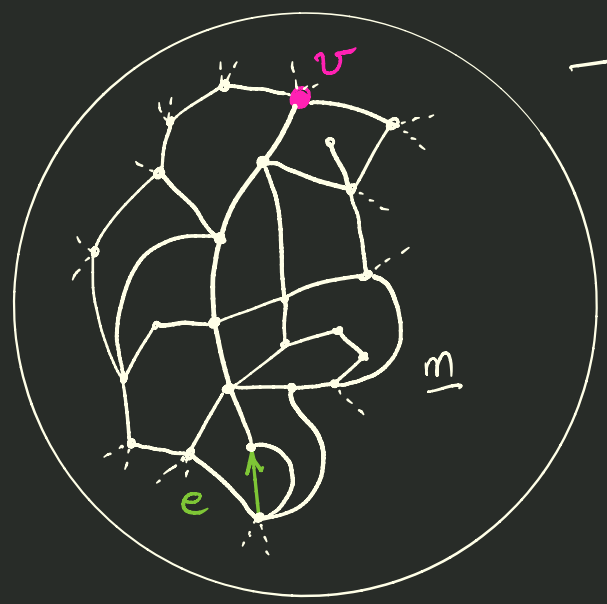
Let $R = R(t, t_2, t_4, t_6, \dots)$ $\quad (t_{2i+1} = 0 \ \forall i \geq 0)$.

$$= Z_{\Phi}(t, t_2, t_4, t_6, \dots)$$

not counted in the weight

where $\Phi = \{$planar rooted pointed positive maps$\}$

$$d_{\underline{m}}(e^-, v) = d_{\underline{m}}(e^+, v) + 1.$$



Then $$\boxed{R = t + \sum_{k \geq 1} \binom{2k-1}{k} t_{2k} R^k}$$

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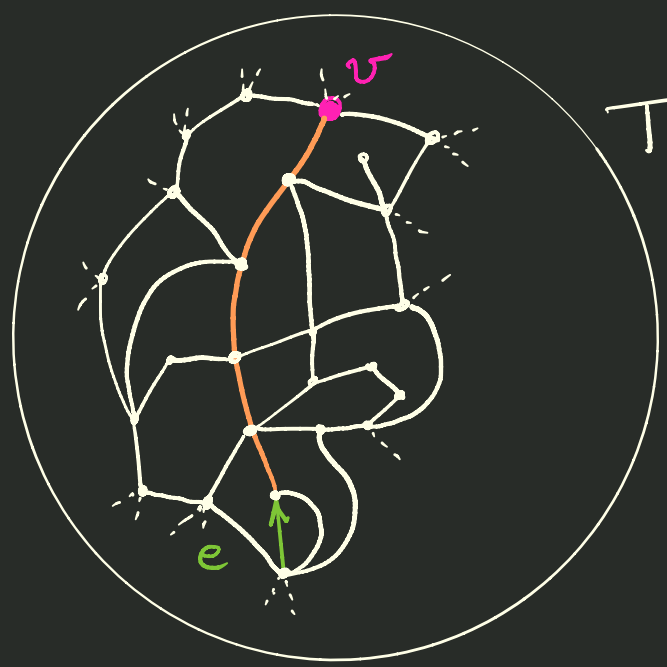

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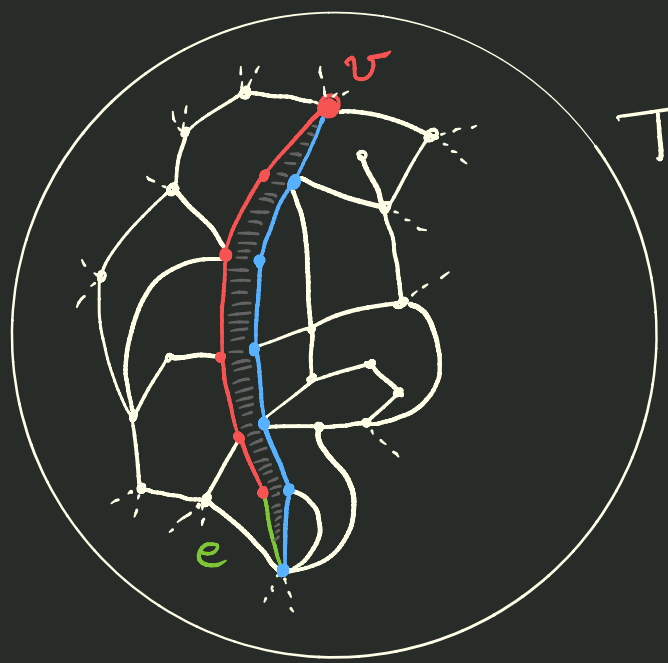
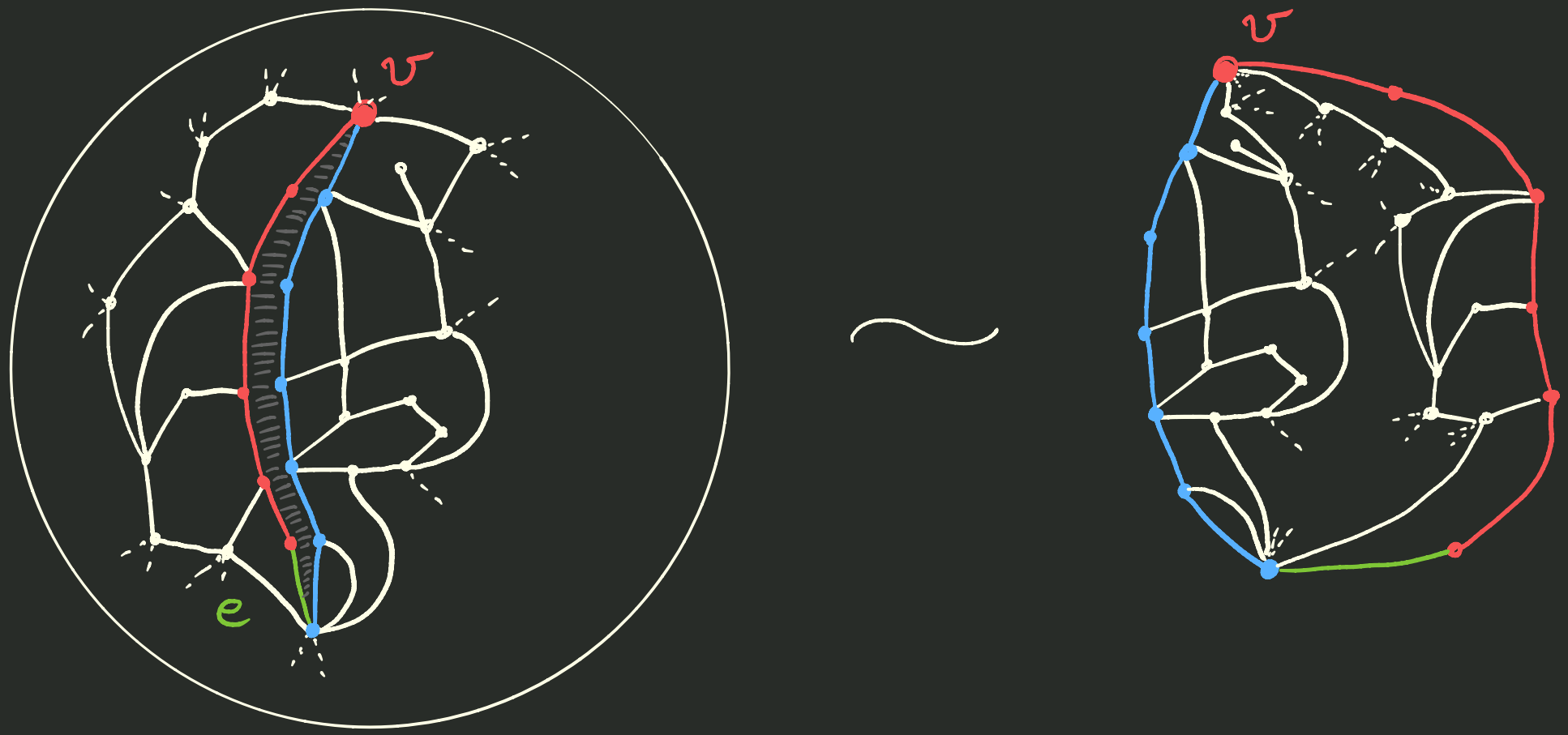


Then $\boxed{R = t + \sum_{k \geq 1} \binom{2k-1}{k} t_{2k} R^k}$

We consider the leftmost geodesic starting from e, ending at v.

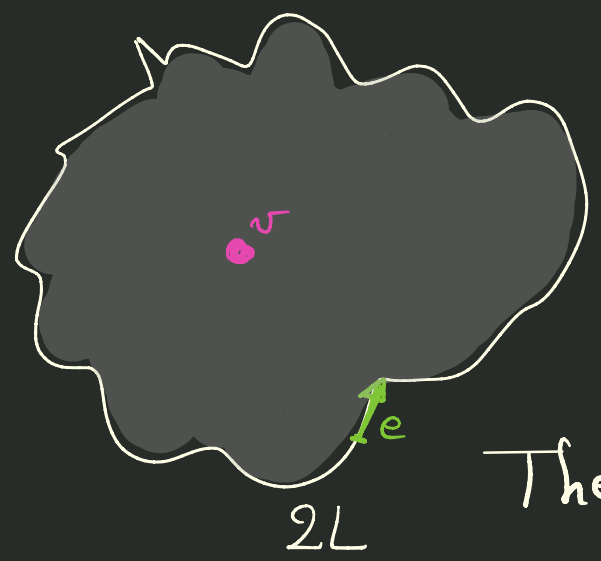
Then we cut open along this path



<u>Fact</u>: Unless we started from a conventional trivial (vertex) map, with weight t, corresponding to the empty slice, this yields an elementary non-empty slice, and this is one-to-one.

b. The disk formula

Fix $L > 0$ and $\Phi = \left\{ \begin{array}{l} \text{rooted, pointed plane bipartite} \\ \text{maps with exterior face of deg} = 2L \end{array} \right\}$



"disks of perimeter $2L$".

Set $D = Z_\Phi(t, t_2, t_4, t_6, \dots)$

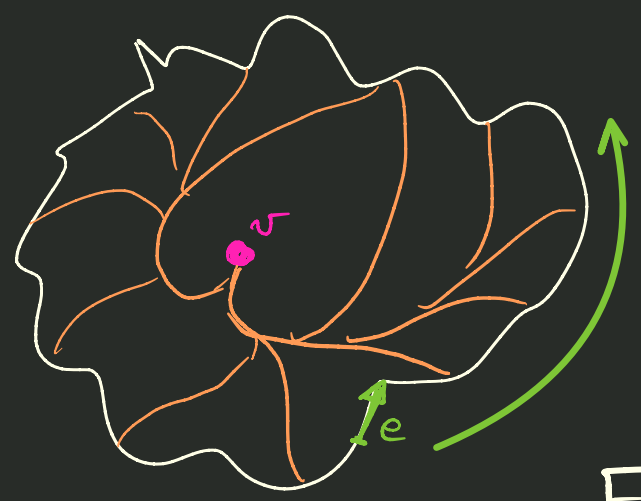
(the root face is not included as a $t_{\deg f}$ factor).

Then $\boxed{D = \binom{2L}{L} R^L}$

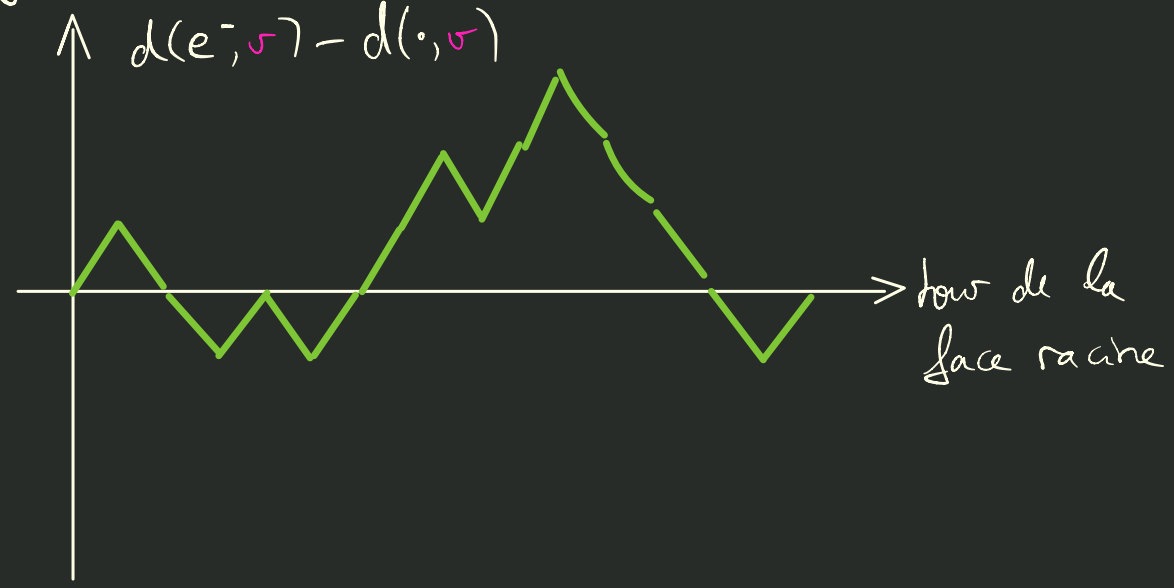
Bouttier–Guitter

Proof by a slice decomposition

Turning around the root face, the distance variations to v form a path of length $2L$ with steps ± 1



$2L$



From every (up) step we start a leftmost geodesic from the corresponding boundary corner to v.

This results (bijectively) in a collection of L non trivial elementary slices glued on the up-steps of a lattice path of length $2L$ from 0 to 0.

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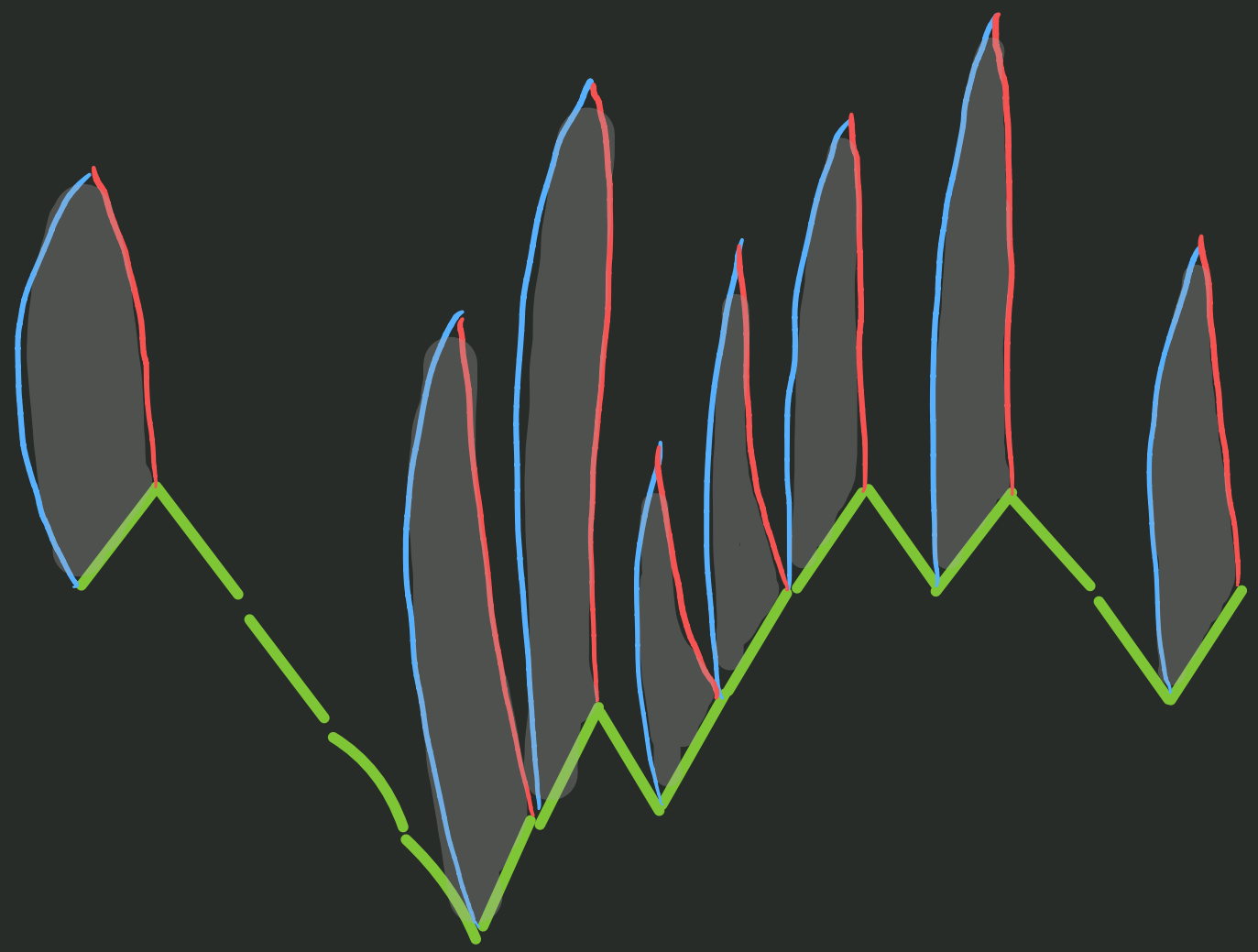

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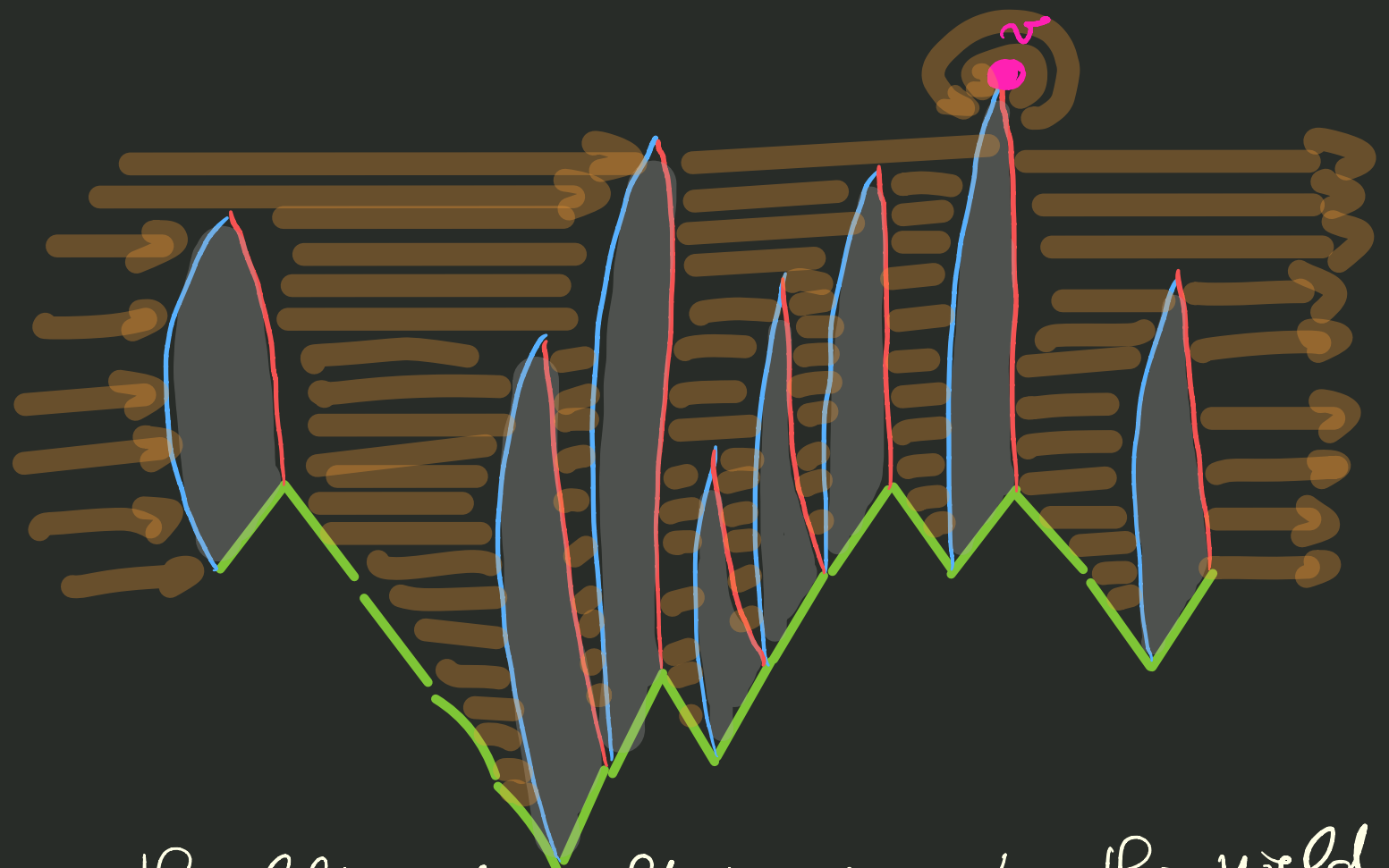
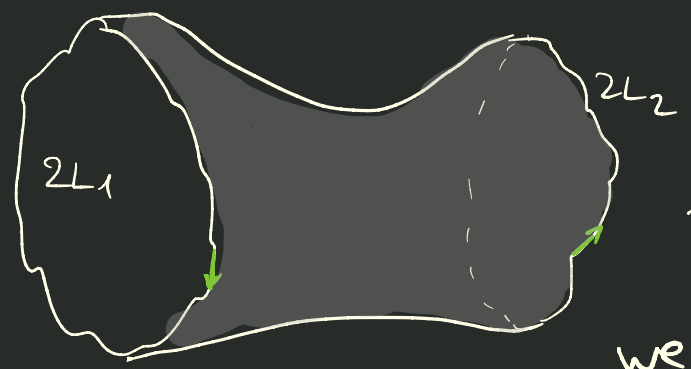


Gluing the left and right boundaries together yield the wanted disk.

c. The cylinder formula

Let $L_1, L_2 > 0$. Let $\Phi = \left\{ \begin{array}{l} \text{Planar maps with two} \\ \text{distinguished rooted faces of} \\ \text{degrees } 2L_1 \quad 2L_2 \end{array} \right\}$



Set $\alpha(L) = \dfrac{2L!}{L!(L-1)!}$

Then if $C = Z_\Phi(t, t_2, t_4, \dots)$

we have

$$\boxed{C = \alpha(L_1)\,\alpha(L_2)\,\frac{R^{L_1+L_2}}{L_1+L_2}}$$

Eynard, Collet-Fusy

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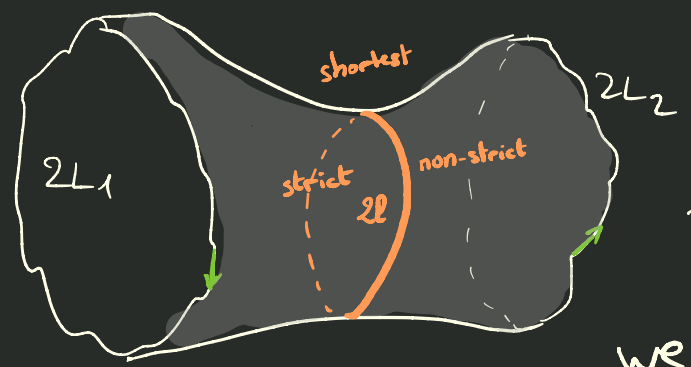

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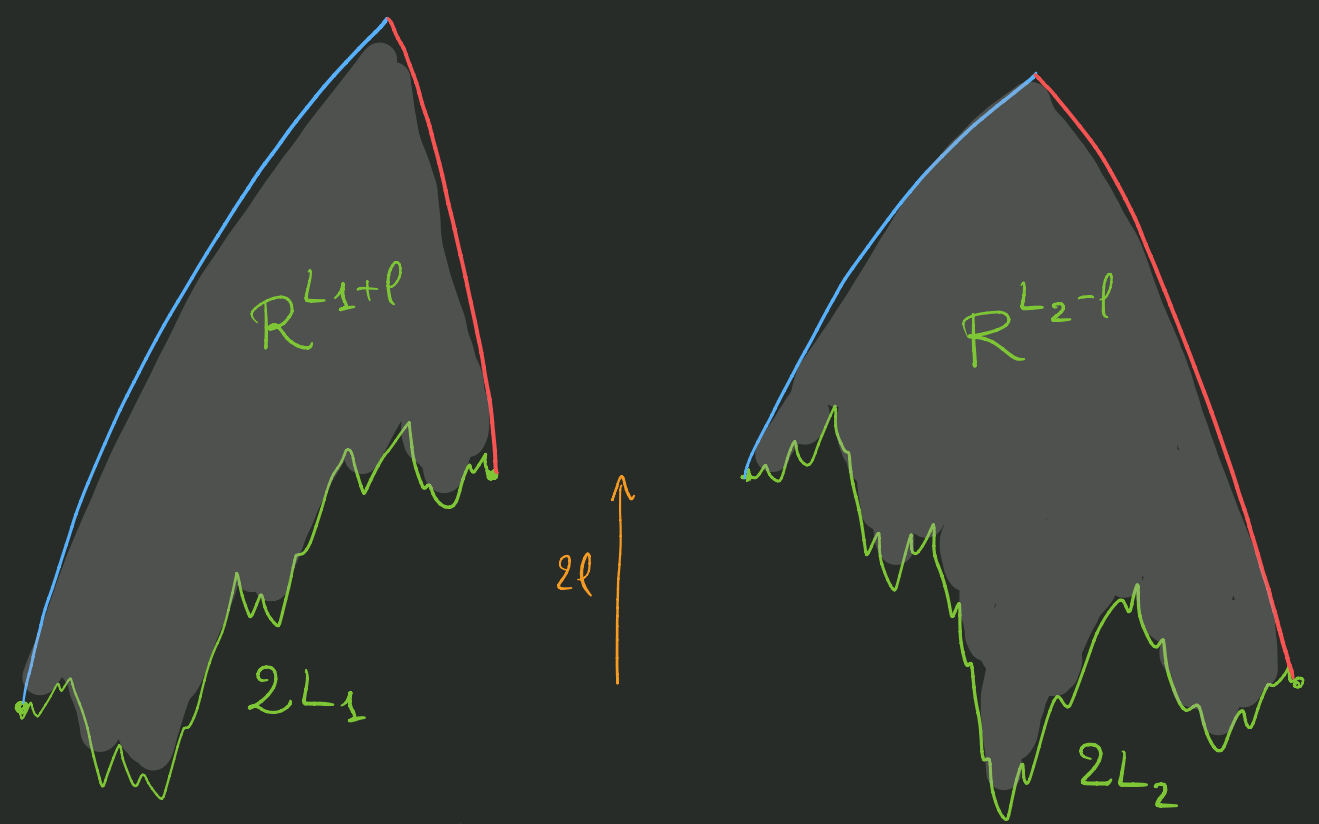
Eynard, Collet-Fusy

One can understand this formula by using <u>two</u> decompositions into "slices set on paths" similar to the case of disks (Bouttier - Guitter)

This time, the two paths do not end up at their starting values



number of paths

$$\binom{2L_1}{L_1+\ell} \qquad \binom{2L_2}{L_2-\ell}$$

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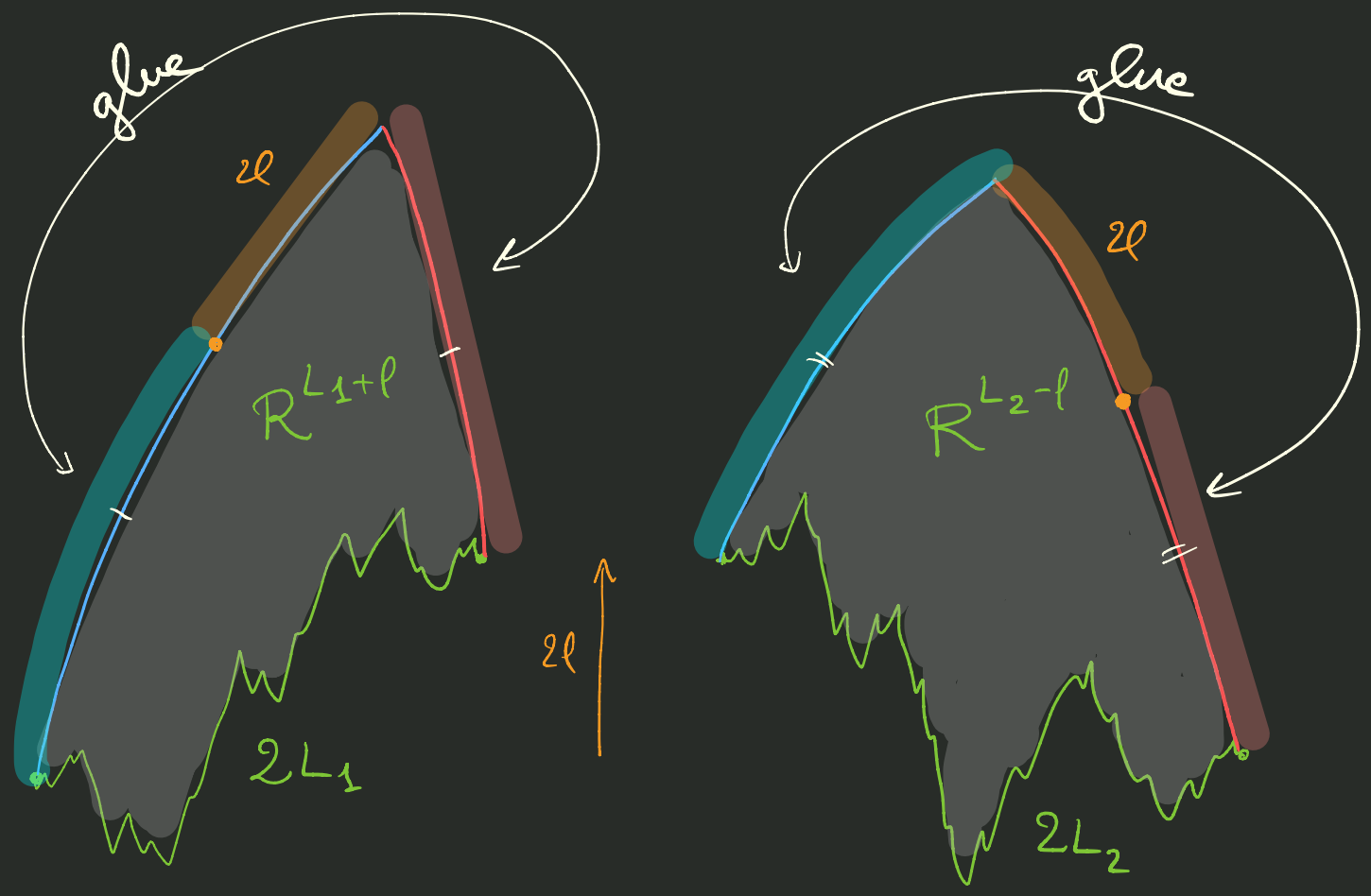

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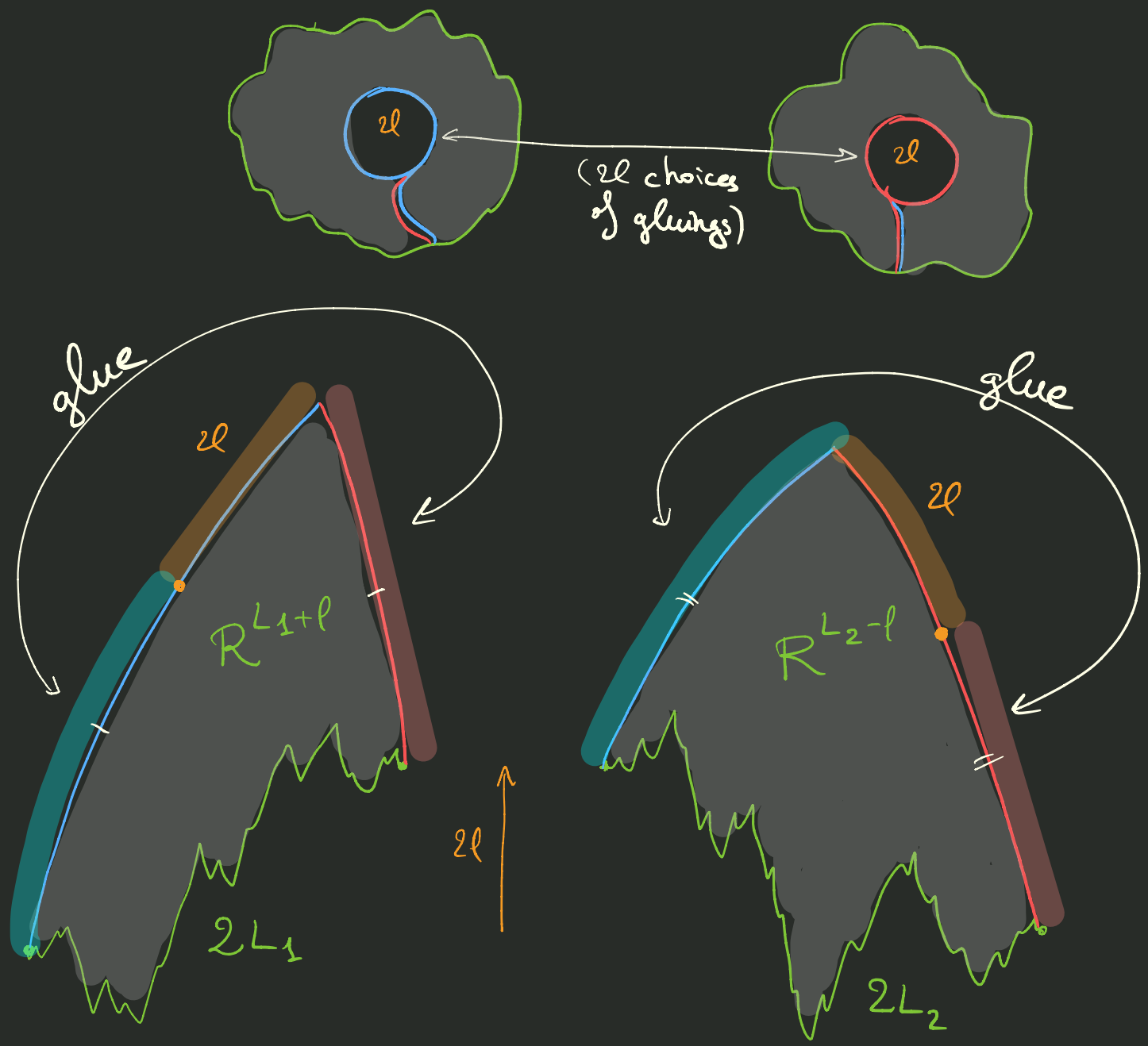


$$\sum_{\ell} 2\ell \binom{2L_1}{L_1+\ell}\binom{2L_2}{L_2-\ell} = \frac{\alpha(L_1)\alpha(L_2)}{L_1+L_2}$$

d. The pairs of pants formula

Fix $L_1, L_2, L_3 > 0$.

Let $\Phi = \{$Rooted maps with 3 distinguished faces, and degrees $2L_1, 2L_2, 2L_3\}$

Then $P = Z_\Phi(t; t_2, t_4, \dots)$ satisfies:

$$\boxed{P = \alpha(L_1)\,\alpha(L_2)\,\alpha(L_3)\, R^{L_1+L_2+L_3} \frac{d \ln R}{dt}}$$

We give a geometrical // combinatorial proof (Eynard, Collet-Fusy) of this identity, and a related one. By a decomposition analogous to cylinders, we may focus on the case of "tight pants", where each boundary is of minimal length in its free homotopy class.

Bouttier – Guitter – M.

One gets:

$$\boxed{TP_{\text{tight}} = R^{\ell_1+\ell_2+\ell_3} \frac{d \ln R}{dt}}$$

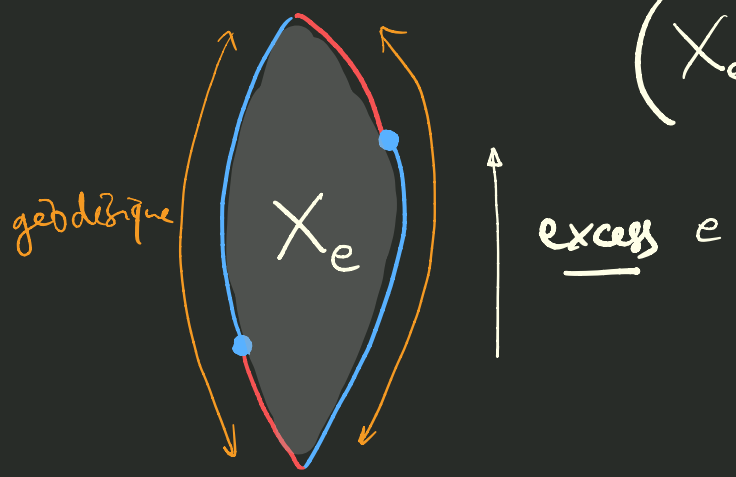
if $\ell_1+\ell_2+\ell_3 > 0$

$2\ell_1, 2\ell_2, 2\ell_3$ degrees of tight boundaries.

An important novelty (essentially coming from the term $\frac{d \ln R}{dt}$) consists in introducing new elementary pieces:

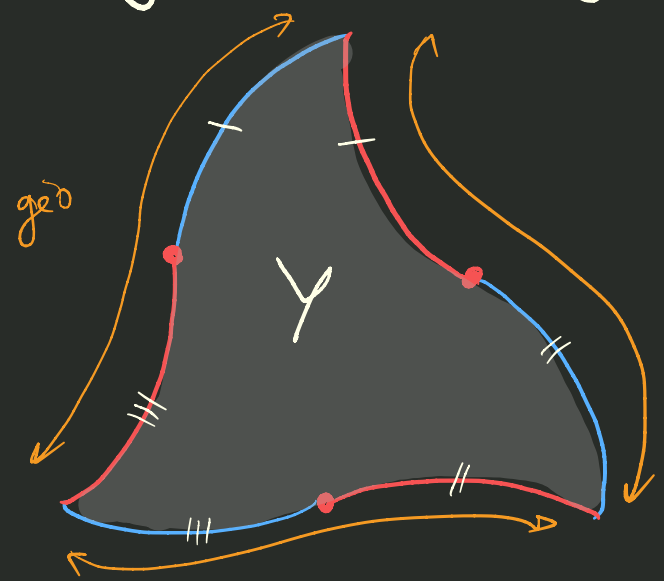
geodesic diangles

$(X_e, Y$ f.g.$)$



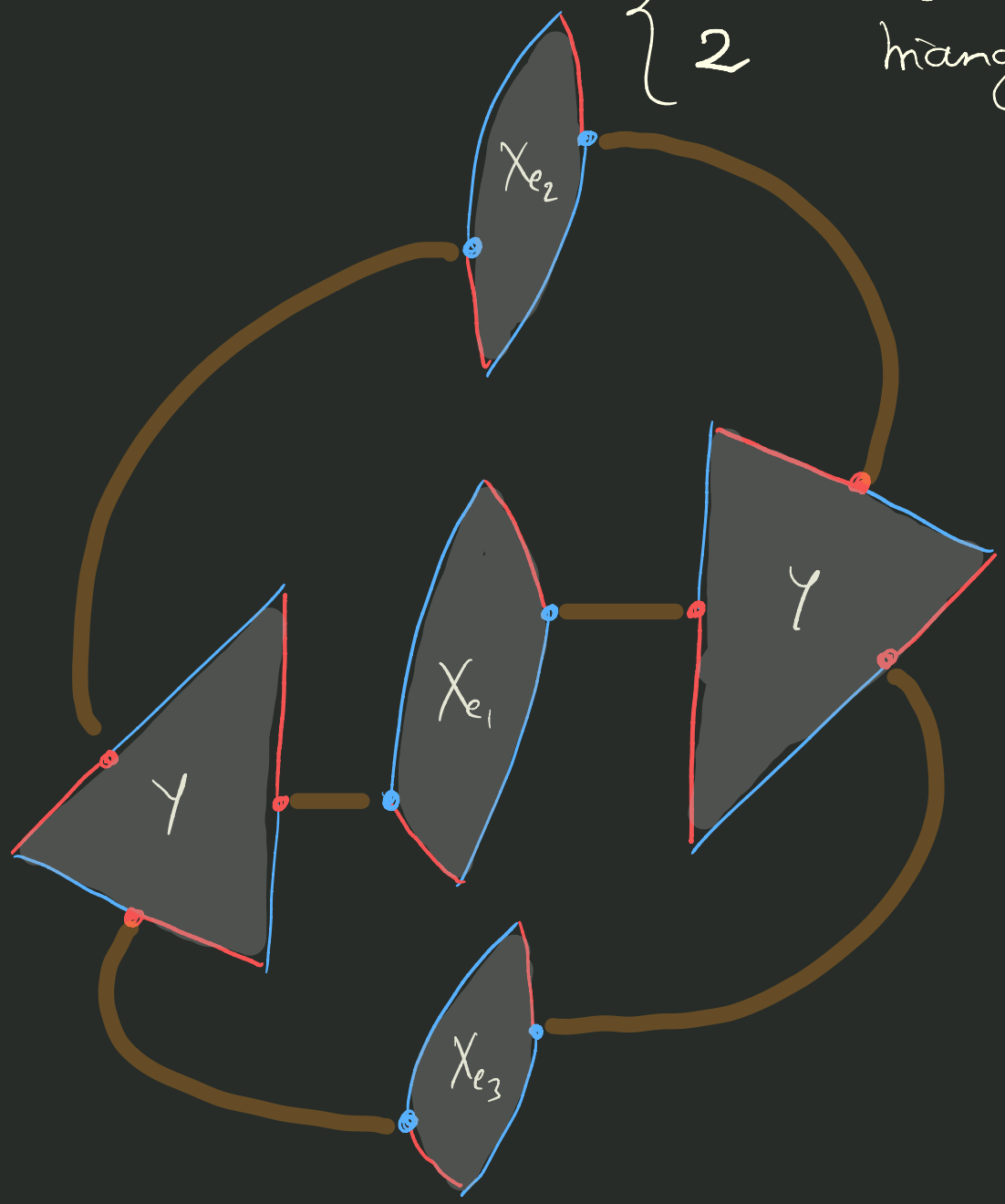
Red = unique geodesic

geodesic triangles



Red dots = mandatory passage points for geodesics.

We build a tight pairs of pants with perimeters $2\ell_1, 2\ell_2, 2\ell_3$ with five pieces : $\begin{cases} 3 \text{ diangles of excesses } e_1, e_2, e_3 \\ 2 \text{ triangles} \end{cases}$



$$\begin{cases} e_1 + e_2 = 2\ell_1 \\ e_2 + e_3 = 2\ell_2 \\ e_3 + e_1 = 2\ell_3 \end{cases}$$

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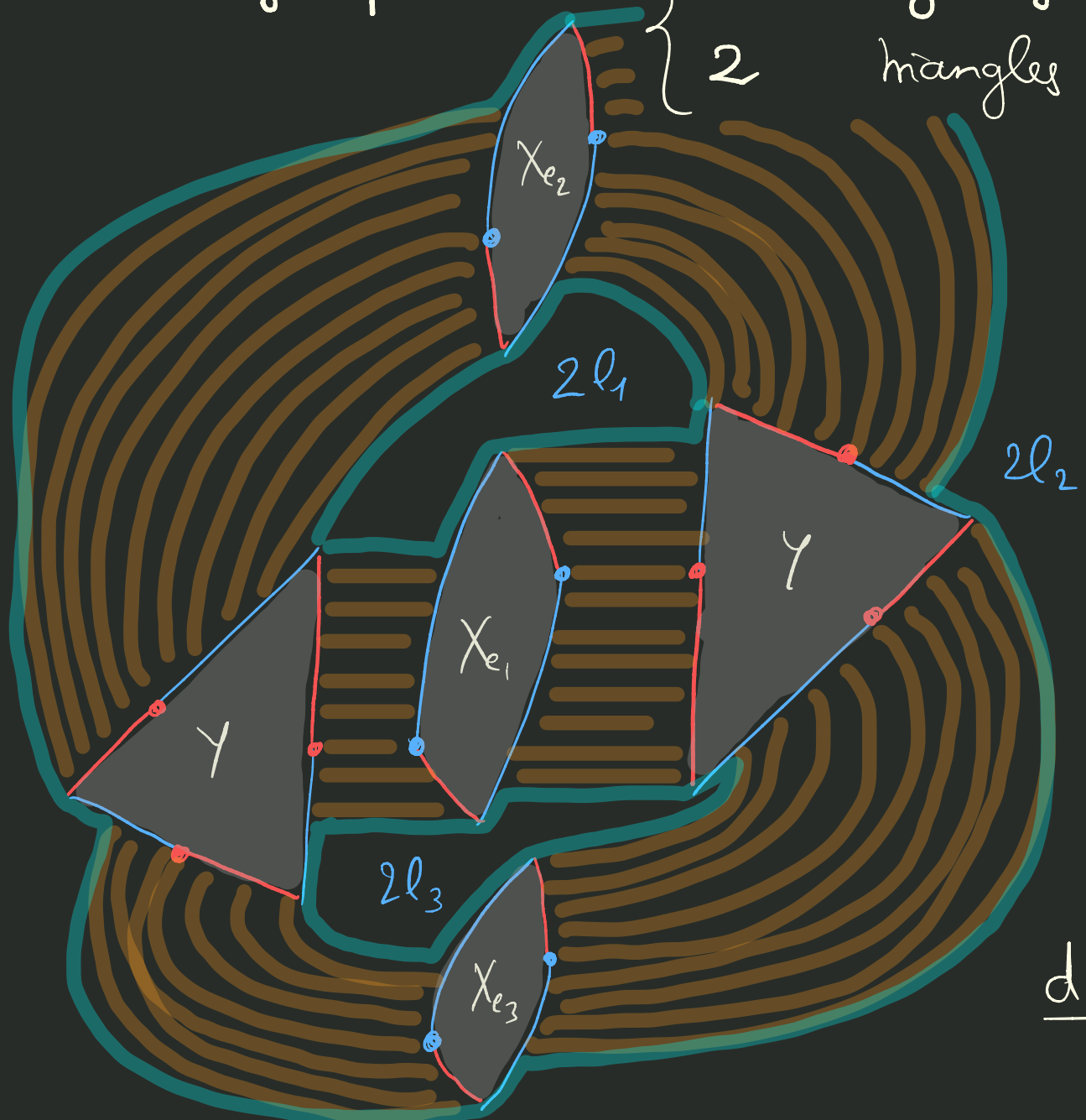

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We conclude with :

<u>Fact 1</u> :

$$X_e = R^e X_0$$

<u>Fact 2</u> :

$$\frac{d \ln R}{dt} = \frac{X_0^3 Y^2}{t^6}$$

3. Some probabilistic consequences

Goal: We want to discuss limits of plane maps (with boundaries)

Let $q = (q_1, q_2, \dots)$ be an admissible sequence, so one can define

$$\mathbb{P}_q(\underline{m}) = \frac{\prod_{f \in \mathrm{Faces}(\underline{m})} q_{\deg(f)/2}}{W_q},$$

q-Boltzmann law on bipartite, plane maps.

and

$$\mathbb{P}_q^{(p)}(\underline{m}) = \frac{\prod_{f \in \mathrm{Faces}(\underline{m}) \setminus \mathrm{root}} q_{\deg(f)/2}}{W_q^{(p)}},$$

q-Boltzmann law on bipartite, plane maps with root face degree $2p$

(similarly, $\mathbb{P}_q^{(p_1, p_2, \dots, p_k)}$).

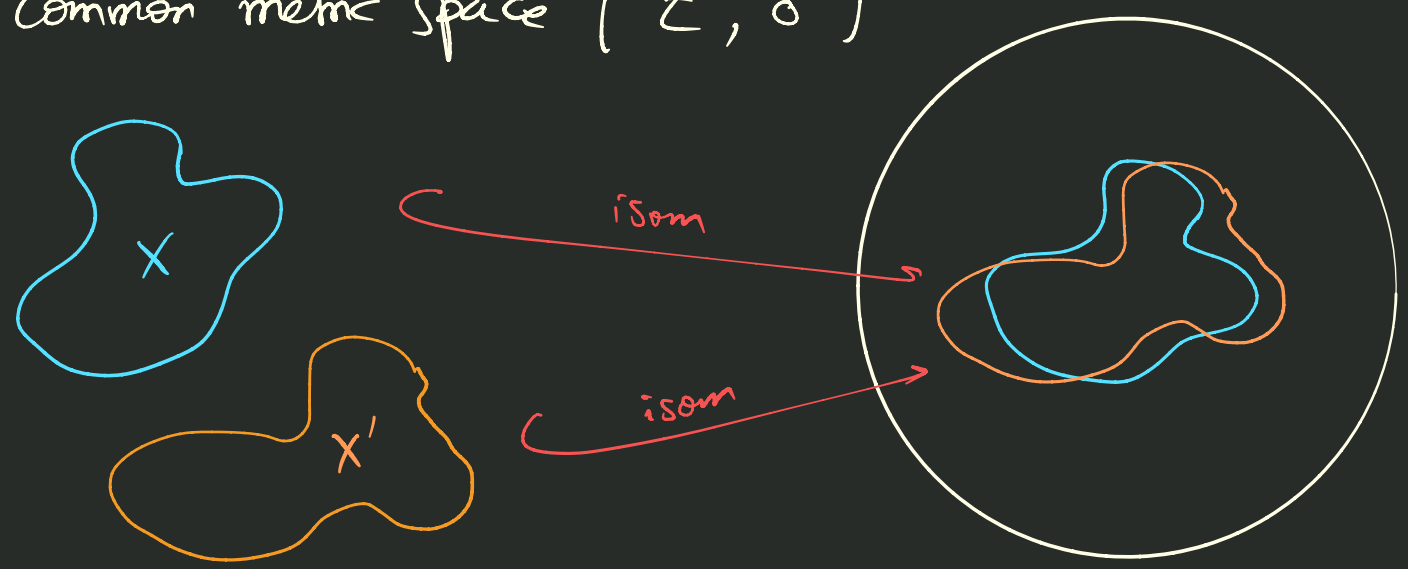
What can we say of $\mathbb{P}_q^{(p)}$ as $p \to \infty$?

(Scaling) limits of metric spaces:

The Gromov-Hausdorff-Prokhorov distance between (X,d,μ), (X',d',μ') compact, metric, measured spaces, is

$$\inf_{\phi,\phi'}\left(\delta_{Haus}(\phi(X),\phi'(X')) \vee \delta_{Pro}(\phi_*\mu,\phi'_*\mu')\right)$$

where ϕ, ϕ' are isometric embeddings of (X,d), (X',d') in a common metric space (Z,δ)



a. Scaling limits for random plane maps and slices

Thm (M. for $q = a\delta_2$, Le Gall): Let q be finitely supported.

Let M_n have law $\mathbb{P}_q(\cdot \mid \# V(\underline{m}) = n)$.

Then for an appropriate constant $c = c_q$

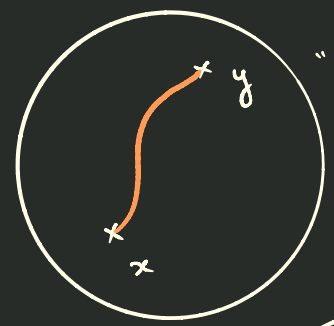
$$\left(V(M_n), \frac{c}{n^{1/4}} d_{M_n}, \frac{1}{n}\sum_{u \in V(M_n)} \delta_u\right) \xrightarrow[n\to\infty]{(d)} (S, D_S, \lambda)$$

in distribution for the GHP topology, where the limit is a random measured compact metric space (the Brownian map, or Brownian sphere), independent of q.

Remark: One can improve a lot the assumptions on q (Marzouk).

Under the same context as above,

let S_n be a random slice obtained by cutting along the leftmost geodesic of the map M_n as above.



x, y "uniformly chosen" in (S, D)

<u>Thm</u> (Le Gall): One has the joint convergence

$$\left(V(M_n), cn^{-1/4} d_{M_n}, e^-, v\right) \xrightarrow[n\to\infty]{(d)} \left(S, D_S, x, y\right)$$

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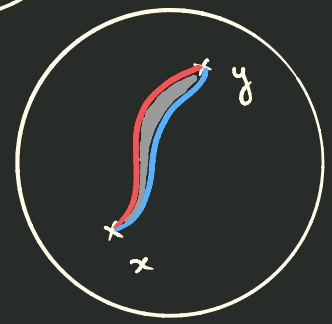


For pointed GH topology

The limit (S', D_S') is the Brownian slice with unit area

The Brownian slice with area A has the law of $(S', A^{1/4} D_S')$.

b. Scaling limits of disks

Thm (Bettinelli–M.): Assume q is a finitely supported weight sequence + criticality condition... Then if M_p has law $\mathbb{P}_q^{(p)}$, one has

$$\left(V(M_p), \left(\frac{2p}{3}\right)^{-1/2} d_{M_p}, \frac{1}{p^2}\sum_{v\in V(M_p)} \delta_v\right) \xrightarrow[p\to\infty]{(d)} (FBD_1, d, \mu)$$

in distribution for the GHP topology

The limit is called the free Brownian disk with perimeter 1.

Its area $\mu(FBD_1)$ has law $\frac{f_{1/2}(x)}{x}\,dx$, where $f_{1/2}$ is a stable $(\frac{1}{2})$ probability density function.

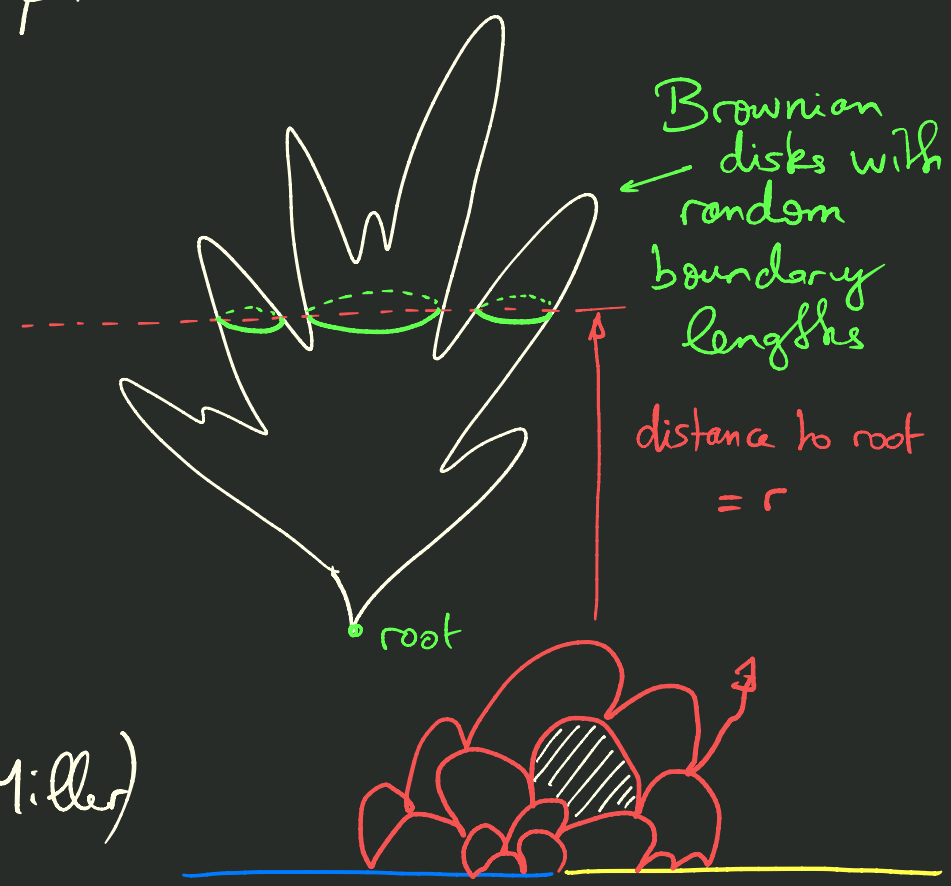
We let $FBD_L \overset{(d)}{=} (FBD_1, L^{1/2} d, L^2 \mu)$.

A curiosity: q does not appear in the rescaling (why??)

Thm (Bettinelli): A.s. FBD_L is homeomorphic to $\{z \in \mathbb{C} : |z| \le 1\}$ with $\dim_{\mathcal{H}aus}(FBD_L) = 4$, $\dim_{\mathcal{H}aus}(\partial FBD_L) = 2$

Brownian disks are known to apear naturally in the study of random maps and the Brownian map:

- Metric structure of (S, D) (Miller-Sheffield, Le Gall)
- SAW on random maps (Gwynne-Miller)
- Percolation exploration / SLE_6 on $LQG_{\sqrt{8/3}}$ surfaces disconnect pieces with law FBD_L (Gwynne-Miller)
- Also relevant in Holden-Sun's Cardy embedding papers.



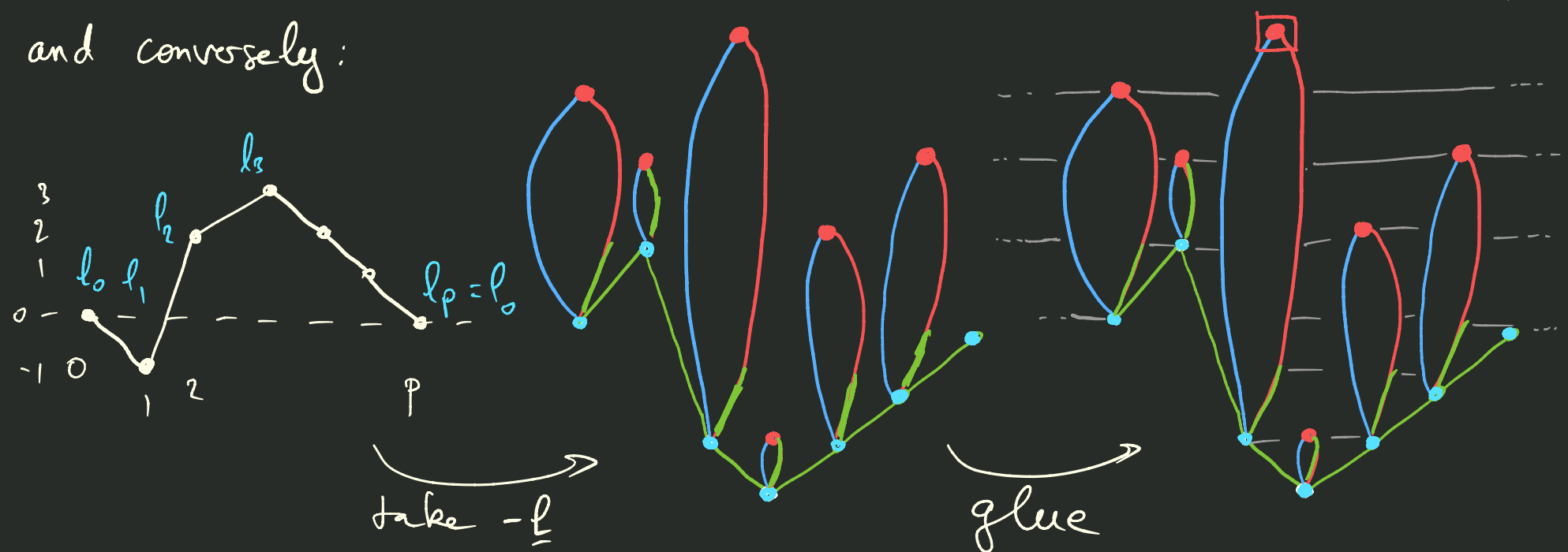
Slicing maps with boundaries

Let $\mathcal{Q}^{\bullet}_{n,p} = \{$pointed quadrangulations with a boundary perimeter $2p$ and n non-root faces$\}$

Then there is a bijection

$$\mathcal{Q}^{\bullet}_{n,p} \longleftrightarrow \{(\underline{\ell}, \underline{s}) : \underline{\ell} = (\ell_0, \dots, \ell_p) \text{ a skip-free walk of length } p \text{ from } 0 \text{ to } 0,\ \underline{s} = (s_1, \dots, s_p) \text{ slices}\}$$

consisting in taking leftmost geodesics from boundary to the distinguished point and conversely:



Slicing maps with boundaries

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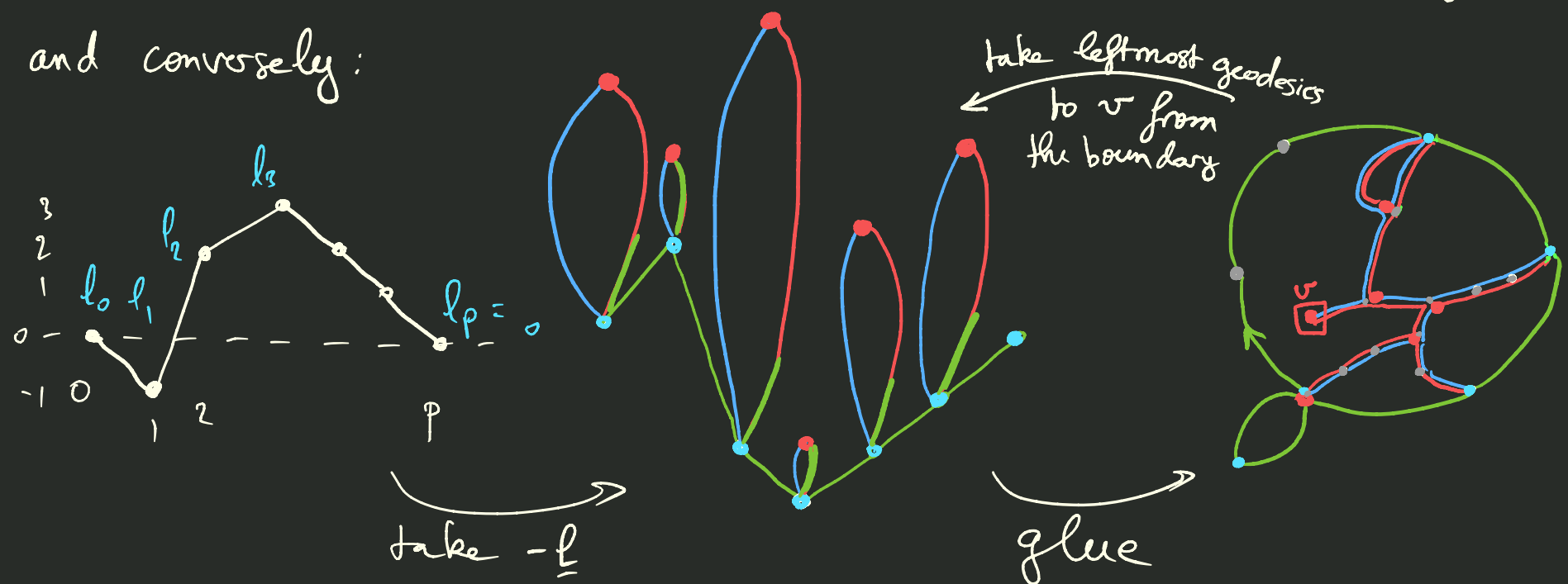


We now want to choose M_p with law $\mathbb{P}^{(p)}_{\frac{1}{12}\delta_2}$ pointed, whose image under the slice decomposition above is the law of $(L_p, (\mathcal{S}_{(1)}, \dots, \mathcal{S}_{(p)}))$ where :

- L_p is a uniform skip-free bridge from 0 to 0 of length p
- $\mathcal{S}_{(1)}, \mathcal{S}_{(2)}, \dots, \mathcal{S}_{(p)}$ are iid slices of $\mathbb{P}^{\bullet}_{\frac{1}{12}\delta_2}$ - distributed quadrangulations.

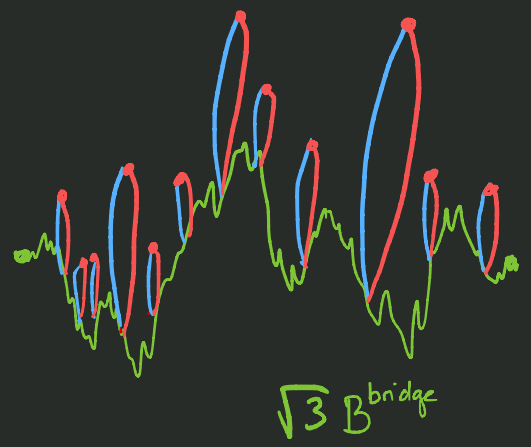
Recall : $\#\{\text{quadrangulations with } n \text{ faces}\} \underset{n\to\infty}{\sim} C\, 12^n\, n^{-5/2}$

So $\#\{\text{Slices with } n \text{ faces}\} \sim \frac{C}{2}\, 12^n\, n^{-3/2}$

Which implies that $\mathbb{P}^{\bullet}_{\frac{1}{12}\delta_2}(\#F(\mathcal{S}_1) > n) \sim C'/n^{1/2}$,

$$\boxed{\text{that is, } \frac{\#F(\mathcal{S}_{(1)}) + \dots + \#F(\mathcal{S}_{(p)})}{p^2} \xrightarrow[p\to\infty]{(d)} \mathcal{A} \quad \text{where } \mathcal{A} \text{ has a Stable}(1/2) \text{ law } f_{1/2}(x)\,dx.}$$

(this explains why the area of FBD_1 has density $\frac{f_{1/2}(x)}{x}\,dx$: one has to <u>unbias</u> the marked vertex).



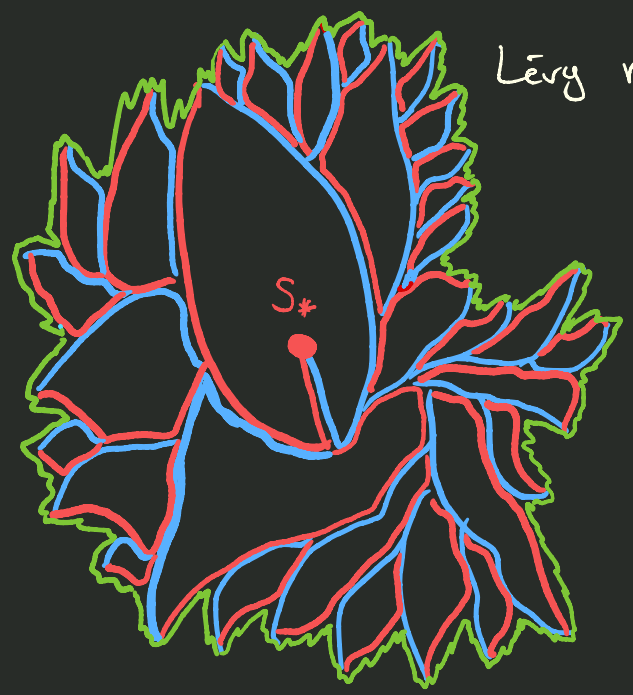
Then : $\left(\dfrac{L_p(\lfloor pt \rfloor)}{\sqrt{2p/3}} \right)_{t \in [0,1]} \xrightarrow{(d)} \left(\sqrt{3}\, B_t^{\text{bridge}} \right)_{t \in [0,1]}$

a standard Brownian bridge

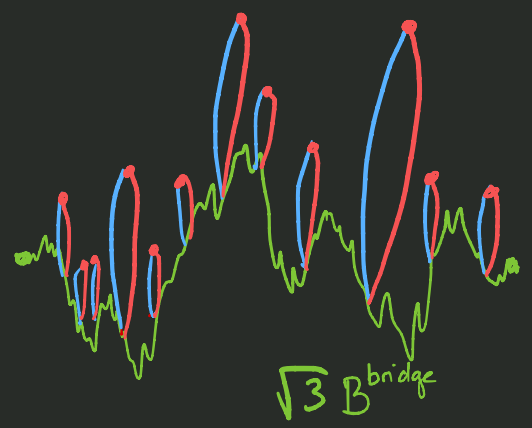
and $\displaystyle\sum_{i=1}^{p} \delta_{\left(i/p,\ \frac{y_{(i)}}{\sqrt{p}} \right)} \longrightarrow$ Poisson RM $(dt \otimes \lambda\, d\Theta)$

λ constant > 0.

where $d\Theta = \displaystyle\int_0^\infty \underbrace{\frac{dA}{2\sqrt{2\pi}\, A^{3/2}}}_{\text{Lévy measure of } \mathcal{J}_{1/2}} \underbrace{\mathrm{Slice}_A(ds)}_{\text{law of a slice of area } A}$



⚠ Gluing along geodesics is well-behaved wrt GH distance, but gluing infinitely many spaces is not!



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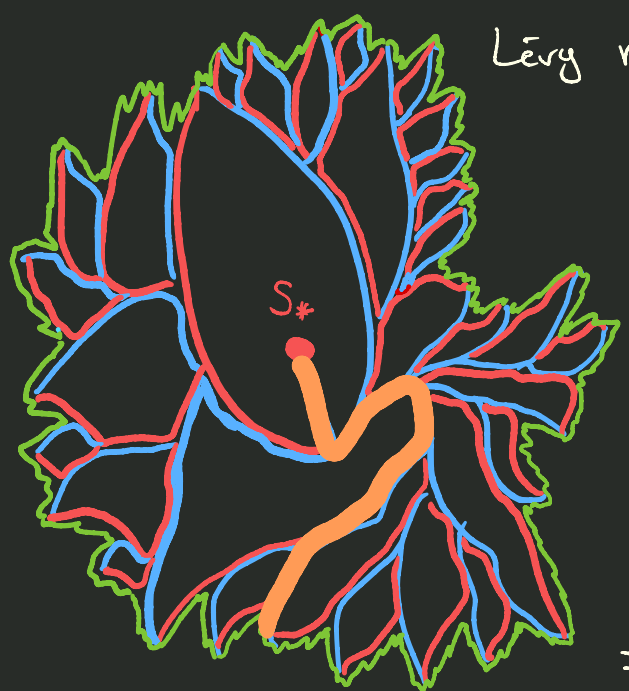
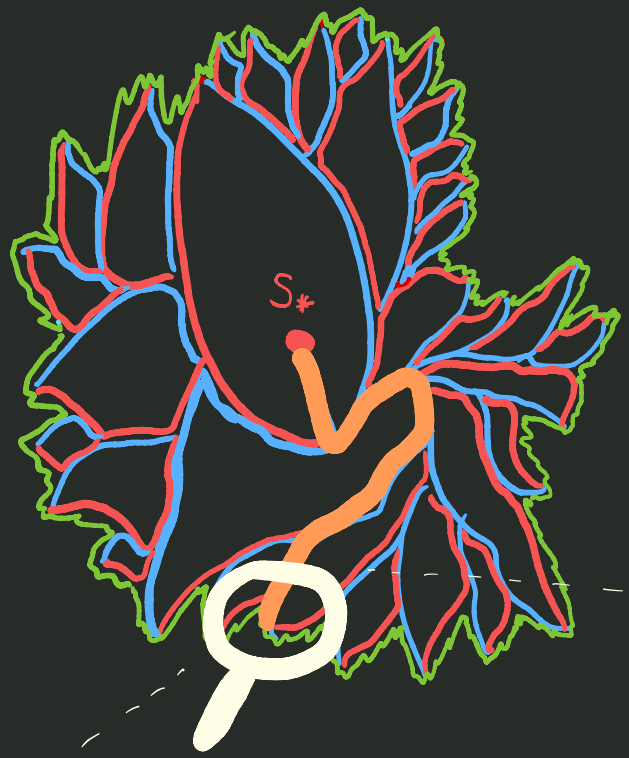


Key lemma : Geodesics to s_* do not intersect the boundary (except maybe at the starting point).

$\Longrightarrow$ Typical geodesics intersect finitely many slices and the metric gluing is well-behaved.

$\Longrightarrow \left(\sqrt{\frac{2p}{3}}\right)^{-1} M_p \xrightarrow[p\to\infty]{d}$ this gluing.



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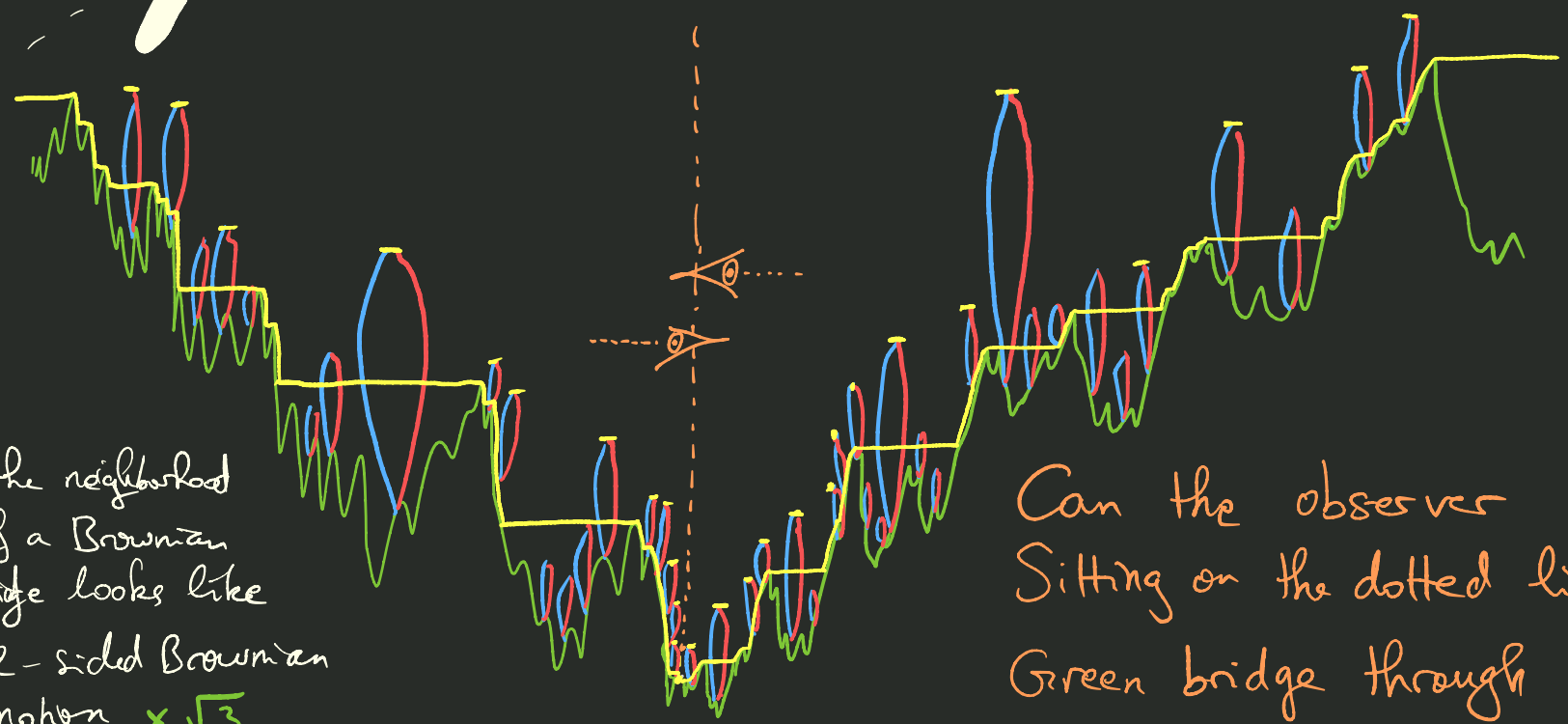
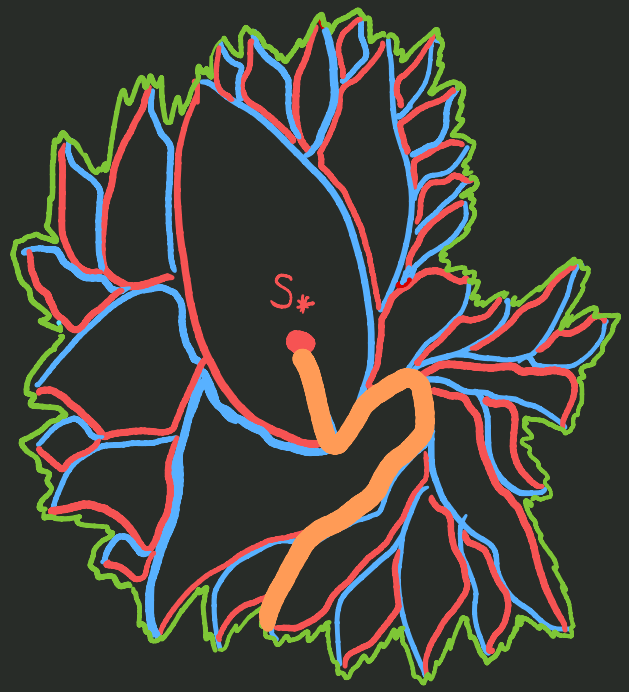
Proved via Shepp's covering theorem for Poisson random measures [Bettinelli]



the neighborhood of a Brownian bridge looks like 2-sided Brownian motion $\times \sqrt{3}$

Can the observer sitting on the dotted line see the Green bridge through the slices?

No!



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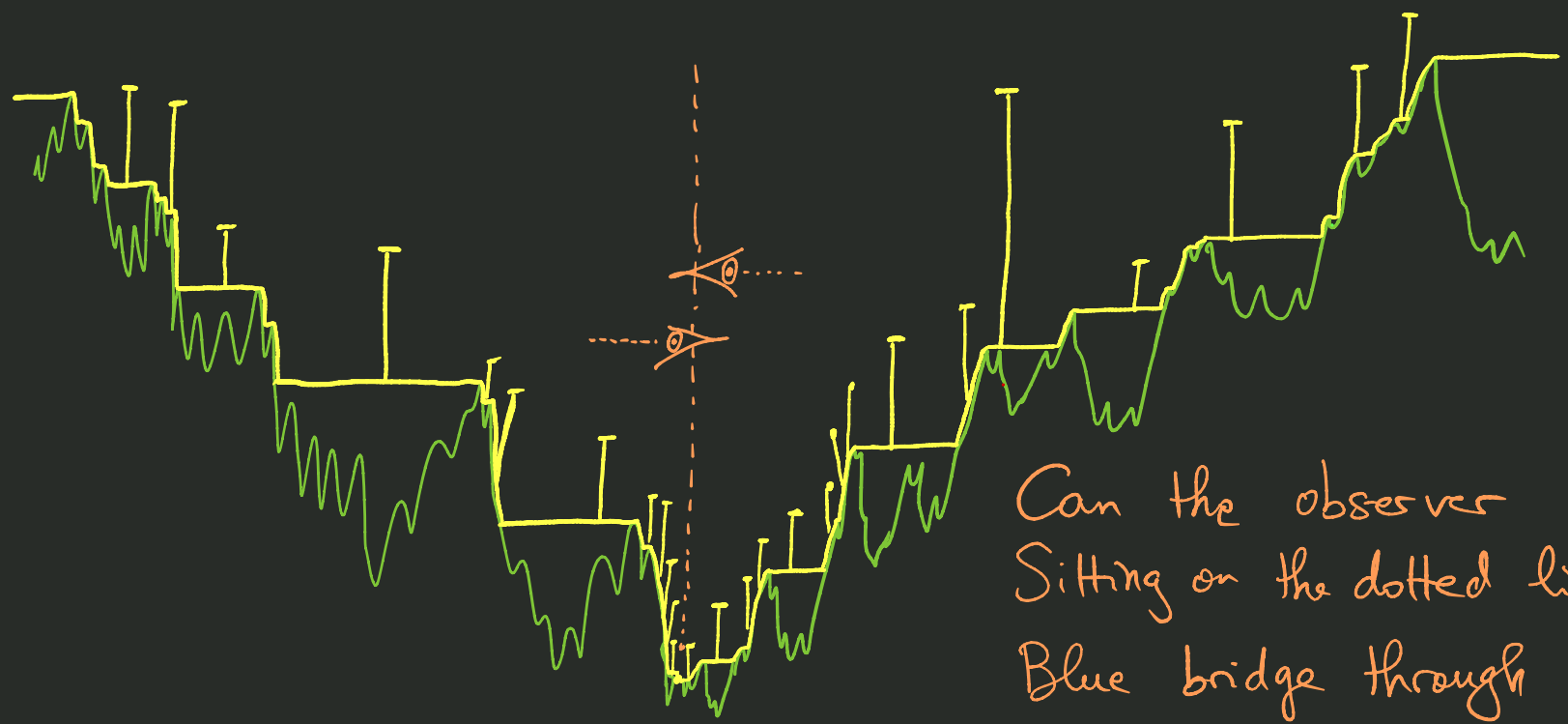
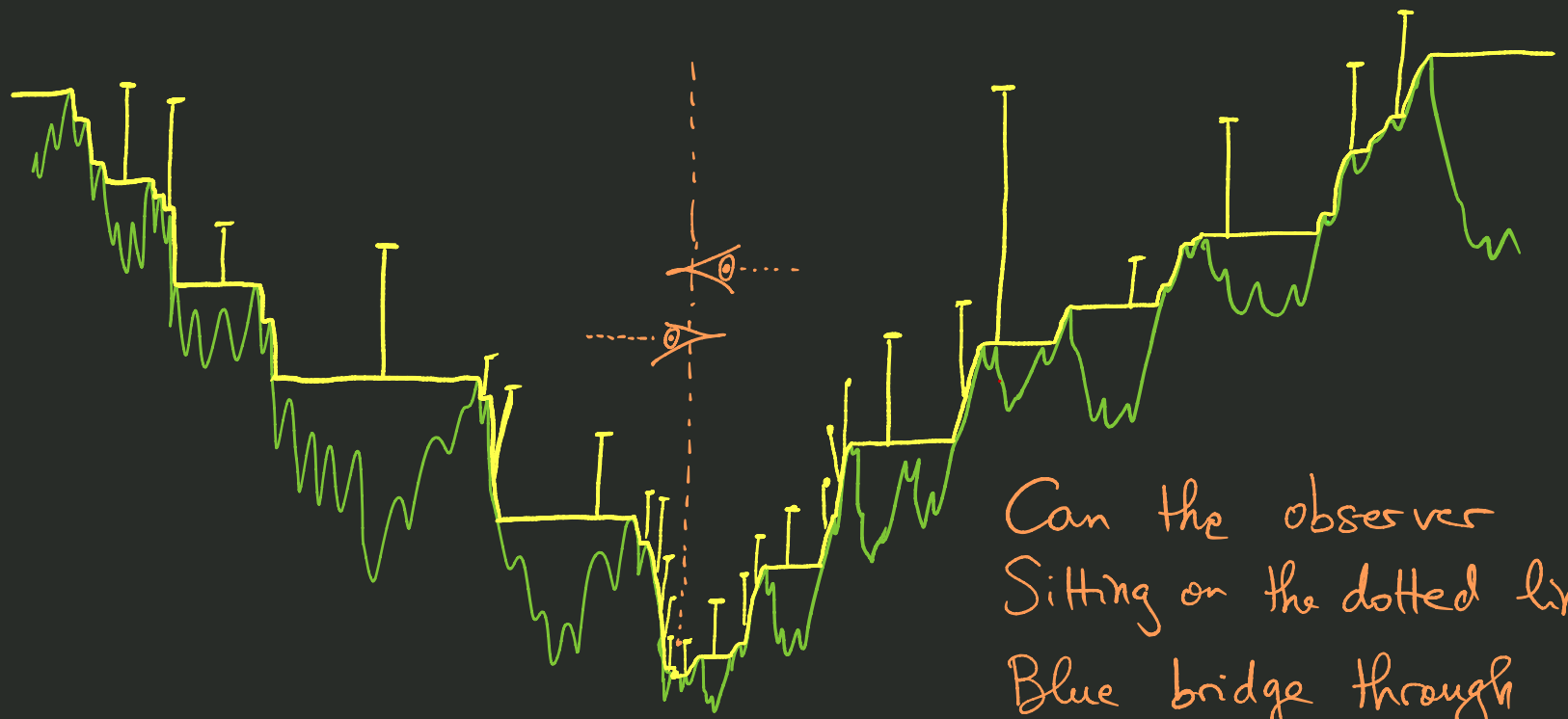


Can the observer sitting on the dotted line see the blue bridge through the ~~slices~~ yellow bars?

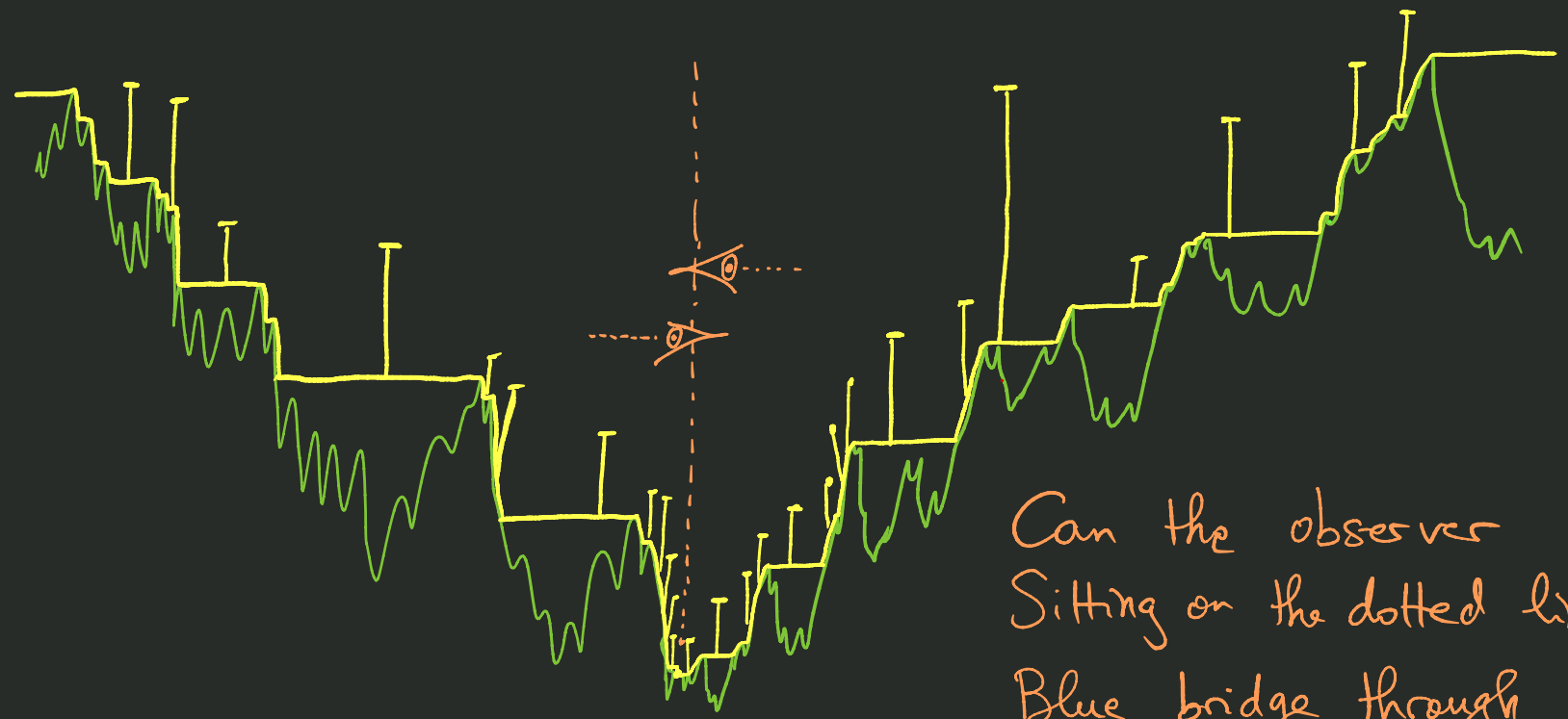
No!



Can the observer sitting on the dotted line see the Blue bridge through the ~~slices~~ yellow bars?

No!

In this form, using excursion theory, the problem becomes a Shepp covering problem of $(0,\infty)$ by intervals $(t_i, t_i + x_i)$, where $\sum_{i \geq 1} \delta_{(t_i, x_i)}$ is a Poisson random measure with intensity $dt \otimes \frac{\nu\, dx}{x^2}$, where ν happens to be 1 (critical case!)



Can the observer sitting on the dotted line see the Blue bridge through the ~~slices~~ yellow bars?

NO! BUT ALMOST!

In this form, using excursion theory, the problem becomes a Shepp covering problem of $(0,\infty)$ by intervals $(t_i, t_i + x_i)$, where $\sum_{i \geq 1} \delta_{(t_i, x_i)}$ is a Poisson random measure with intensity $dt \otimes \frac{\nu\, dx}{x^2}$, where ν happens to be 1 (critical case!)

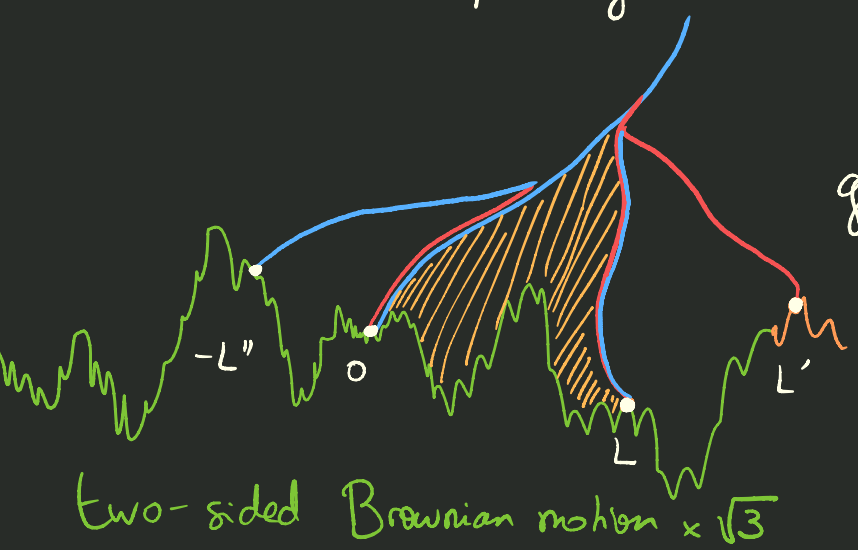
A word on non-compact limits: slices and Brownian half-plane.

The local picture of the Brownian disk near a boundary point (the root say) is described by the Brownian half-plane (Baur–M.–Ray, also Gwynne–Miller):

$$\left(\mathrm{FBD}_L, \frac{1}{\varepsilon} D, \text{root}\right) \xrightarrow[\varepsilon \downarrow 0]{(d)} \mathrm{BHP}, \quad L > 0 \text{ fixed.}$$

for the local Gromov–Hausdorff topology.

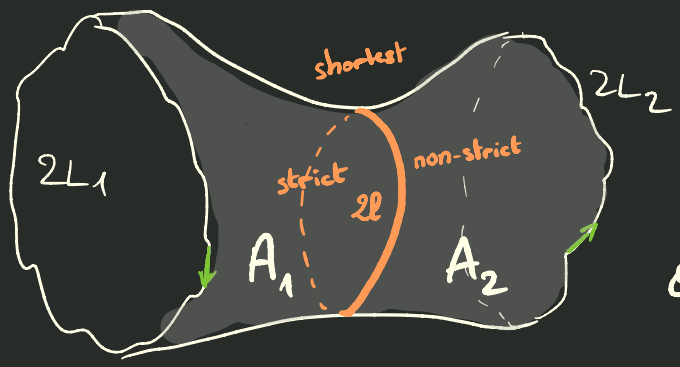
The description of BHP in terms of slices is nicer than for FBD_L.



Gluing all slices between times L and L' gives a "thick slice" of width $L'-L$, and those have a (homogeneous) semigroup property w.r.t gluing.

c. Minimal separating curves in cylinders and pairs of pants.

Let Q_n be a uniformly chosen quadrangulation with two boundaries of perimeters $2L_1^{(n)}, 2L_2^{(n)}$, where $L_i^{(n)} \sim \sqrt{2n}\,\mathcal{L}_i$. Let $2\ell^{(n)}$ be the length of the shortest path separating the boundaries, and $A_1^{(n)}, A_2^{(n)}$ the number of quadrangles on each side.



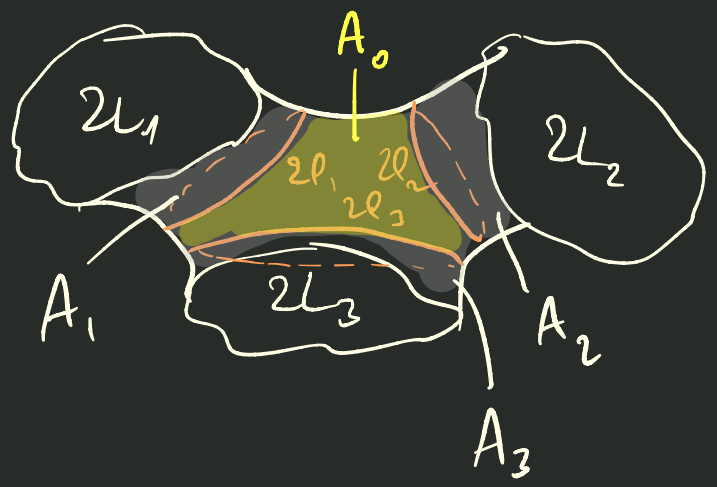
Thm (Bouttier–Guitter–M.): One has

$$\left(A_1^{(n)}/n\,,\ \left(\frac{2}{n}\right)^{1/4}\ell^{(n)}\right) \xrightarrow[n\to\infty]{(d)} (\mathcal{A}_1, \mathcal{R})$$

where $\mathcal{A}_1$ and ℓ are independent, $\mathcal{A}_1$ has a conditioned stable(1/2) distribution, and $\mathcal{R} \sim \text{Rayleigh}\left(\frac{1}{\mathcal{L}_1}+\frac{1}{\mathcal{L}_2}\right)^{-1}$.

Let Q_n be a uniformly chosen quadrangulation with three boundaries of perimeters $2L_1^{(n)}, 2L_2^{(n)}, 2L_3^{(n)}$

$$L_i^{(n)} \sim \sqrt{2n}\,\mathcal{L}_i .$$



Let $2\ell_i^{(n)}$ be the length of the shortest path homotopic to the i-th boundary. And $A_0^{(n)}, A_1^{(n)}, A_2^{(n)}, A_3^{(n)}$ the "interior" and "exterior" areas.

Thm (Bouttier–Guitter–M.): One has

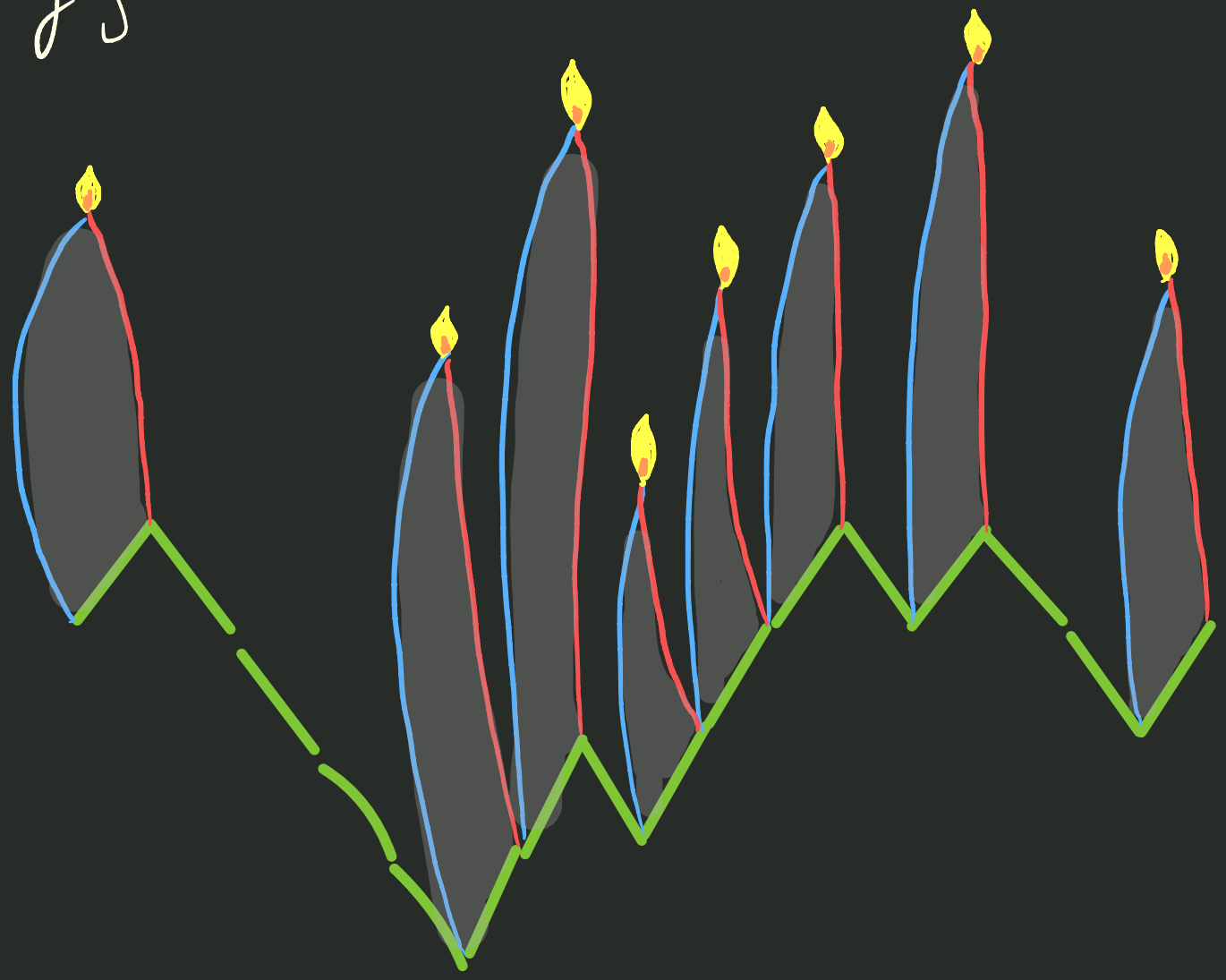
$$\left(\left(A_i^{(n)}/n\right)_{0\le i\le 4}, \left(\left(\frac{2}{n}\right)^{1/4} \ell_i^{(n)}\right)_{1\le i\le 4}\right) \xrightarrow[n\to\infty]{(d)} \left((A_i),(R_i)\right)$$

where A, R are independent, and $R_i \sim \mathrm{Rayleigh}(\mathcal{L}_i)$.

Note: $\min(R_1, R_2, R_3) \sim \mathrm{Rayleigh}\left(\frac{1}{\mathcal{L}_1} + \frac{1}{\mathcal{L}_2} + \frac{1}{\mathcal{L}_3}\right)^{-1}$.

Thanks for listening

Joyeux anniversaire !



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