Exceptional Complex Structures and K-Stability

David Tennyson, Imperial College London

1910.04795
1912.09981
2104.09900

A. Ashmore, C. Strickland-Constable, D. Waldram

April 13, 2022

Background

We have seen that string theory provides a rich arena in which geometry and physics interplay.

e.g. Geometric engineering - Minkowski $\times$ M for M special holonomy

$$\text{Cycles/singularities of } M \quad \leftrightarrow \quad \text{BPS states}$$

Flux Backgrounds

String backgrounds have more degrees of freedom than just the metric

$$g, \quad \phi, \quad F$$

We choose special holonomy M for convenience.

String theory has no preference.

Flux backgrounds are hard

$$\exists \text{ global spinor } \xi \qquad \nabla\xi + F \cdot \xi = 0$$

Unlike with special holonomy

- ξ defines a local G-structure
- KSE translates to complicated differential constraints on G-structure

These complicated questions are made easier in Generalised Geometry

Results

Supersymmetric backgrounds of string theory are described by global 'integrable' generalised G-structures

These G-structures are defined by a complex vector bundle $L \to M$ satisfying

- Complexity - $L \cap \bar{L} = 0$
- Isotropy - $L \times_N L = 0$
- Positivity - positive definite metric
- Maximality - rank L is maximal
- Involutivity - $[\![L, L]\!] \subset L$

In fact, we need a refined structure ψ

Results

We call the geometry defined by L an Exceptional Complex Structure

Outline

Generalised Geometry

Exceptional Complex Structures

K-Stability

Generalised Geometry

Leibniz Algebroids

A Leibniz algebroid is a set $\{E, M, a, \mathbb{L}\}$ such that

- $E \to M$ is a smooth vector bundle
- $a : E \to TM$ is a smooth bundle map
- $\mathbb{L} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ an $\mathbb{R}$-linear map such that
  1. $\mathbb{L}_V \mathbb{L}_W X = \mathbb{L}_{\mathbb{L}_V W} X + \mathbb{L}_W \mathbb{L}_V X$
  2. $\mathbb{L}_V(fW) = a(V)(f)W + f\mathbb{L}_V W$

$a =$ 'anchor' $\quad \mathbb{L} =$ 'Dorfman derivative'

Closed form Leibniz algebroids [Baraglia]

A special type of Leibniz algebroid where

$$E \sim T \oplus \bigoplus_i [\wedge^{p_i} T^* \otimes \mathfrak{g}_i] \oplus \mathfrak{X}$$

Clearly

$$\bigoplus_i [\wedge^{p_i+1} T^* \otimes \mathfrak{g}_i] \subset \mathrm{End}(E)$$

Then the Dorfman derivative

$$\mathbb{L}_V \sim \mathcal{L}_v - d\lambda \qquad V = v + \lambda + k \in \Gamma(E)$$

We define an antisymmetric bracket

$$[\![V, W]\!] = \tfrac{1}{2}(\mathbb{L}_V W - \mathbb{L}_W V)$$

Generalised Geometry

Particular closed form Leibniz algebroids such that

- $\wedge^{p_i} T^*$ generate the form field gauge symmetries
- E can be equipped with a G-structure that corresponds to the global symmetry of the corresponding supergravity theory

Examples

- Type 1/NSNS - $B \in \Omega^2$
  $E = T \oplus T^*$ $\quad$ $O(d,d)$ structure - η

- Heterotic on 6d - $B \in \Omega^2$, $A \in \Omega^1 \otimes \mathfrak{g}$
  $E = T \oplus \text{ad}\, P_G \oplus T^*$ $\quad$ $O(6, 6+n)$ structure - η

- M-Theory on 6d - $A \in \Omega^3$, $\tilde{A} \in \Omega^6$
  $E = T \oplus \wedge^2 T^* \oplus \wedge^5 T^*$ $\quad$ $E_{6(6)} \times \mathbb{R}^+$ structure - c

- M-Theory on 7d - $A \in \Omega^3$, $\tilde{A} \in \Omega^6$
  $E = T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*)$
  $E_{7(7)} \times \mathbb{R}^+$ structure - q, s

Minor technicality

Really these are extensions

$$0 \longrightarrow \wedge^{\bullet} T^* \longrightarrow E \longrightarrow T \longrightarrow 0$$

We locally patch by

$$0 \longrightarrow \Omega_{cl}^{\bullet+1} \longrightarrow \mathrm{GDiff} \longrightarrow \mathrm{Diff} \longrightarrow 0$$

Choosing a global isomorphism $E = T \oplus \ldots$ is equivalent to choosing flux F

$$\mathbb{L}_V \sim \mathcal{L}_v - d\lambda + \underbrace{(V \cdot F)}_{\mathrm{End}(E)}$$

Another minor technicality

There exists some $N \subset S^2 E$, an equivariant map

$$\times_N : S^2 E \longrightarrow N$$

and a differential

$$d_E : \Gamma(N) \longrightarrow \Gamma(E)$$

Such that the following holds

$$\mathbb{L}_V W + \mathbb{L}_W V = d_E(V \times_N W)$$

Generalised Complex Geometry [Hitchin, Gualtieri]

$$E = T \oplus T^*$$

A generalised $\mathbb{C}$ structure is a reduction of the structure group

$$O(2n, 2n) \to U(n, n)$$

It is equivalent to a choice of $L \subset E_{\mathbb{C}}$ such that

- Complexity - $L \cap \bar{L} = 0$
- Isotropy - $\eta(L, L) = 0$
- Maximality - rank L is maximal

It is integrable if it is involutive

$$[\![L, L]\!] \subset L$$

Generalised Calabi-Yau [Hitchin]

$$E = T \oplus T^*$$

A generalised Calabi-Yau is a refined G-structure

$$U(n,n) \to SU(n,n)$$

A G$\mathbb{C}$S defines a complex line bundle

$$\mathcal{U}_L \subset \Omega^{\bullet}_{\mathbb{C}}(M)$$

A generalised Calabi-Yau is defined by a global non-vanishing

$$\Phi \in \Gamma(\mathcal{U}_L)$$

It is integrable if

$$d\Phi = 0$$

Example

$$E = T \oplus T^*$$

$\mathbb{C}$-structures

$$L = T^{1,0} \oplus T^{*0,1} \qquad \mathcal{U}_L = \Omega^{0,3}(M)$$

Integrability

$$[\![L, L]\!] \subset L \qquad \Leftrightarrow \qquad [T^{1,0}, T^{1,0}] \subset T^{1,0}$$

Symplectic structures

$$L = e^{i\omega} \cdot T \qquad \mathcal{U}_L = \left\langle e^{i\omega} \right\rangle$$

Integrability

$$[\![L, L]\!] \subset L \qquad \Leftrightarrow \qquad d\omega = 0$$

More generally

Using isotropy, we can show L must take the form

$$L = e^{\varepsilon}[\Delta \oplus \mathrm{Ann}_1(\Delta)]$$

for $\Delta \subset T_{\mathbb{C}}$ and $\varepsilon \in \Omega^2(M)_{\mathbb{C}}$

Integrability then tells us

$$[\![L, L]\!] \subset L \qquad \Leftrightarrow \qquad \begin{cases} [\Delta, \Delta] \subset \Delta \\ d_{\Delta}\varepsilon = 0 \end{cases}$$

Applications

- Describe certain flux backgrounds of string theory (GMPT)
- Used to study the topological B-model
- Used to study mirror symmetry

Exceptional Complex Structures

General String Backgrounds

Let G be our (non-compact) structure group
Let $H \subset G$ be the maximally compact subgroup

It is known that the bosonic fields define a point in the coset

$$\{\text{Bosons}\} \in \frac{G}{H}$$

In this sense, a choice of background defines a reduction of the structure group

$$G \to H$$

SUSY backgrounds [Waldram, Pacheco, Strickland-Constable, Coimbra,...]

Suppose we can lift the structure to $\tilde{H}$, the universal cover.

- The global spinor field

$$\xi \in \Gamma(S) \qquad S \sim \text{fundamental of } \tilde{H}$$

  Hence the structure group $\tilde{H} \to \tilde{H}_\xi$

- The Killing spinor equation

$$\nabla\xi + F \cdot \xi = 0 \qquad \Leftrightarrow \qquad \text{Integrable } \tilde{H}_\xi \text{ structure}$$

G-structures

G	H	$\tilde{H}$	$\tilde{H}_\xi$
$O(6,6+n)$	$O(6)\times O(6+n)$	$\mathrm{SU}(4)\times O(6+n)$	$\mathrm{SU}(3)\times O(6+n)$
$E_{6(6)}\times\mathbb{R}^+$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{USp}(8)$	$\mathrm{USp}(6)$
$E_{7(7)}\times\mathbb{R}^+$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{SU}(8)$	$\mathrm{SU}(7)$

General idea

Decompose $E_{\mathbb{C}}$ under $\mathrm{U}(1) \times \tilde{H}_\xi$

$$E_{\mathbb{C}} = L \oplus \dots$$

We find L satisfies

- Complexity - $L \cap \bar{L} = 0$
- Isotropy - $L \times_N L = 0$
- Positivity - positive definite metric
- Maximality - rank L is maximal

A choice of L satisfying above defines a $\mathrm{U}(1) \times \tilde{H}_\xi$ structure

$\tilde{H}_\xi$ Structures

A choice of L defines a complex line bundle

$$\mathcal{U}_L \subset \wedge^3 E_{\mathbb{C}}$$

The $\tilde{H}_\xi$ structure is defined by a global, non-vanishing

$$\psi \in \Gamma(\mathcal{U}_L)$$

Integrability is given by

$$[\![L, L]\!] \subseteq L \qquad \mu(V) = \int_M \frac{\langle \mathbb{L}_V \psi, \bar{\psi} \rangle}{\langle \psi, \bar{\psi} \rangle^\alpha} \equiv 0$$

Example - Heterotic

$$E = T \oplus \mathrm{ad}\, P_G \oplus T^*$$

We can classify all possible ECS

$$L = e^{i\omega} e^{A} T^{1,0} \qquad \mathcal{U}_L = \left\langle e^{i\omega} e^{A} \Omega^{0,3} \right\rangle$$

Integrability then gives us

$$[\![L, L]\!] \subset L \qquad \Leftrightarrow \qquad \begin{cases} [T^{1,0}, T^{1,0}] \subset T^{1,0} \\ H = d^J \omega \\ F_{0,2} = 0 \end{cases}$$

$$\mu \equiv 0 \qquad \Leftrightarrow \qquad \begin{cases} d(e^{-2\phi}\Omega) = 0 \\ F \wedge \omega \wedge \omega = 0 \\ d(e^{-2\phi}\omega \wedge \omega) = 0 \end{cases}$$

Example - M-Theory 6d

$$E = T \oplus \wedge^2 T^* \oplus \wedge^5 T^*$$

We can find the most general E$\mathbb{C}$S. Isotropy tells us

$$L = e^{\alpha+\beta}[\Delta \oplus \mathrm{Ann}_2(\Delta)] \qquad \mathcal{U}_L = e^{\alpha+\beta}\det(\mathrm{Ann}_1(\Delta))$$

for $\alpha \in \Omega^3(M)_{\mathbb{C}}$, $\beta \in \Omega^6(M)_{\mathbb{C}}$, $\Delta \subset T_{\mathbb{C}}$ where $\mathrm{Codim}_{\mathbb{C}}\,\Delta = 0, 3$

Involutivity gives us

$$[\![L, L]\!] \subset L \qquad \Leftrightarrow \qquad \begin{cases} [\Delta, \Delta] \subset \Delta \\ d_\Delta \alpha = 0 \end{cases}$$

Example - M-theory 6d

$$E = T \oplus \wedge^2 T^* \oplus \wedge^5 T^*$$

Let's take the following example

$$L = e^{i\rho} \cdot T_{\mathbb{C}} \qquad \mathcal{U}_L = \left\langle e^{i\rho} \right\rangle$$

Positivity constraint tells us

$$H(\rho) < 0 \qquad \exists\, \Omega = \rho + i\hat{\rho}$$

Integrability tells us

$$\begin{aligned} [\![L, L]\!] \subset L \qquad &\Leftrightarrow \qquad d\rho = 0 \\ \mu \equiv 0 \qquad &\Leftrightarrow \qquad d\hat{\rho} = 0 \end{aligned}$$

Example - M-theory 7d

$$E = T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*)$$

We can find the most general E$\mathbb{C}$S. Isotropy tells us

$$L = e^{\alpha+\beta}[\Delta \oplus \mathrm{Ann}_2(\Delta)] \qquad \mathcal{U}_L = e^{\alpha+\beta}\det(\mathrm{Ann}_1(\Delta))$$

for α, β, Δ as before

Involutivity gives us

$$[\![L, L]\!] \subset L \qquad \Leftrightarrow \qquad \begin{cases} [\Delta, \Delta] \subset \Delta \\ d_\Delta \alpha = 0 \end{cases}$$

Example - M-theory 7d

$$E = T \oplus \wedge^2 T^* \oplus \wedge^5 T^* \oplus (T^* \otimes \wedge^7 T^*)$$

Lets take the following example

$$L = e^{i\varphi} \cdot T_{\mathbb{C}} \qquad \mathcal{U}_L = \langle e^{i\varphi} \rangle$$

The positivity constraint says

$$(\imath_v \varphi) \wedge (\imath_w \varphi) \wedge \varphi = g(v, w) \, \mathrm{vol}_g \quad \text{positive definite}$$

Integrability gives

$$\begin{aligned} [\![L, L]\!] \subset L \qquad &\Leftrightarrow \qquad d\varphi = 0 \\ \mu \equiv 0 \qquad &\Leftrightarrow \qquad d * \varphi = 0 \end{aligned}$$

Summary

- All supersymmetric backgrounds are described by E$\mathbb{C}$S
- They have properties very similar to $\mathbb{C}$S and G$\mathbb{C}$S
- We have classified all E$\mathbb{C}$S along with integrability conditions
- In certain cases they give rise to $SL(3,\mathbb{C})$ structures and G_2 structures

Applications

- Unifies $\mathbb{C}$-structures, G_2, G$\mathbb{C}$S, hyperkähler, SE,...
- Moduli
- AdS_5/CFT_4 - chiral ring, superconformal indices, RG flows, a-maximisation
- Topological theories - AKSZ, Hitchin functionals
- Geometric engineering?
- K-stability

K-Stability

Moduli

$$\mathcal{M}_{\mathsf{phys}} = \mathcal{M}_\psi / \mathbb{C}^*$$

$$\mathcal{M}_\psi = \{\psi \,|\, L \text{ involutive}, \quad \mu = 0\}/\mathrm{GDiff}$$

The space $\mathcal{Z} = \{\psi \,|\, L \text{ involutive}\}$ is Kähler with Kähler potential

$$\mathcal{K} = \int_M \langle \psi, \bar{\psi} \rangle^{1-\alpha}$$

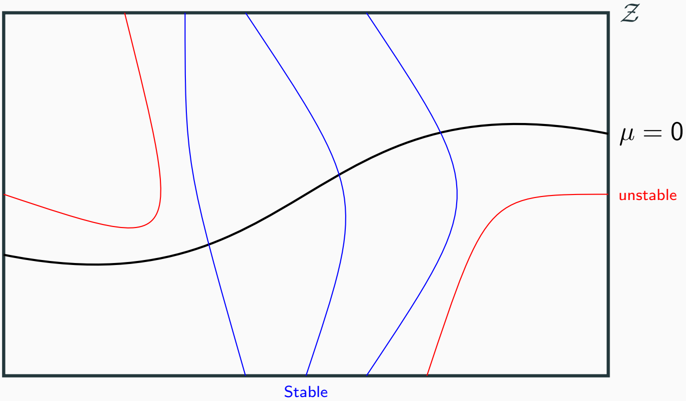
Formally, μ is a moment map for GDiff on $\mathcal{Z}$

$$\mathcal{M}_\psi = \mathcal{Z}//\mathrm{GDiff}$$

If we were in finite dimensions then the Kemp-Ness Theorem would tell us

$$\mathcal{M}_\psi = \mathcal{Z}//\mathrm{GDiff} = \mathcal{Z}^{ps}/\mathrm{GDiff}_{\mathbb{C}}$$

Picture



The Hilbert-Mumford Criterion

Consider some $U(1) \subset \mathrm{GDiff}$ and complexify to $\mathbb{C}^*$ action

$$\psi \to \psi(\nu) \qquad \nu \in \mathbb{C}^*$$

Consider the limit $\nu \to 0$

If $\mathcal{Z}/\mathbb{C}^*$ is compact then we have a fixed point

$$\lim_{\nu \to 0} \psi(\nu) = \nu^{w(\psi,V)} \psi_0 \quad \Rightarrow \quad \lim_{\nu \to 0} \mathcal{K} = |\nu|^{2w(\psi,V)} \mathcal{K}_0$$

some $w(\psi, V) \in \mathbb{Z}$

We define

if $w(\psi) < 0$ for all 1-PS then ψ is stable,

if $w(\psi) \leq 0$ for all 1-PS then ψ is semistable,

if $w(\psi) > 0$ for some 1-PS then ψ is unstable.

The moment map

We have a nice identity

$$\mathcal{L}_{\mathcal{I}\rho_V}\mathcal{K} = -2\mu(V)$$

This then tells us the following

- $mu \equiv 0 \quad \leftrightarrow \quad$ stationary points of $\mathcal{K}$
- $w(\psi, V)\mathcal{K}_0 = -2\mu(\psi_0, V)$

Picture

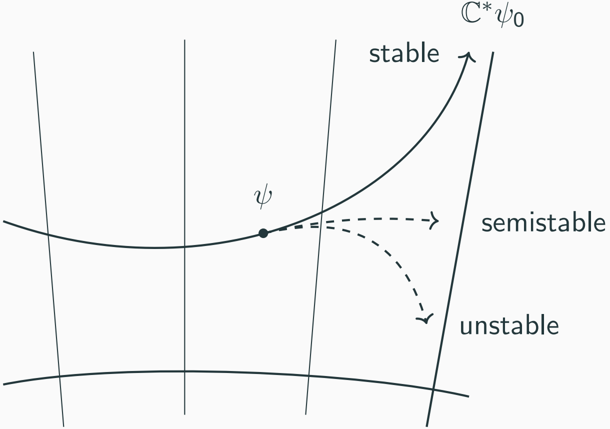


Figure 1: Stability for a 1-PS orbit of ψ.

Problems

- The space $\mathcal{Z}/\mathbb{C}^*$ is not compact
- $\mathcal{K}$ is not convex along $\mathrm{GDiff}_{\mathbb{C}}$ flows

Heterotic

$$E = T \oplus \mathrm{ad}\, P_G \oplus T^*$$

We have a normal subgroup

$$\Omega^2_{cl} \triangleleft \mathrm{GDiff} \qquad \widetilde{\mathrm{GDiff}} := \mathrm{GDiff}/\Omega^2_{cl}$$

We can first do the quotient by Ω^2_{cl} and then the rest

$$\tilde{\mathcal{Z}} = \mathcal{Z}//\Omega^2_{cl} \qquad \mathcal{M}_\psi = \tilde{\mathcal{Z}}^{ps}/\widetilde{\mathrm{GDiff}}_{\mathbb{C}}$$

Note we have $G \subset \mathcal{G} \triangleleft \widetilde{\mathrm{GDiff}}$. The moment map for this action on $\tilde{\mathcal{Z}}$ is

$$\mu(\theta) \sim \int_M e^{-2\phi} Tr(\theta F) \wedge \omega \wedge \omega$$

Heterotic

Taking $\mathbb{C}^* \subset G_{\mathbb{C}}$. The limit $\nu \to 0$ we get

$$\nu G_{\mathbb{C}} \nu^{-1} \to G_{\text{par}}$$

Then

$$\lim_{\nu \to 0} \psi \quad \to \quad \text{holomorphic reduction ad } P_G \to \text{ad } P_{G_{\text{par}}}$$

We then have

$$w(\psi, \theta) = -2\mu(\psi_0, \theta) \sim \int_M e^{-2\psi} c_1(\text{ad } P_{G_{\text{Par}}}) \wedge \omega \wedge \omega$$

The statement that this is negative for all parabolic subgroups is equivalent to slope stability

Heterotic ($H = 0$)

Now consider the normal subgroup

$$\mathcal{G} \ltimes \Omega^2_{cl} \triangleleft \mathrm{GDiff} \qquad \mathrm{Symp} \sim \mathrm{GDiff}/(\mathcal{G} \ltimes \Omega^2_{cl})$$

Hence, by first doing

$$\hat{\mathcal{Z}} = \mathcal{Z}//(\mathcal{G} \ltimes \Omega^2_{cl})$$

We find that $\mathcal{K}$ is convex on Symp flows on $\hat{\mathcal{Z}}$ and

$\mu \sim$ Futaki invariant $\qquad \mathcal{K} \sim$ *M*abuchi functional

K-Stability [Yau, Tian, Donaldson]

A Fano manifold (X, L) admits a Kähler-Einstein metric if and only if it is K-stable

A test configuration $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ is

- $\pi : \mathcal{X} \to \mathbb{C}$ is a flat family with $\pi^{-1}(1) = X$
- $\mathcal{L} \to \mathcal{X}$ is a line bundle which is ample on the fibres, and the restriction to $\pi^{-1}(1)$ is L^m, some $m > 0$
- $\exists$ a $\mathbb{C}^*$ action on $\mathcal{L}, \mathcal{X}$ which lifts the $\mathbb{C}^*$ action on $\mathbb{C}$

A Fano manifold is K-stable if, for all (non-trivial) test configurations

$$\mathrm{Fut}(\pi^{-1}(0)) > 0$$

A reasonable starting point?

A test configuration of $(M, \mathcal{U}_L)$ is given by

$$\mathcal{L} \to \mathcal{X} \to \mathbb{C}$$

- $\pi : \mathcal{X} \to \mathbb{C}$ is some flat bundle with $\pi^{-1}(1) = M$
- $\mathcal{L} \to \mathcal{X}$ is a line bundle with the restriction to $\pi^{-1}(1)$ being $\mathcal{U}_L$
- $\exists$ a smooth $\mathbb{C}^*$ action on $\mathcal{L}, \mathcal{X}$ which lifts the $\mathbb{C}^*$ action on $\mathbb{C}$

Could $(M, \mathcal{U}_L)$ be stable if for all non-trivial test configurations we have

$$\mu(\pi^{-1}(0), V) > 0$$

By studying this in the $E_{7(7)}$-case, could this define a notion of K-stability for G_2