THÈSE
présentée
à l'UNIVERSITÉ DE BORDEAUX I
pour l'obtention du titre de
DOCTEUR EN MATHEMATIQUES
mention : Informatique théorique
par
Bernard VAUQUELIN

UN TEST D'ISOMORPHISME POUR LES CARTES
DESSINEES DANS LE PLAN

Soutenue le 15 Mars 1976 devant la Commission d'Examen :

Président : François DRESS
Examinateurs : Richard CASTANET
              Robert CORI
              Marcel Paul SCHUTZENBERGER

- 1976 -
La face distinguée est la face infinie.
THÈSE
PRÉSENTÉE À
L'UNIVERSITÉ BORDEAUX I
ÉCOLE DOCTORALE DE MATHEMATIQUES ET INFORMATIQUE
Par Gilles SCHAEFFER
POUR OBTENIR LE GRADE DE
DOCTEUR
SPECIALITÉ : INFORMATIQUE

Conjugaison d'arbres
et
cartes combinatoires aléatoires

Soutenue le jeudi 3 décembre 1998

Après avis de MM. Philippe Flajolet, Directeur de recherche
David M. Jackson, Professeur
Antonio Machi, Professeur

Devant la commission d'examen formée de MM.
Xavier Viennot, Directeur de recherche
Bernard Chazelle, Professeur
Philippe Flajolet, Directeur de recherche
Michel Habib, Professeur
Jean-Marc Steyaert, Professeur
Alexandra Zvonkin, Professeur
Robert Cori, Professeur

--- 1998 ---
Do hypermaps have spanning (hyper)-trees?

Robert Cori

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Joint work with Gábor Hetyei
Spanning trees

**Definition**  In a graph \((V, E)\), each edge \(e \in E\) has two incident vertices. An edge may be incident twice to the same vertex (loops) and two edges may have the same two incident vertices (multiple edges).

*The graphs considered are supposed to be connected*

**Definition** A spanning tree is defined as a subset \(T\) of \(E\) containing \(|V| - 1\) elements and such that \((V, T)\) is connected.
Enumerative properties of spanning trees

Proposition  The number of spanning trees of a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is equal to the determinant of any $(n - 1)$ minor of the Laplacian matrix.

Definition  The Tutte Polynomial of a graph is the sum of monomials in two variables $x$ and $y$ one for each spanning tree of the graph.

The monomial $x^a y^b$ corresponds to a spanning tree with internal activity $a$ and external activity $b$. 
Enumerative properties of spanning trees

**Proposition**  For any edge \( e \), the number of spanning trees of the graph \( G \) is equal to the sum of the number of spanning trees of the graph \( G - e \) obtained by deleting \( e \) and the graph \( G_e \) obtained by contracting the edge \( e \).

The contraction of the edge \( e \) in \( G \) consists in replacing the vertices \( v_i \) and \( v_j \) incident to \( e \) in \( G \) by a new vertex \( v_k \) which is incident to all the edges incident to \( v_i \) or to \( v_j \) in \( G \).
Maps

Definition A combinatorial map consists of two permutations $\sigma$ and $\alpha$ on the set $\{0, 1, \ldots, 2m - 1\}$ such that all the cycles of $\alpha$ have length 2 and such that each half edge $i$ may be attained from 0 by a sequence of $\sigma$ and $\alpha$.

Example Consider the combinatorial map consisting of the two following permutations defined by their cycles:

$$\sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \quad \alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$$

Proposition For a combinatorial map $M = (\sigma, \alpha)$ the faces are represented by the permutation $\alpha \sigma$. 
\[ \sigma = (0,2,4)(1,6,8)(3,9,10)(5,11,7) \quad \alpha = (0,1)(2,3)(4,5)(6,7)(8,9)(10,11) \]
σ = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7)  \quad \alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)

ασ = (0, 3, 8)(1, 7, 4)(2, 5, 10)(6, 9, 11)
Hypermaps

Definition A hypermap consists of two permutations $\sigma$ and $\alpha$ on the set $\{1, 2, \ldots, n\}$ such that each point $i$ may be attained from 1 by a sequence of $\sigma$ and $\alpha$.

Example Consider the hypermap consisting of the two following permutations defined by their cycles:

$$\sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9, 10)(11, 12) \quad \alpha = (1, 6)(2, 11, 9, 5)(3, 7)(4, 10)(8, 12)$$

Proposition For a hypermap $M = (\sigma, \alpha)$ the faces are represented by the permutation $\alpha^{-1}\sigma$. 
\[ \sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9, 10)(11, 12) \quad \alpha = (1, 6)(2, 11, 9, 5)(3, 7)(4, 10)(8, 12) \]
\(\sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9, 10)(11, 12)\) \(\alpha = (1, 6)(2, 11, 9, 5)(3, 7)(4, 10)(8, 12)\)

\(\alpha^{-1}\sigma = (1, 5)(2, 7, 12)(3, 6, 10)(4, 9)(8, 11)\)
Genus

**Theorem** (Alain Jacques, 1968) The genus of a hypermap given by the formula below is a non negative integer:

\[ z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g(\sigma, \alpha) \]

\( z(\alpha) \) denotes the number of cycles of the permutation \( \alpha \).
Spanning trees of maps

**Definition** A map $T = (\sigma, \theta)$ is a tree if it is genus is equal to 0 and if the permutation $\theta \sigma$ has one cycle. A spanning tree of the map $M = (\sigma, \alpha)$ is a tree $T = (\sigma, \theta)$ such that $\theta$ is obtained from $\alpha$ by deleting some edges, that is replacing an edge $(i, j)$ by $(i)(j)$.

**Example** In the map above:

$\sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7)$  
$\alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$

A spanning tree is given by:

$\theta = (4, 5)(6, 7)(10, 11)$

since

$\theta \sigma = (0, 2, 5, 10, 3, 9, 11, 6, 8, 1, 7, 4)$
\[ \sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \quad \alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11) \]
\[ \sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \quad \theta = (0)(1)(2)(3)(4, 5)(6, 7)(8)(9)(10, 11) \]

\[ \theta \sigma = (0, 2, 5, 10, 3, 9, 11, 6, 8, 1, 7, 4) \]
Bernardi tour

The permutation $\theta \sigma$ is sometimes called Bernardi tour

References


Farbod Shokrieh and Cameron Wright Torsor Structures on spanning trees, arXiv: 2103.10370v1 [math.CO]

Refinements of permutations

Definition Let $\zeta$ be a cyclic permutation the permutation $\theta$ is a refinement of the permutation $\zeta$ if $g(\zeta, \alpha) = 0$; equivalently the cycles of $\theta$ determine a non crossing partition of the cycle $\zeta$.

Consider the permutation:

$\zeta = (1, 2, 3, 4, 5)$

Then $\theta = (1, 3)(2)(4, 5)$ is a refinement of $\zeta$ but $\theta' = (1, 3)(2, 4)(5)$ is not.

Remark Notice that the number of refinements of a cycle of length $n$ is the Catalan number $C_n$. 
Refinements of permutations

**Definition** The permutation $\theta$ is a refinement of the permutation $\alpha$ if two elements in the same cycle of $\theta$ are also in the same cycle of $\alpha$. And for each cycle of $\alpha$ the cycles $\theta_1, \theta_2, \ldots, \theta_k$ of $\theta$ contained in it determine a refinement of this cycle.

Consider the permutation:

$$\alpha = (1, 7, 3)(4, 6, 9, 8)(2, 5, 11, 12, 10)$$

Then $\theta = (1, 3)(7)(4, 9)(6)(8)(2, 12, 10)(5, 11)$ is a refinement of $\alpha$ but $\theta' = (1, 3)(7)(2)(4, 9)(6, 8)(2, 12, 10)(5, 11)$ is not.
**Hypertrees**

**Definition** A hypertree is a hypermap \((σ, θ)\) such that \(g(σ, θ) = 0\) and \(z(θ^{-1}σ) = 1\) (it has only one face).

**Does not correspond to Wikipedia definition** In mathematics, a hypergraph \(H\) is called a hypertree if it admits a host graph \(T\) such that \(T\) is a tree. In other words, \(H\) is a hypertree if there exists a tree \(T\) such that every hyperedge of \(H\) is the set of vertices of a connected subtree of \(T\).

**Corresponds to those considered in Béatrice Delcroix-Oger thesis**
Hypertrees

A hypertree with 8 vertices and 5 hyperedges.
Spanning hypertrees

**Definition** The hypermap \((\sigma, \theta)\) is a spanning hypertree of the hypermap \((\sigma, \alpha)\) if it is a hypertree and if \(\theta\) is a refinement of \(\alpha\).

**Remark** The spanning trees of a map are spanning hypertrees
First result

**Definition** The number of spanning hypertrees of a planar hypermap \((\sigma, \alpha)\) is equal to the number of that of its dual: \((\alpha^{-1}\sigma, \alpha^{-1})\).

**Proof** If \((\sigma, \theta)\) is a spanning hypertree of the planar hypermap \((\sigma, \alpha)\) then \((\alpha^{-1}\sigma, \alpha^{-1}\theta)\) is a spanning hypertree of the dual \((\alpha^{-1}\sigma, \alpha^{-1})\)
Meanders

Definition A meander of order n is a self-avoiding plane closed curve which intersects a straight line in 2n points. The meander is defined up to an isotopy of the plane which preserves the intersection points.

Notice that a meander can be defined by two involutions \( \alpha_1, \alpha_2 \) such that 
\[
g(\zeta, \alpha_1) = 0 \quad \text{and} \quad g(\zeta, \alpha_2) = 0
\]
where \( \zeta \) is the circular permutation
\[
\zeta = (1, 2, \cdots, 2n)
\]
and such that the group generated by \( \alpha_1, \alpha_2 \) is transitive.
Meanders
Meanders
Meanders
Meanders
Meanders

\[ \alpha_1 = (1, 4)(2, 3)(5, 10)(6, 7)(8, 9)(11, 12) \]
Meanders

\[ \alpha_2 = (1, 12)(2, 11)(3, 6)(4, 5)(7, 8)(9, 10) \]
Main result 1

Theorem  The number of meanders of order $n$ is equal to the number of spanning hypertrees of the hypermap:

$$\sigma = (1, 2n)(2, 3) \ldots (2i, 2i+1) \ldots (2n-2, 2n-1) \quad \alpha = (1, 3, \ldots, 2n-1)(2, 4, \ldots, 2n)$$

Remarks

$$\alpha \sigma = (1, 2)(3, 4) \ldots (2i - 1, 2i) \ldots (2n - 1, 2n)$$

This hypermap is the reciprocal of a map, namely the dipole with $n$ edges.
The 8 meanders of order 3
The 8 hypertrees of the hypermap reciprocal of a dipole with 3 edges

\[ \alpha = (1, 3, 5)(2, 4, 6) \]

\[ \sigma = (1, 6)(2, 3)(4, 5) \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \theta^{-1}\sigma )</th>
</tr>
</thead>
</table>
| \( (1, 3, 5) \) | \( (1, 6, 5, 4, 3, 2) \) | x 
| \( (2, 4, 6) \) | \( (1, 2, 3, 4, 5, 6) \) | x 
| \( (1, 3)(2, 4) \) | \( (1, 6, 3, 4, 5, 2) \) | x 
| \( (1, 5)(2, 4) \) | \( (1, 6, 5, 2, 3, 4) \) | x 
| \( (3, 5)(2, 4) \) | \( (1, 6)(3, 4)(5, 6) \) | - 
| \( (1, 3)(2, 6) \) | \( (1, 2)(3, 6)(4, 5) \) | - 
| \( (1, 5)(2, 6) \) | \( (1, 2, 3, 6, 5, 4) \) | x 
| \( (3, 5)(2, 6) \) | \( (1, 2, 5, 4, 3, 6) \) | x 
| \( (1, 3)(4, 6) \) | \( (1, 4, 5, 6, 3, 2) \) | x 
| \( (1, 5)(4, 6) \) | \( (1, 4)(2, 3)(5, 6) \) | - 
| \( (3, 5)(4, 6) \) | \( (1, 4, 3, 2, 5, 6) \) | x 

Robert Cori  
Fête de la Bijection LaBRI, October 11, 2021
The 8 meanders of order 3

$\theta^{-1}\sigma = (1, 2, 5, 4, 3, 6), \quad \theta^{-1}\sigma = (1, 4, 5, 6, 3, 2) \quad \theta^{-1}\sigma = (1, 6, 5, 2, 3, 4)$
Hyperdeletion

**Definition** A hyperdeletion is the operation of replacing a hypermap \((\sigma, \alpha)\) with the hypermap \((\sigma, \alpha \tau)\) where \(\tau = (i, j)\) is a transposition such that \(i, j\) are two points in the same cycle of \(\alpha\).

We call the hyperdeletion topological if \(i, j\) are two points in different cycles of \(\alpha^{-1}\sigma\). Such a deletion does not modify the genus of a hypermap.

**Remark** Notice that the hypermap \((\sigma, \alpha \tau)\) obtained by a deletion is such that \(\alpha \tau\) is a refinement of \(\alpha\).

**Proposition** Let \(H = (\sigma, \alpha)\) be a hypermap and let \(i, j\) be two elements of a cycle of \(\alpha\) belonging to different cycles of \(\alpha^{-1}\sigma\). Let \(H'\) be the hypermap obtained by deleting \((i, j)\) in \(H\). Then any spanning hypertree of \(H'\) is also a spanning hypertree of \(H\).
Hypercontraction

**Definition**  A *hypercontraction* is the operation of replacing a hypermap \((\sigma, \alpha)\) with the hypermap \((\tau \sigma, \tau \alpha)\) where \(\tau = (i, j)\) is a transposition such that \(i, j\) are two points in the same cycle of \(\alpha\) and in two different cycles of \(\sigma\).

**Proposition**  Let \(H = (\sigma, \alpha)\) be a hypermap and let \(i, j\) be two elements of a cycle of \(\alpha\) belonging to the different cycles of \(\sigma\). Let \(H'\) be the hypermap obtained by contracting \((i, j)\) in \(H\). Then any spanning hypertree \((\tau \sigma, \theta)\) of \(H'\) is such that \((\sigma, \tau \theta)\) is a spanning hypertree of \(H\).
A recurrence for the number of spanning hypertrees (1)

**Proposition** Let $H = (\sigma, \alpha)$ be a hypermap such that $(1, 2)$ is a cycle of $\alpha$, then the number of spanning hypertrees of $H = (\sigma, \alpha)$ is equal to the sum of the numbers of that of $\text{delete}(H, 1, 2)$ and of that of $\text{contract}(H, 1, 2)$. 
A recurrent formula for the number of spanning hypertrees (2)

**Proposition** Let $H = (\sigma, \alpha)$ be a hypermap such that $(1, 2, 3)$ is a cycle of $\alpha$. Let $H_1$ be the hypermap obtained by deleting $(1, 3)$ in $H$, $H_2$ be the hypermap obtained by contracting $(1, 2)$ in $H$ and $H_3$ be the hypermap obtained by first contracting $(1, 3)$ in $H$ then deleting $(1, 2)$. Then the number of spanning hypertrees of $H$ is equal to the sum of the numbers of that of $H_1$, $H_2$ and $H_3$. 
\[ H_1 = \text{del}(1, 3, H), \ H_2 = \text{contr}(1, 2, H), \ H_3 = \text{del}(1, 2, \text{contr}(1, 3, H)) \]

- The hypertrees \((\sigma, \theta)\) of \(H\) with cycles \((1)(2)(3)\) in \(\theta\) come from those with the same cycles in \(H_1\).
- The hypertrees \((\sigma, \theta)\) of \(H\) with cycles \((1, 2)(3)\) in \(\theta\) come from those in \(H_2\) with cycles \((1)(2)(3)\)
- The hypertrees \((\sigma, \theta)\) of \(H\) with cycles \((1, 3)(2)\) in \(\theta\) come from those in \(H_3\)
- The hypertrees \((\sigma, \theta)\) of \(H\) with cycles \((1)(2, 3)\) in \(\theta\) come from those in \(H_1\) with the same cycles.
- The hypertrees \((\sigma, \theta)\) of \(H\) with cycles \((1, 2, 3)\) in \(\theta\) come from those in \(H_2\) with cycles \((1)(2, 3)\)
Main result 2.

Proposition  Let $H = (\sigma, \alpha)$ be a hypermap such that $(1, 2, \cdots, k)$ is a cycle of $\alpha$. Let $H_1$ be the hypermap obtained by deleting $(1, k)$ in $H$, $H_2$ be the hypermap obtained by contracting $(1, 2)$ in $H$ and $H_i$ for $i = 3, \ldots, k$ be the hypermap obtained by first contracting $(1, i)$ in $H$ then deleting $(1, i - 1)$. Then the number of spanning hypertrees of $H$ is equal to the sum of the numbers of that of $H_1, H_2, \cdots, H_k$.

Proof  Use the decomposition of non crossing partitions into disjoint subsets given by Simion et Ullmann