THÈSE

présentée

à l'UNIVERSITE DE BORDEAUX I

pour l'obtention du titre de

DOCTEUR EN MATHEMATIQUES

mention : Informatique théorique

par

Bernard VAUQUELIN
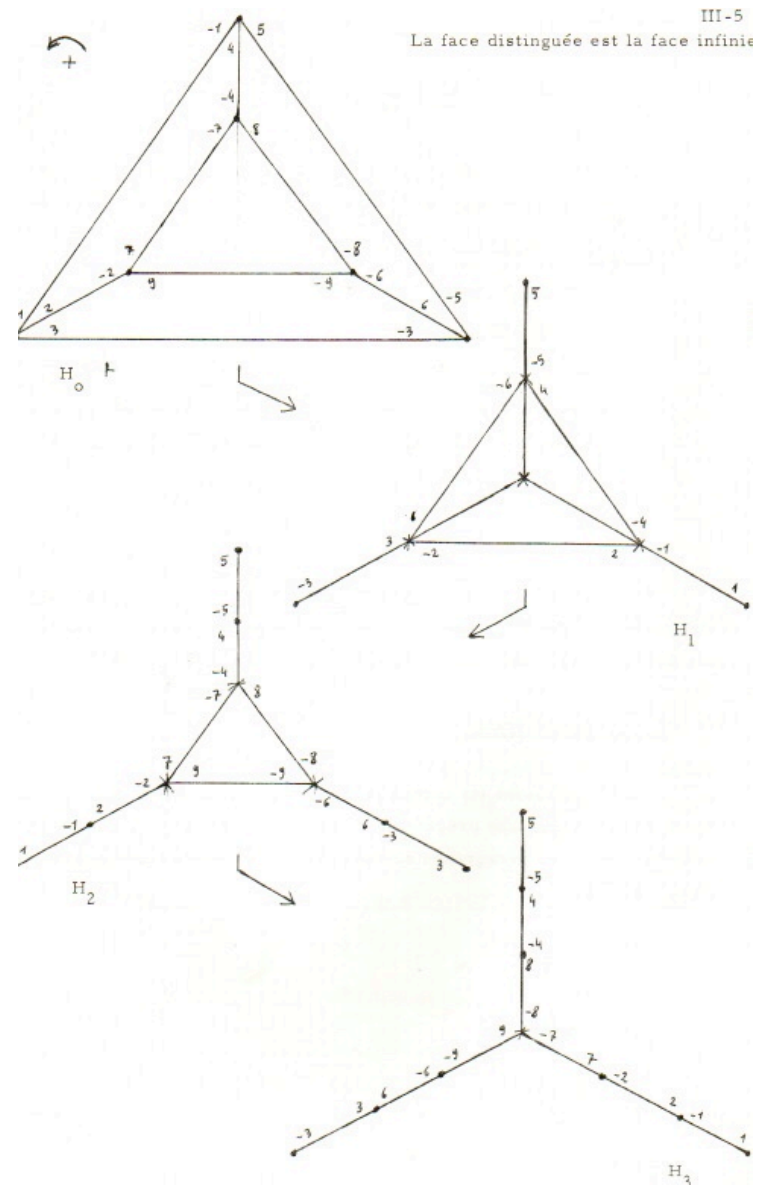
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UN TEST D'ISOMORPHISME POUR LES CARTES

DESSINEES DANS LE PLAN

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Soutenue le 15 Mars 1976 devant la Commission d'Examen :

Président :   François DRESS

Examinateurs :   Richard CASTANET
                 Robert CORI
                 Marcel Paul SCHUTZENBERGER

- 1976 -

III-5
La face distinguée est la face infinie

H_0

H_1

H_2

H_3

```
'BEGIN' 'INTEGER' M;
    'PROCEDURE' ECLATE(SD,A,SR,ETI); 'INTEGER' 'ARRAY' SD,SR,ETI;
                                     'INTEGER' A; 'BEGIN'
        'INTEGER' 'ARRAY' SL,PL(/-M:M+1/); 'INTEGER' B,C,X,Y,K;
        'PROCEDURE' RESTR(SS,U,V); 'INTEGER' 'ARRAY' SS;
                                   'INTEGER' U,V; 'BEGIN'
            'INTEGER' R,A,B,C;
            R:=V;
            'FOR' A:=SL(/U/) 'WHILE' A 'NE' U 'DO' 'BEGIN' B:=A;
                'FOR' C:=A, B 'WHILE' C 'NE' A 'DO' 'BEGIN'
                    'FOR' B:=SS(/B/) 'WHILE' ETI(/B/) < C 'DO' 'BEGIN'
                        ETI(/B/):=K;
                        SL(/R/):=B; PL(/B/):=R; R:=B 'END';
                    SR(/C/):=B;
                    SL(/PL(/B/)/):=SL(/B/); PL(/SL(/B/)/):=PL(/B/) 'END';
                'END';
            SL(/R/):=V; PL(/V/):=R 'END' RESTR;
        X:=M+1; Y:=0;
        'FOR' B:=-M 'STEP' 1 'UNTIL' M 'DO' 'BEGIN'
            SR(/B/):=-SD(/B/); ETI(/B/):=-1 'END';
    C:=A;
        'FOR' B:=SD(/C/) 'WHILE' B 'NE' A 'DO' 'BEGIN'
            SL(/C/):=B; PL(/B/):=C; ETI(/B/):=0, C:=B 'END';
        SL(/X/):=A; SL(/C/):=X; PL(/A/):=X; PL(/X/):=C; ETI(/A/):=0;
        SL(/Y/):=PL(/Y/):=Y;
        K:=0;
        BOUCLE: K:=K+1; RESTR(SR,X,Y);
            'IF' SL(/Y/)=Y 'THEN' 'GOTO' FIN;
            K:=K+1; RESTR(SD,Y,X);
            'IF' SL(/X/)=X 'THEN' 'GOTO' FIN;
            'GOTO' BOUCLE;
        FIN: 'END' ECLATE;
    'PROCEDURE' ELAGUE(SD,A,TYPE); 'INTEGER' 'ARRAY' SD;
                                   'INTEGER' A,TYPE; 'BEGIN'
        'INTEGER' 'ARRAY' LBP(/1:1/); 'INTEGER' 'ARRAY' SA,PA(/-M:M/);
        'INTEGER' I,J,K,B,C;
        J:=0; SD(/0/):=1;
        'FOR' B:=-M 'STEP' 1 'UNTIL' M 'DO' 'BEGIN'
            'IF' SD(/B/)=B 'THEN' 'BEGIN' J:=J+1; LBP(/J/):=B 'END';
            SA(/B/):=SD(/B/); PA(/SA(/B/)/):=B 'END';
        BOUCLE: 'IF' LBP(/1/)=LBP(/2/) 'THEN' 'BEGIN'
            TYPE:=0; A:=LBP(/1/); 'GOTO' FIN 'END';
            'IF' LBP(/1/)=-LBP(/2/) 'THEN' 'BEGIN'
            TYPE:=1; A:=LBP(/1/); 'GOTO' FIN 'END';
            K:=J; J:=0;
            'FOR' I:=1 'STEP' 1 'UNTIL' K 'DO' 'BEGIN'
                B:=-LBP(/I/); A:=PA(/B/); C:=SA(/B/);
                'IF' A=C 'THEN' 'BEGIN' J:=J+1; LBP(/J/):=A 'END';
                PA(/C/):=A; SA(/A/):=C 'END';
            'GOTO' BOUCLE;
        FIN: 'END' ELAGUE;
    'PROCEDURE' CODE(SA,A,ETI,MOT); 'INTEGER' 'ARRAY' SA,ETI,MOT;
                                    'INTEGER' A; 'BEGIN'
        'INTEGER' 'ARRAY' VAL(/-M:M/); 'INTEGER' B,I;
        'FOR' B:=-M 'STEP' 1 'UNTIL' M 'DO' VAL(/B/):=ETI(/-B/)
```

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ BORDEAUX I

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE

Par Gilles SCHAEFFER

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : INFORMATIQUE

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Conjugaison d'arbres

et

cartes combinatoires aléatoires

Soutenue le jeudi 3 décembre 1998

Après avis de MM. Philippe Flajolet,   Directeur de recherche   ··········   Rapporteurs
David M. Jackson,   Professeur
Antonio Machí,   Professeur

Devant la commission d'examen formée de MM.

Xavier Viennot,   Directeur de recherche   ···············   Président
Bernard Chazelle,   Professeur   ···············   Examinateurs
Philippe Flajolet,   Directeur de recherche
Michel Habib,   Professeur
Jean-Marc Steyaert,   Professeur
Alexandre Zvonkin,   Professeur   ···············   Rapporteur
Robert Cori,   Professeur   ···············   Directeur

— 1998 —

Do hypermaps have spanning (hyper)-trees?

Robert Cori

LaBRI, Université de Bordeaux

Joint work with Gábor Hetyei

Spanning trees

Definition In a graph (V, E), each edge $e \in E$ has two incident vertices. An edge may be incident twice to the same vertex (loops) and two edges may have the same two incident vertices (multiple edges)

The graphs considered are supposed to be connected

Definition A spanning tree is defined as a subset T of E containing $|V| - 1$ elements and such that (V, T) is connected.

Enumerative properties of spanning trees

Proposition The number of spanning trees of a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is equal to the determinant of anyl $(n-1)$ minor of the Laplacian matrix.

Definition The Tutte Polynomial of a graph is the sum of monomials in two variables x and y one for each spanning tree of the graph.

The monomial $x^a y^b$ corresponds to a spanning tree with internal activity a and external activity b.

Enumerative properties of spanning trees

Proposition For any edge e, the number of spanning trees of the graph G is equal to the sum of the number of spanning trees of the graph $G - e$ obtained by deleting e and the graph G_e obtained by contracting the edge e.

The contraction of the edge e in G consists in replacing the vertices v_i and v_j incident to e in G by a new vertex v_k which is incident to all the edges incident to v_i or to v_j in G.

Maps

Definition A combinatorial map consists of two permutations σ and α on the set $\{0, 1, \ldots, 2m - 1\}$ such that all the cycles of α have length 2 and such that each half edge i may be attained from 0 by a sequence of σ and α.

Example Consider the combinatorial map consisting of the two following permutations defined by their cycles:

$$\sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \quad \alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$$

Proposition For a combinatorial map $M = (\sigma, \alpha)$ the faces are represented by the permutation $\alpha\sigma$.
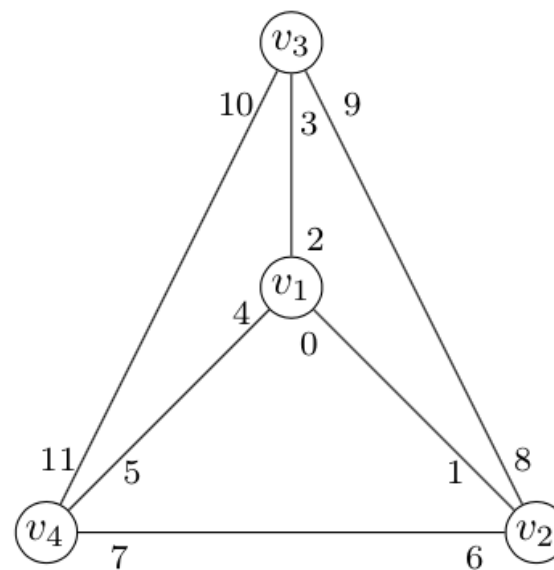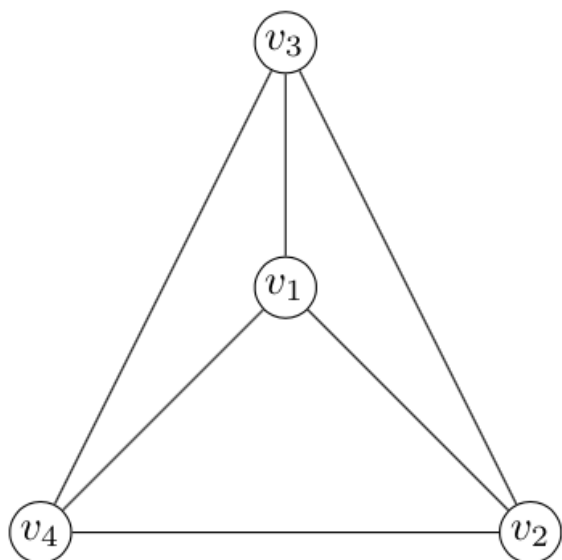
$$\sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \quad \alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$$
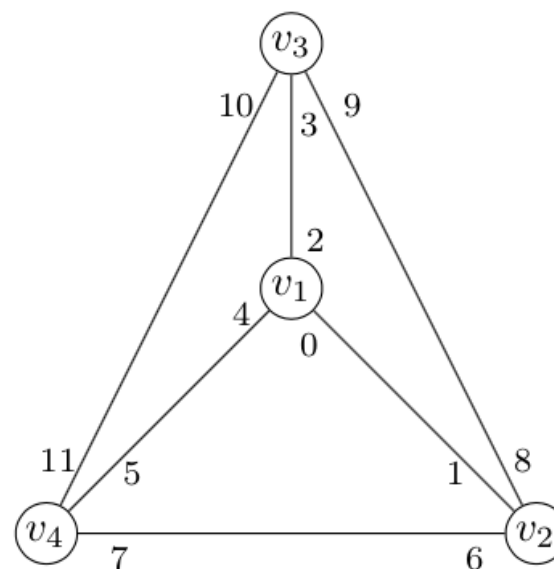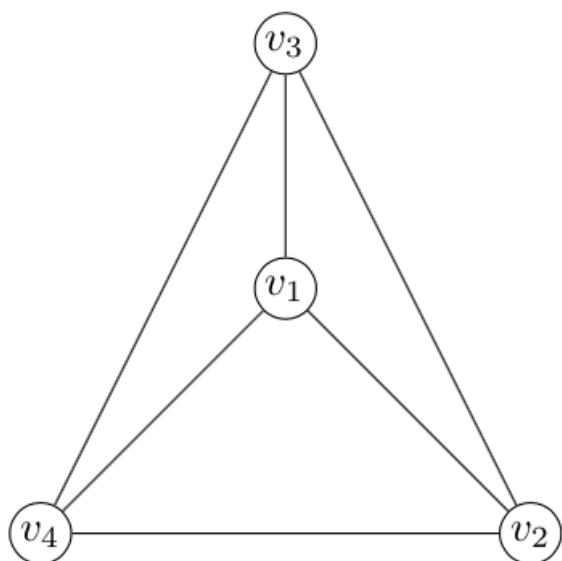
$$\sigma = (0,2,4)(1,6,8)(3,9,10)(5,11,7) \quad \alpha = (0,1)(2,3)(4,5)(6,7)(8,9)(10,11)$$

$$\alpha\sigma = (0,3,8)(1,7,4)(2,5,10)(6,9,11)$$

Hypermaps

Definition A hypermap consists of two permutations σ and α on the set $\{1, 2, \ldots, n\}$ such that each point i may be attained from 1 by a sequence of σ and α.
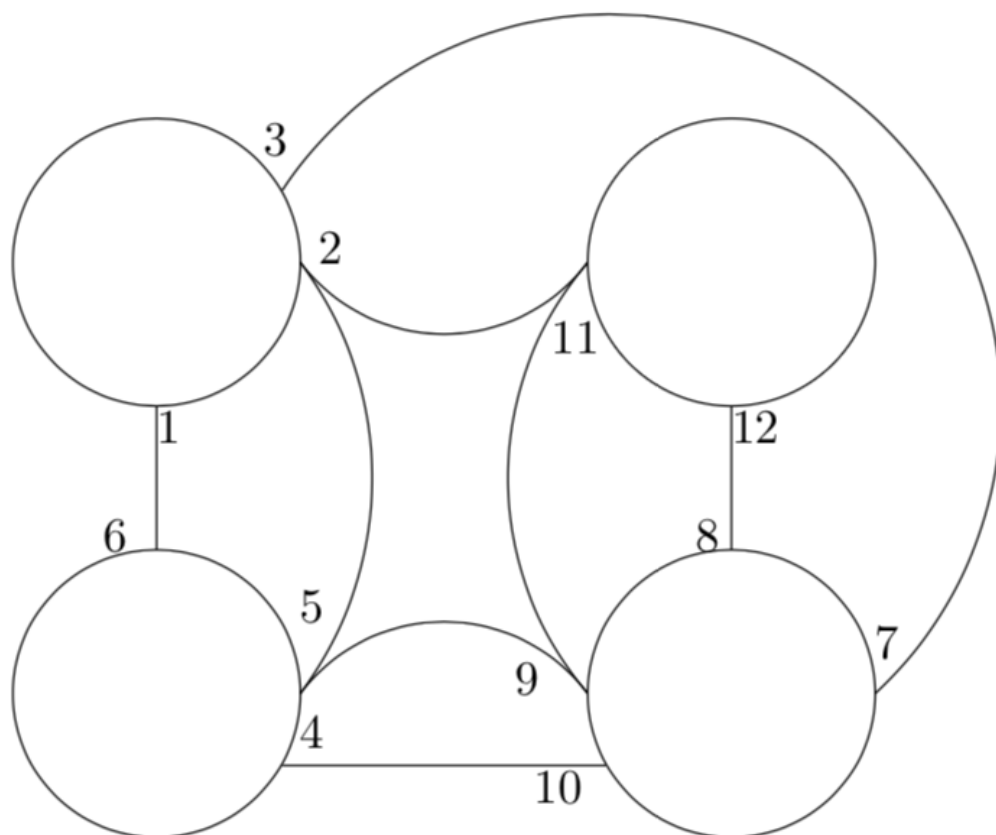
Example Consider the hypermap consisting of the two following permutations defined by their cycles:

$$\sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9, 10)(11, 12) \; \alpha = (1, 6)(2, 11, 9, 5)(3, 7)(4, 10)(8, 12)$$

Proposition For a hypermap $M = (\sigma, \alpha)$ the faces are represented by the permutation $\alpha^{-1}\sigma$.
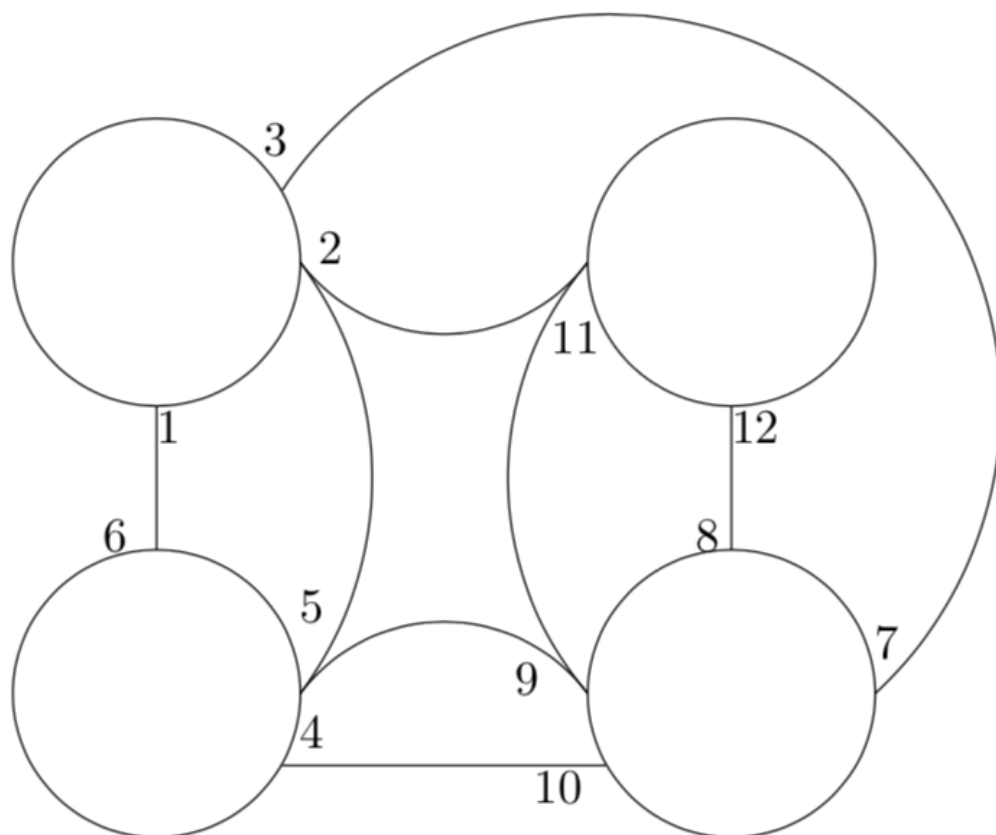
$$\sigma = (1,2,3)(4,5,6)(7,8,9,10)(11,12) \quad \alpha = (1,6)(2,11,9,5)(3,7)(4,10)(8,12)$$

$$\sigma = (1,2,3)(4,5,6)(7,8,9,10)(11,12) \quad \alpha = (1,6)(2,11,9,5)(3,7)(4,10)(8,12)$$

$$\alpha^{-1}\sigma = (1,5)(2,7,12)(3,6,10)(4,9)(8,11)$$

Genus

Theorem (Alain Jacques, 1968) The genus of a hypermap given by the formula below is a non negative integer:

$$z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g(\sigma, \alpha)$$

$z(\alpha)$ denotes the number of cycles of the permutation α.

Spanning trees of maps

Definition A map $T = (\sigma, \theta)$ is a tree if it is genus is equal to 0 and if the permutation $\theta\sigma$ has one cycle. A spanning tree of the map $M = (\sigma, \alpha)$ is a tree $T = (\sigma, \theta)$ such that θ is obtained from α by deleting some edges , that is replacing an edge (i, j) by $(i)(j)$
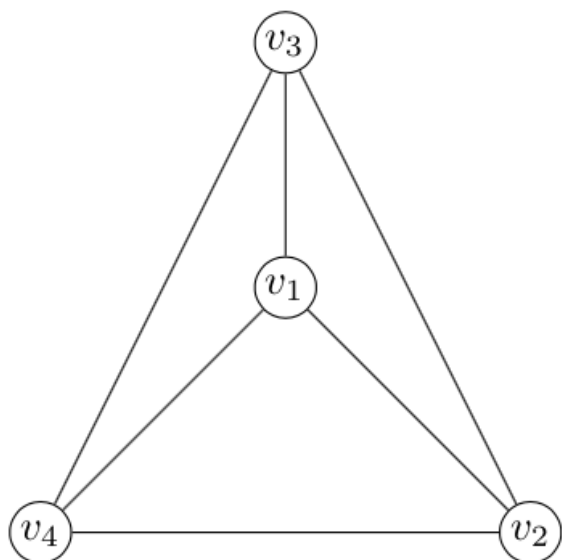
Example In the map above :

$$\sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \quad \alpha = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$$
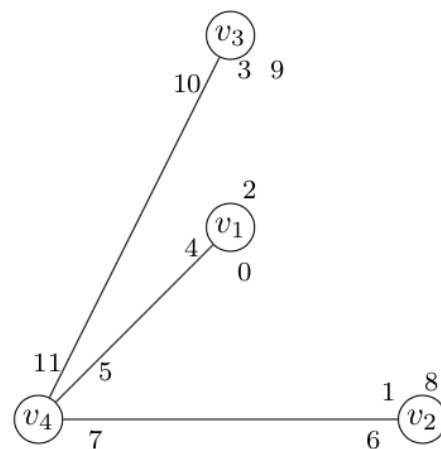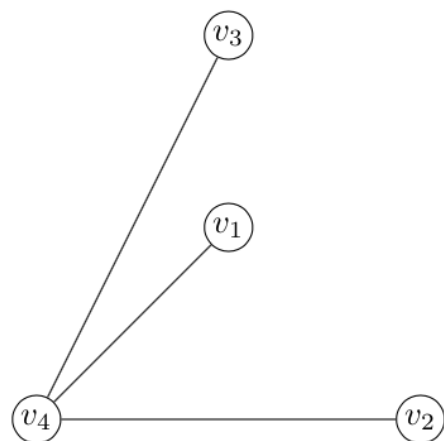
A spanning tree is given by :

$$\theta = (4, 5)(6, 7)(10, 11)$$

since

$$\theta\sigma = (0, 2, 5, 10, 3, 9, 11, 6, 8, 1, 7, 4)$$
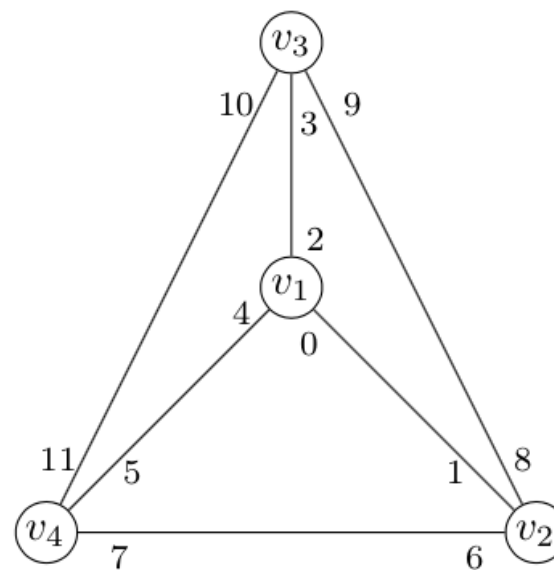
$$\sigma = (0,2,4)(1,6,8)(3,9,10)(5,11,7) \qquad \alpha = (0,1)(2,3)(4,5)(6,7)(8,9)(10,11)$$

$$\sigma = (0, 2, 4)(1, 6, 8)(3, 9, 10)(5, 11, 7) \qquad \theta = (0)(1)(2)(3)(4, 5)(6, 7)(8)(9)(10, 11)$$

$$\theta\sigma = (0, 2, 5, 10, 3, 9, 11, 6, 8, 1, 7, 4)$$

Bernardi tour

The permutation $\theta\sigma$ is sometimes called Bernardi tour

References

M. Baker and Y. Wang (2017) "The Bernardi Process and Torsor Structures on Spanning Trees," International Mathematics Research Notices, Vol. 2018, No. 16, pp. 5120–5147

Tamás Kálmán and Lilla Tóthmérész Hypergraph polynomials and the Bernardi process, Algebraic Combinatorics, Volume 3, issue 5 (2020), p. 1099-1139.

Farbod Shokrieh and Cameron Wright Torsor Structures on spanning trees, arXiv: 2103.10370v1 [math.CO]

Olivier Bernardi, Tamas Kalman, Alex Postnikov Universal Tutte polynomial, arXiv:2004.00683 [math.CO]

Refinements of permutations

Definition Let ζ be a cyclic permutation the permutation θ is a refinement of the permutation ζ if $g(\zeta, \alpha) = 0$; equivalently the cycles of θ determine a non crossing partition of the cycle ζ

Consider the permutation:

$$\zeta = (1, 2, 3, 4, 5)$$

Then $\theta = (1, 3)(2)(4, 5)$ is a refinement of ζ but $\theta' = (1, 3)(2, 4)(5)$ is not.

Remark Notice that the number of refinements of a cycle of length n is the Catalan number C_n

Refinements of permutations

Definition The permutation θ is a refinement of the permutation α if two elements in the same cycle of θ are also in the same cycle of α. And for each cycle of α the cycles $\theta_1, \theta_2, \cdots, \theta_k$ of θ contained in it determine a refinement of this cycle

Consider the permutation:

$$\alpha = (1, 7, 3)(4, 6, 9, 8)(2, 5, 11, 12, 10)$$

Then $\theta = (1, 3)(7)(4, 9)(6)(8)(2, 12, 10)(5, 11)$ is a refinement of α but
$\theta' = (1, 3)(7)(2)(4, 9)(6, 8)(2, 12, 10)(5, 11)$ is not.

Hypertrees

Definition A hypertree is a hypermap (σ, θ) such that $g(\sigma, \theta) = 0$ and $z(\theta^{-1}\sigma) = 1$ (it has only one face).

Does not correspond to Wikipedia definition In mathematics, a hypergraph H is called a hypertree if it admits a host graph T such that T is a tree. In other words, H is a hypertree if there exists a tree T such that every hyperedge of H is the set of vertices of a connected subtree of T.

Corresponds to those considered in Béatrice Delcroix-Oger thesis

Hypertrees
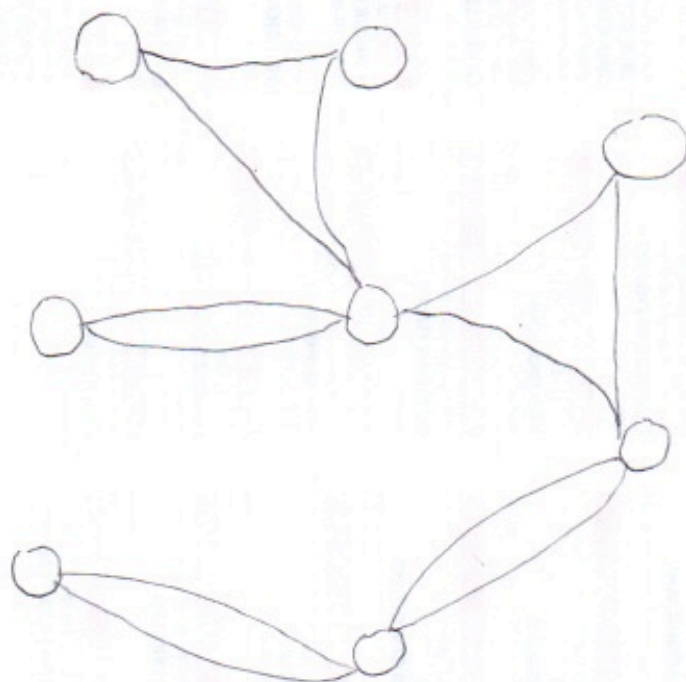
A hypertree with 8 vertices and 5 hyperedges.

Spanning hypertrees

Definition The hypermap (σ, θ) is a spanning hypertree of the hypermap (σ, α) if it is a hypertree and if θ is a refinement of α.

Remark The spanning trees of a map are spanning hypertrees

First result

Definition The number of spanning hypertrees of a planar hypermap (σ, α) is equal to the number of that of its dual: $(\alpha^{-1}\sigma, \alpha^{-1})$.

Proof If (σ, θ) is a spanning hypertree of the planar hypermap (σ, α) then $(\alpha^{-1}\sigma, \alpha^{-1}\theta)$ is a spanning hypertree of the dual $(\alpha^{-1}\sigma, \alpha^{-1})$
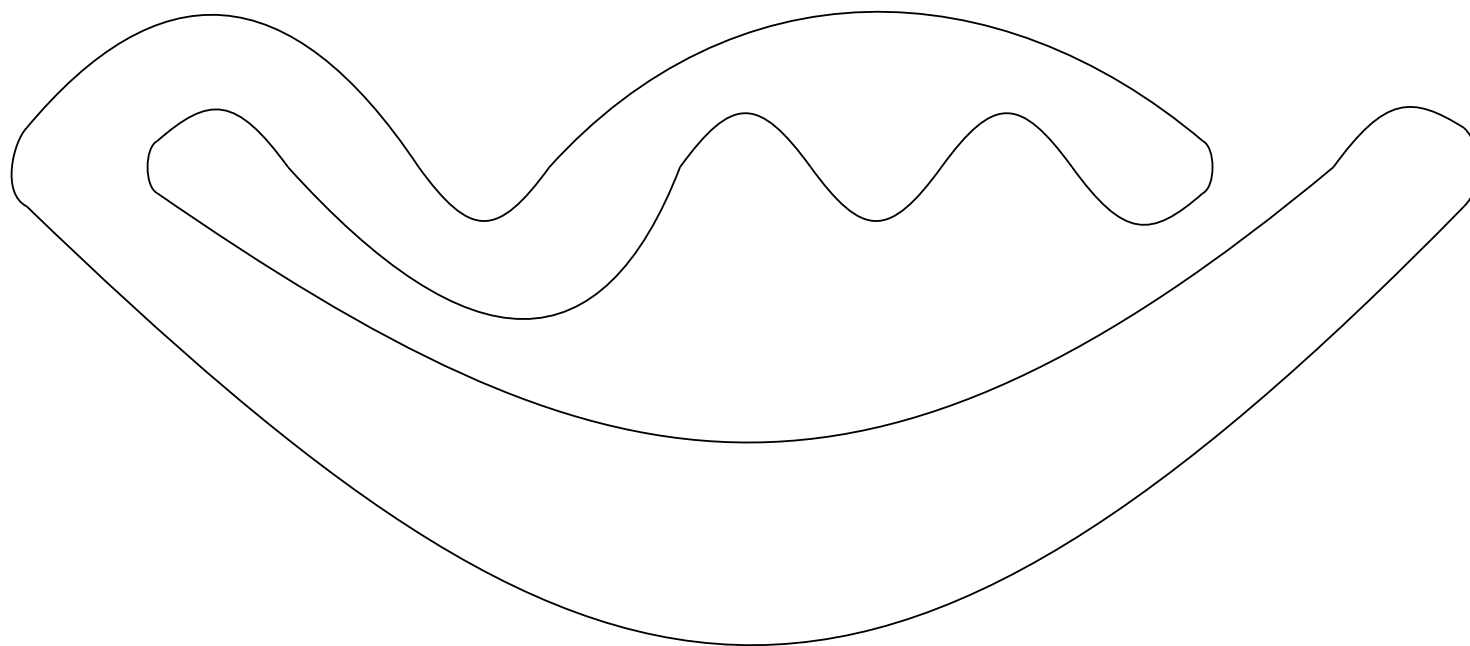
Meanders

Definition A meander of order n is a self-avoiding plane closed curve which intersects a strait line in 2n points. The meander is defined up to an isotopy of the plane which preserves the intersection points.

Notice that a meander can be defined by two involutions α_1, α_2 such that $g(\zeta, \alpha_1) = 0$ and $g(\zeta, \alpha_2) = 0$ where ζ is the circular permutation
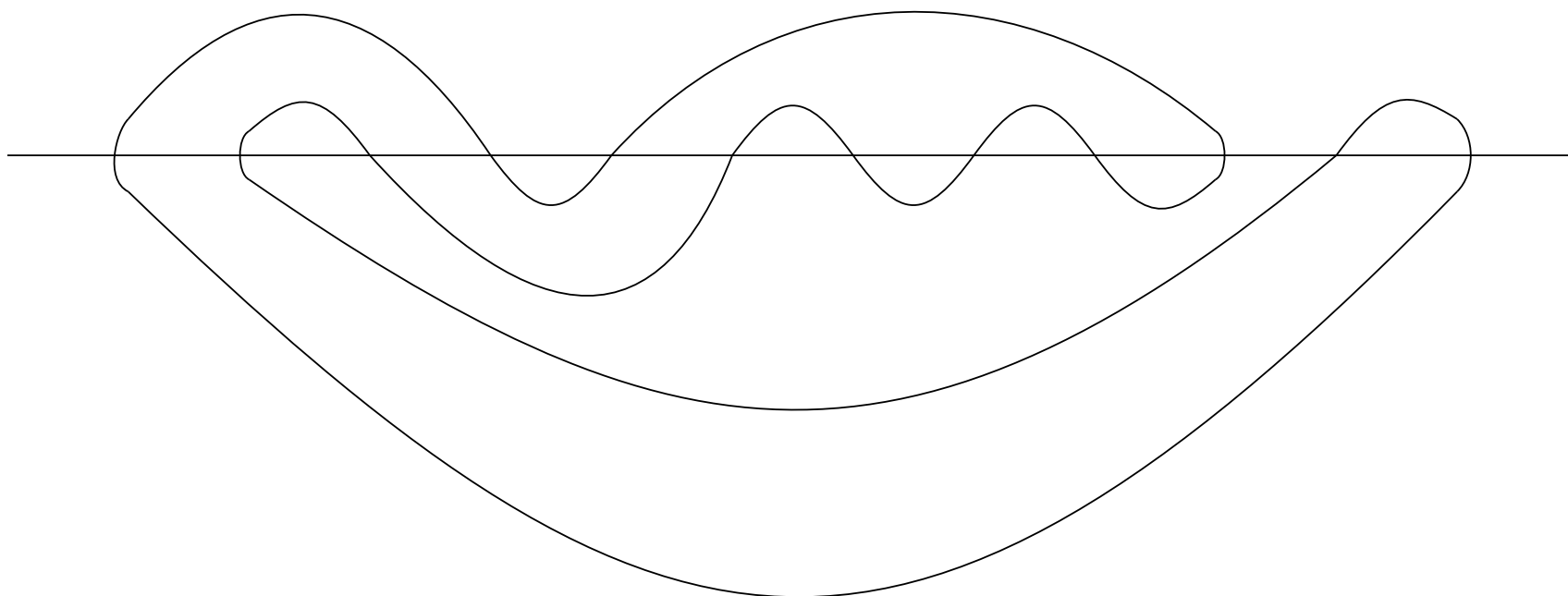
$$\zeta = (1, 2, \cdots, 2n)$$

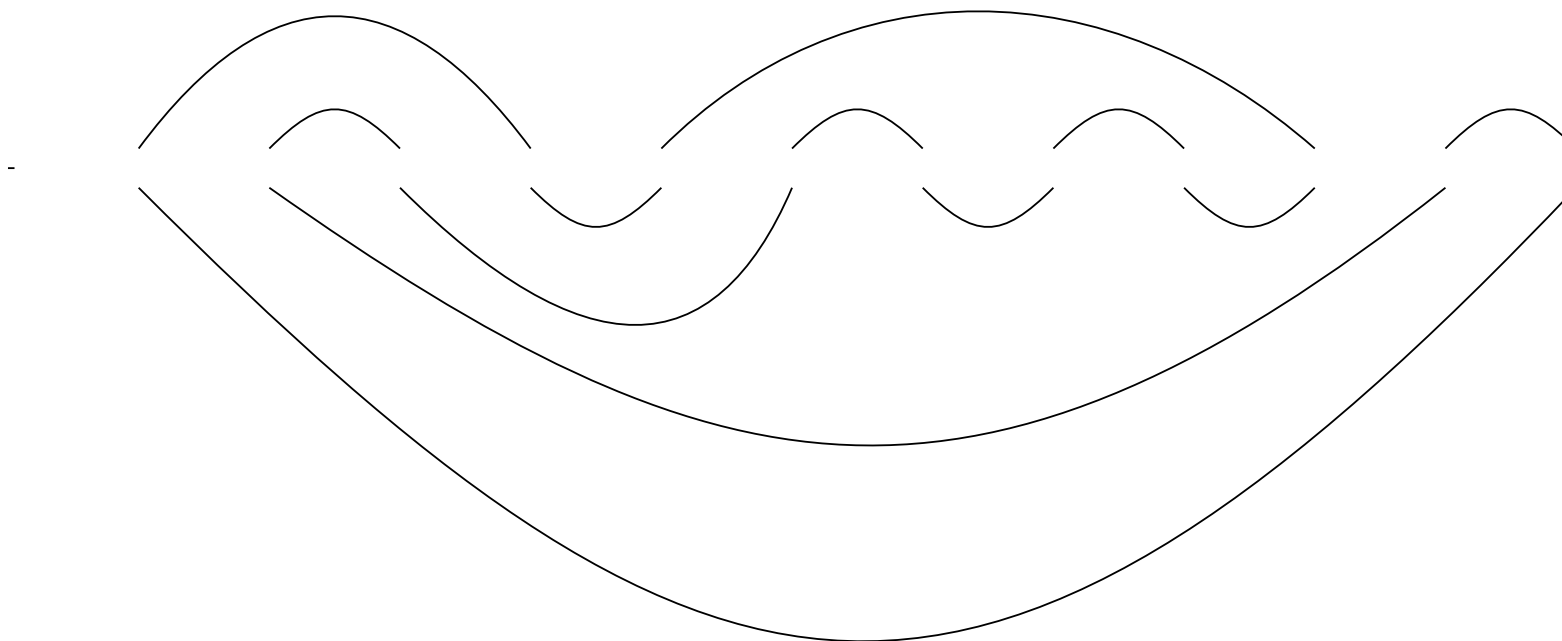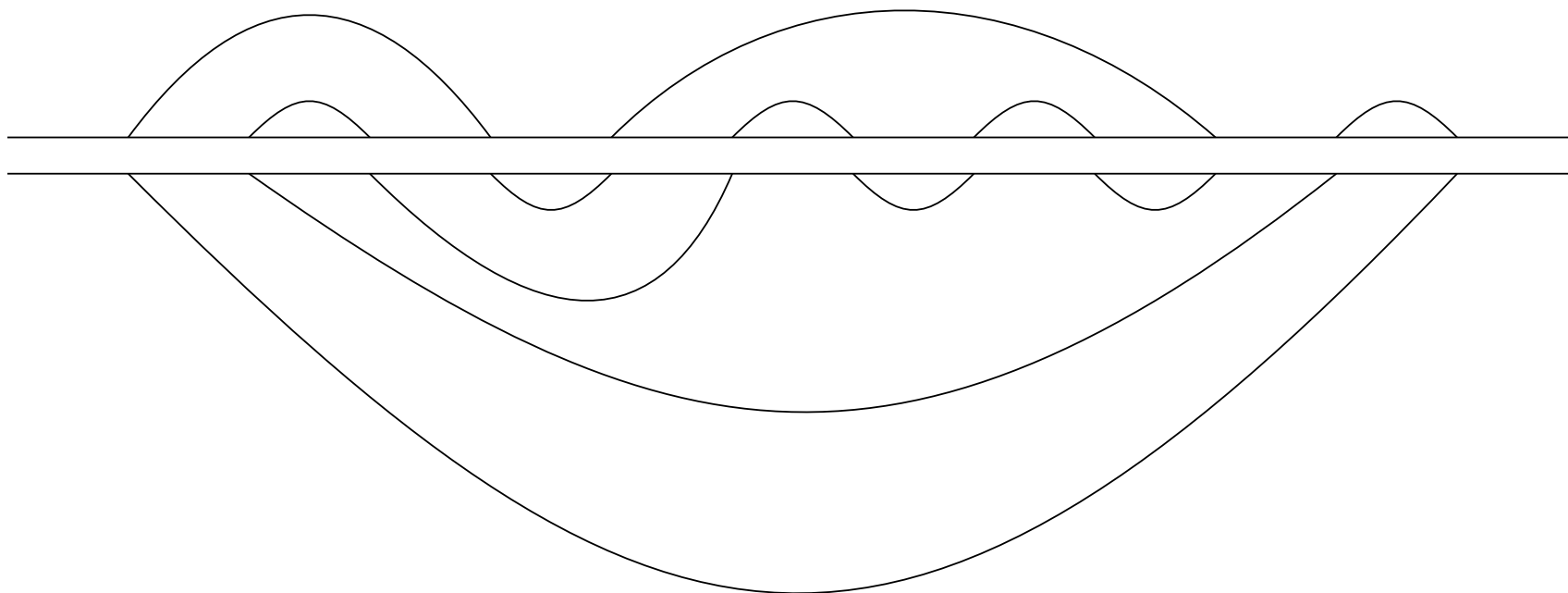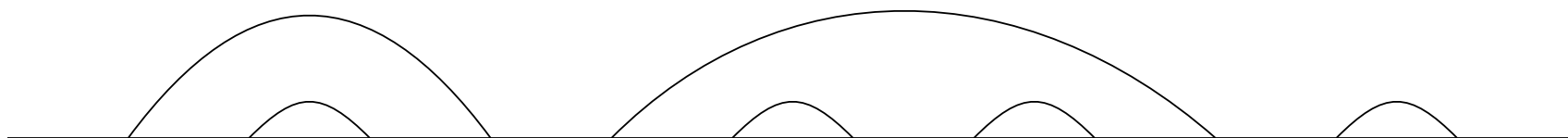and such that the group generated by α_1, α_2 is transitive.
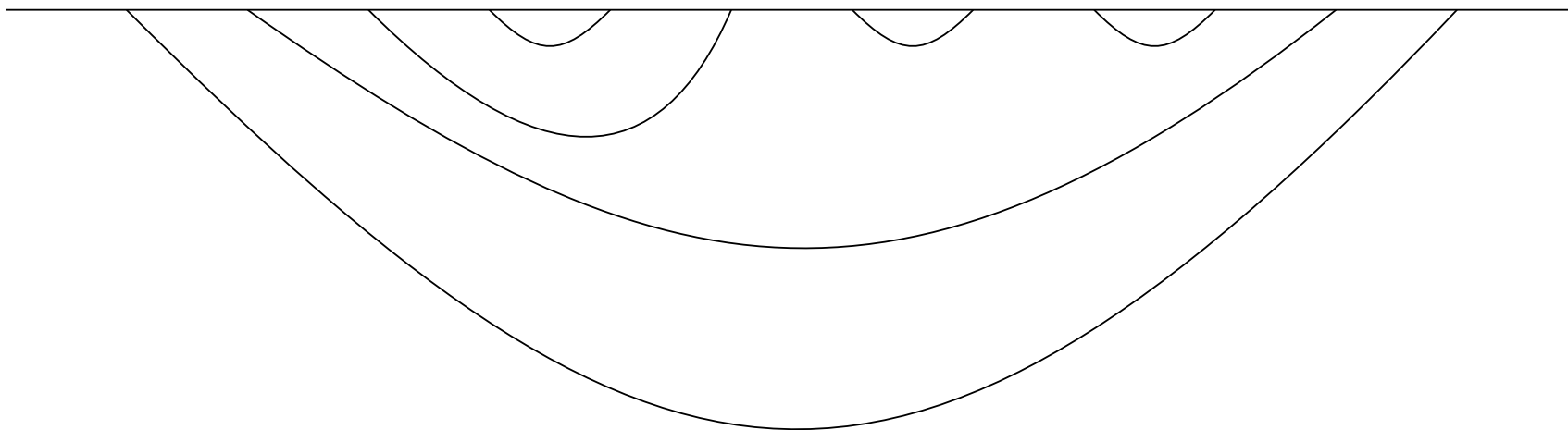
Meanders

Meanders

Meanders

Meanders

Meanders

$$\alpha 1 = (1,4)(2,3)(5,10)(6,7)(8,9)(11,12)$$

Meanders

$$\alpha_2 = (1,12)(2,11)(3,6)(4,5)(7,8)(9,10)$$

Main result 1

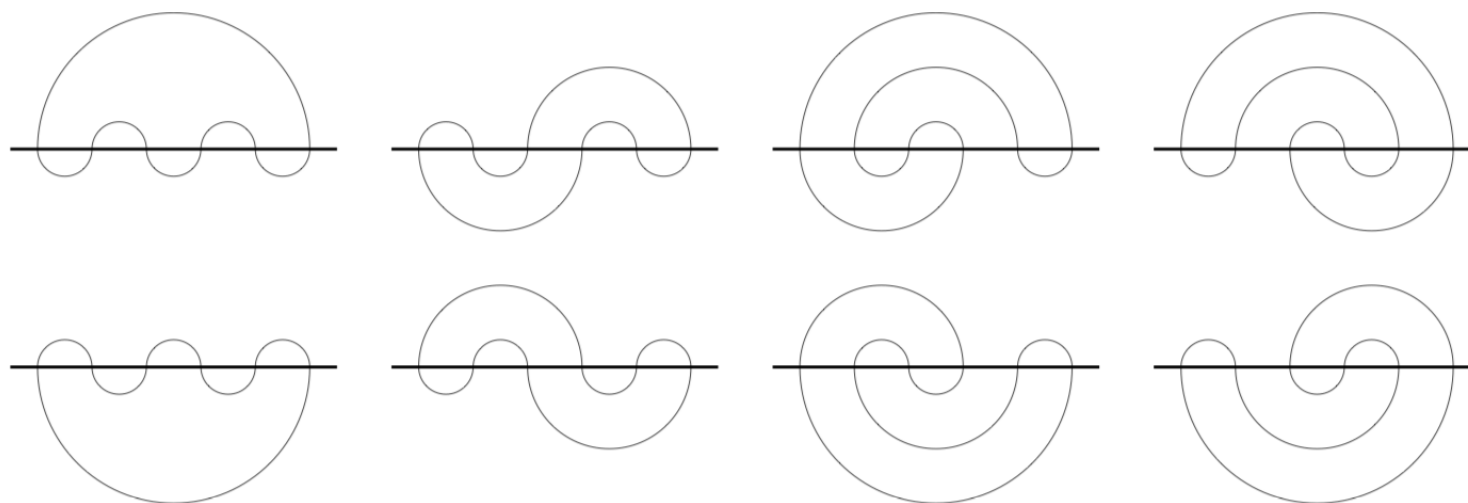**Theorem** The number of meanders of order n is equal to the number of spanning hypertrees of the hypermap:

$$\sigma = (1, 2n)(2, 3) \ldots (2i, 2i+1) \ldots (2n-2, 2n-1) \quad \alpha = (1, 3, \ldots, 2n-1)(2, 4, \ldots, 2n)$$

Remarks

$$\alpha\sigma = (1, 2)(3, 4) \ldots (2i - 1, 2i) \ldots (2n - 1, 2n)$$

This hypermap is the reciprocal of a map, namely the dipole with n edges.
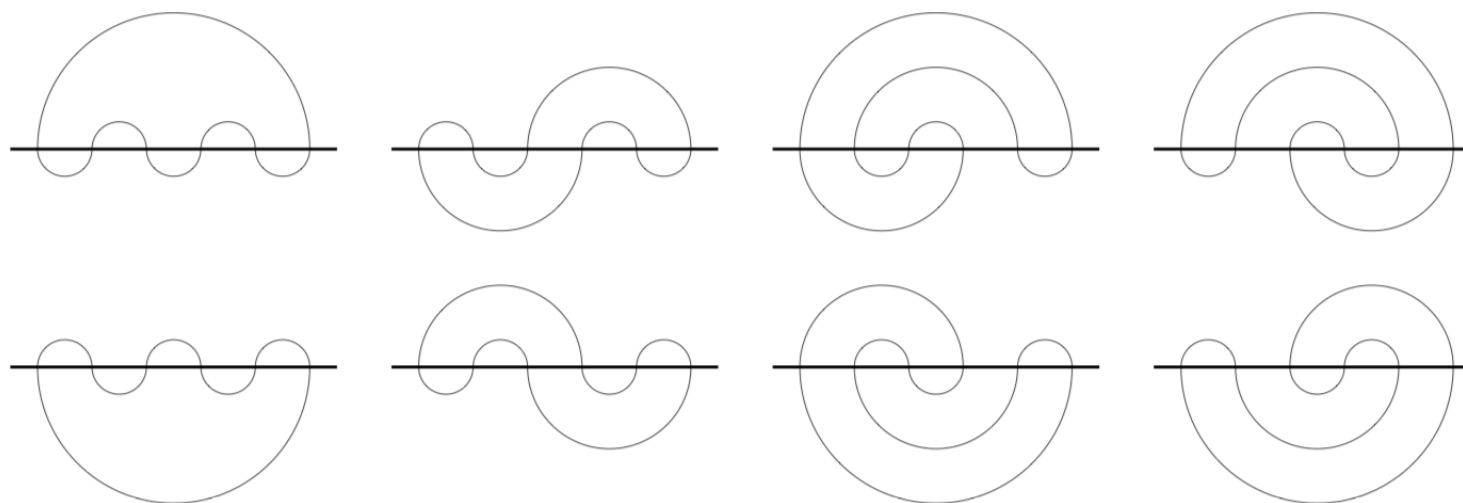
The 8 meanders of order 3

The 8 hypertrees of the hypermap reciprocal of a dipole with 3 edges

$$\sigma = (1,6)(2,3)(4,5) \qquad \alpha = (1,3,5)(2,4,6)$$

$\theta = (1,3,5)$	$\theta^{-1}\sigma = (1,6,5,4,3,2)$	x
$\theta = (2,4,6)$	$\theta^{-1}\sigma = (1,2,3,4,5,6)$	x
$\theta = (1,3)(2,4)$	$\theta^{-1}\sigma = (1,6,3,4,5,2)$	x
$\theta = (1,5)(2,4)$	$\theta^{-1}\sigma = (1,6,5,2,3,4)$	x
$\theta = (3,5)(2,4)$	$\theta^{-1}\sigma = (1,6)(3,4)(5,6)$	-
$\theta = (1,3)(2,6)$	$\theta^{-1}\sigma = (1,2)(3,6)(4,5)$	-
$\theta = (1,5)(2,6)$	$\theta^{-1}\sigma = (1,2,3,6,5,4)$	x
$\theta = (3,5)(2,6)$	$\theta^{-1}\sigma = (1,2,5,4,3,6)$	x
$\theta = (1,3)(4,6)$	$\theta^{-1}\sigma = (1,4,5,6,3,2)$	x
$\theta = (1,5)(4,6)$	$\theta^{-1}\sigma = (1,4)(2,3)(5,6)$	-
$\theta = (3,5)(4,6)$	$\theta^{-1}\sigma = (1,4,3,2,5,6)$	x

The 8 meanders of order 3

$$\theta^{-1}\sigma = (1,2,5,4,3,6), \quad \theta^{-1}\sigma = (1,4,5,6,3,2) \quad \theta^{-1}\sigma = (1,6,5,2,3,4)$$

Hyperdeletion

Definition A *hyperdeletion* is the operation of replacing a hypermap (σ, α) with the hypermap $(\sigma, \alpha\tau)$ where $\tau = (i, j)$ is a transposition such that i, j are two points in the same cycle of α.

We call the hyperdeletion topological if i, j are two points in different cycles of $\alpha^{-1}\sigma$. Such a deletion does not modify the genus of a hypermap.

Remark Notice that the hypermap $(\sigma, \alpha\tau)$ obtained by a deletion is such that $\alpha\tau$ is a refinement of α.

Proposition Let $H = (\sigma, \alpha)$ be a hypermap and let i, j be two elements of a cycle of α belonging to different cycles of $\alpha^{-1}\sigma$. Let H' be the hypermap obtained by deleting (i, j) in H. Then any spanning hypertree of H' is also a spanning hypertree of H .

Hypercontraction

Definition A *hypercontraction* is the operation of replacing a hypermap (σ, α) with the hypermap $(\tau\sigma, \tau\alpha)$ where $\tau = (i, j)$ is a transposition such that i, j are two points in the same cycle of α and in two different cycles of σ.

Proposition Let $H = (\sigma, \alpha)$ be a hypermap and let i, j be two elements of a cycle of α belonging to the different cycles of σ. Let H' be the hypermap obtained by contracting (i, j) in H. Then any spanning hypertree $(\tau\sigma, \theta)$ of H' is such that $(\sigma, \tau\theta)$ is a spanning hypertree of H.

A reccurence for the number of spanning hypertrees (1)

Proposition Let $H = (\sigma, \alpha)$ be a hypermap such that $(1, 2)$ is a cycle of α, then the number of spanning hypertrees of $H = (\sigma, \alpha)$ is equal to the sum of the numbers of that of $delete(H, 1, 2)$ and of that of $contract(H, 1, 2)$.

A recurrent formula for the number of spanning hypertrees (2)

Proposition Let $H = (\sigma, \alpha)$ be a hypermap such that $(1, 2, 3)$ is a cycle of α. Let H_1 be the hypermap obtained by deleting $(1, 3)$ in H, H_2 be the hypermap obtained by contracting $(1, 2)$ in H and H_3 be the hypermap obtained by first contracting $(1, 3)$ in H then deleting $(1, 2)$. Then the number of spanning hypertrees of H is equal to the sum of the numbers of that of H_1, H_2 and H_3.

$$H_1 = del(1,3,H), H_2 = contr(1,2,H), H_3 = del(1,2,contr(1,3,H))$$

- The hypertrees (σ, θ) of H with cycles $(1)(2)(3)$ in θ come from those with the same cycles in H_1.

- The hypertrees (σ, θ) of H with cycles $(1,2)(3)$ in θ come from those in H_2 with cycles $(1)(2)(3)$

- The hypertrees (σ, θ) of H with cycles $(1,3)(2)$ in θ come from those in H_3

- The hypertrees (σ, θ) of H with cycles $(1)(2,3)$ in θ come from those in H_1 with the same cycles.

- The hypertrees (σ, θ) of H with cycles $(1,2,3)$ in θ come from those in H_2 with cycles $(1)(2,3)$

Main result 2.

Proposition Let $H = (\sigma, \alpha)$ be a hypermap such that $(1, 2, \cdots, k)$ is a cycle of α. Let H_1 be the hypermap obtained by deleting $(1, k)$ in H, H_2 be the hypermap obtained by contracting $(1, 2)$ in H and H_i for $i = 3, \ldots k$ be the hypermap obtained by first contracting $(1, i)$ in H then deleting $(1, i - 1)$. Then the number of spanning hypertrees of H is equal to the sum of the numbers of that of $H_1, H_2, \cdots, H_k$.

Proof Use the decomposition of non crossing partitions into disjoint subsets given by Simion et Ullmann