Surface defects, tt* equations and BPS spectra

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Workshop on Quivers, Calabi-Yau threefolds and DT Invariants

Sorbonne U - Campus Pierre et Marie Curie, 11–15 Apr 2022 based on:

G. Bonelli, F. Globlek, A.T. Phys.Rev.Lett. 126 (2021)

G. Bonelli, F. Del Monte, A.T. Ann. Henri Poincare' (2021)

G. Bonelli, F. Globlek, N. Kubo, T. Nosaka, A.T. arXiv:2202.10654

… and work in progress…

Beauty of defects

Defects in QFT are a set of boundary conditions on the fields and boundary couplings that one imposes on sub-manifolds.

They are both useful to model physical systems and also to probe general properties of quantum field theories.

Indeed, the partition function in presence of a defect Z_{defect} ***is not simply a number but rather an element of a vector space*** (for fixed BC) or more in general an object in a category, when considering all admissible boundary conditions.

This makes the study of Z_{defect} a very rich subject naturally related to ***geometry, integrable systems and representation theory.***

Which defects ?

In this talk we will consider surface defects in ***four and five dimensional supersymmetric gauge theories with 8 supercharges*** and show that the related partition functions obey respectively ***a set of differential and q-difference equations which provide very effective and general tools to explore their non-perturbative dynamics and BPS spectrum.***

Main results - 4d :

in four dimensions, Z_{defect} satisfies a set of differential equations that produce ***new recurrence relations*** for multi-instantons of N=2 Super-Yang-Mills theory in a *self-dual Omega background*. These provide a ***new algorithm*** to evaluate them for arbitrary simple groups from A to E .

integrability:

- Z_{defect} for gauge group G are *tau-functions* of *non-autonomous Toda chain of type* $(\hat{G})^{\vee}$, $t \simeq \Lambda$.
- ***4d/2d correspondence*** with ***tt* equations*** and Ising model corr.

random matrices:

- late time = strong coupling expansion of $Z_{\mathrm{defect}}(A_N)$ gives a matrix model presentation of the ***magnetic phase,*** expansion around the ***monopole point.***

Main results - 5d :

- in five dimensions, Z_{defect} satisfies a set of ***q-difference equations*** arising from the symmetry group of the ***BPS quiver*** of the SCFT.

representation theory:

- Z_{defect} are *tau-functions* of the ***cluster algebra*** associated to the quiver.

- reduction to four-dimensional BPS quivers and a new viewpoint on ***Argyres-Douglas*** theory.

Surface defects

Surface defects

Surface defects can be defined by the assignment of singular boundary conditions for the fields in the normal bundle of the surface D

$$A = ad\theta + \dots$$

$z = \rho e^{i\theta}$ normal coordinate $a \in \mathbf{t}$ specifies the residual gauge symmetry on the defect to its commutant

$\mathbb{L} \subset G$ Levi subgroup and determines the ***monodromy*** around D

Another parameter is given by the coupling to the ***magnetic charge*** of the defect

$$\exp(i\frac{b}{2\pi}\int_D F)$$

the magnetic charge $\mathbf{m} \in Q^\vee$ is an element of the coroot lattice. The two parameters are packed into a complex one η

We will consider ***full surface operators***, namely the ones with ***minimal*** residual gauge symmetry

$$\mathbb{L} = \mathbb{T}$$ ← Cartan torus of G

Moreover, we can **twist** their monodromy by a ***central element***

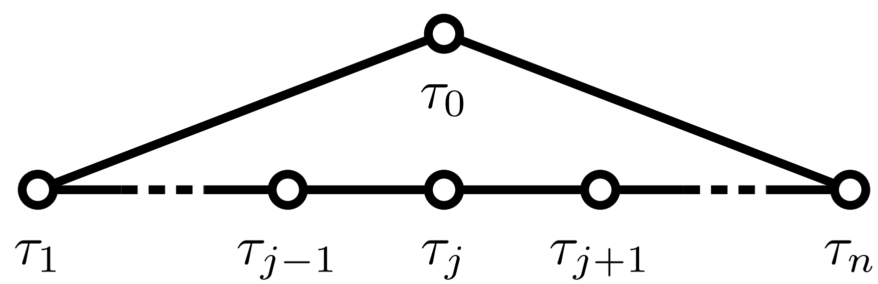
These are the surface operators generating the ***one-form symmetry*** of Yang-Mills valued in the center $\mathcal{Z}(G)$. Introduced by 't Hooft to describe *phases* of gauge theories.

Their vevs depend both on the continuous parameters η and the discrete label of the center element. This can be described in terms of the Dynkin diagram of the affine group $\hat{G}$

Indeed, $\mathcal{Z}(G)$ is the ***automorphism group*** of the affine Dynkin diagram

$\mathfrak{g}$	A_n	B_n	C_n	D_{2n}	D_{2n+1}	E_n	F_4	G_2
$Z(G)$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_{9-n}$	1	1

It acts on the ***affine nodes***. E.g. for A_n



The center is given by the quotient of the affine co-weight lattice by the affine co-root lattice

$$\mathcal{Z}(G) = Q_{\text{aff}}/Q^{\vee}_{\text{aff}}$$

Weyl orbit = orbit of $\mathcal{Z}(G)$

Toda lattice equations for N=2 surface defects

What's special about surface defects in N=2 SYM ?

We propose that in this case Z_{defect} is the **tau-function** of Toda lattice equations

$$D^2(\tau_{\boldsymbol{\beta}}) = -\frac{\boldsymbol{\beta}^\vee \cdot \boldsymbol{\beta}^\vee}{2} t^{1/h^\vee} \prod_{\beta \in \hat{\Delta}, \boldsymbol{\beta} \neq \boldsymbol{\alpha}} [\tau_{\boldsymbol{\alpha}}]^{-\boldsymbol{\alpha} \cdot \boldsymbol{\beta}^\vee}$$

$D^2(f) = f\partial^2_{\log t} f - (\partial_{\log t} f)^2$ second Hirota derivative $\quad t := (\Lambda/\epsilon)^{2h^\vee}$

$\alpha \in \hat{\Delta}$ simple root of affine Lie algebra $\quad \boldsymbol{\alpha}^\vee = 2\boldsymbol{\alpha}/(\boldsymbol{\alpha}, \boldsymbol{\alpha})$ co-root

A special role is played by the tau-functions associated to the ***affine nodes***

$$\tau_{\boldsymbol{\alpha}_{\text{aff}}}(\boldsymbol{\sigma}, \boldsymbol{\eta} | \kappa_{\mathfrak{g}} t) = \sum_{\mathbf{n} \in Q^\vee_{\text{aff}}} e^{2\pi\sqrt{-1}\boldsymbol{\eta}\cdot\mathbf{n}} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2} B(\boldsymbol{\sigma} + \mathbf{n} | t)$$

$\kappa_{\mathfrak{g}} = (-n_{\mathfrak{g}})^{r_{\mathfrak{g},s}}$

number of short simple roots

ratio of squares of long vs. short roots

necessary to implement the correct S-duality for non simply laced groups

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$B(\boldsymbol{\sigma}|t) \;=\; B_0(\boldsymbol{\sigma})\sum_{i\geq 0} t^i Z_i(\boldsymbol{\sigma}) \qquad Z_0(\boldsymbol{\sigma}) \;\equiv\; 1$ convergent power series

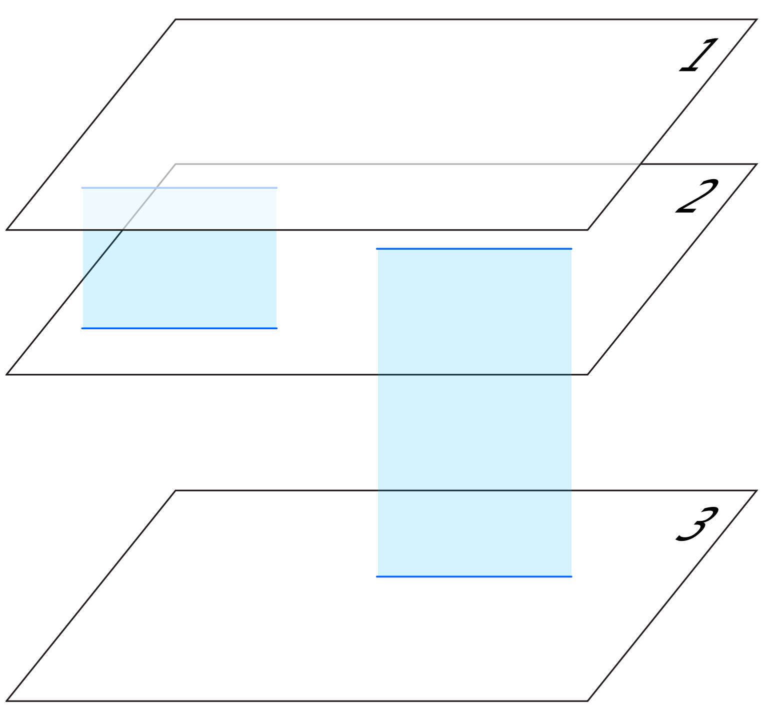
$\boldsymbol{\eta},\ \boldsymbol{\sigma} \in Q^\vee$ integration constants

Toda Lattice equations from M-theory

r M5 branes on $\mathbb{R}^6$ described by A_{r-1} superconformal field theory in six-dimensions with (2,0) supersymmetry

superconformal group $\supset$ SO(6,2) x SO(5)

Coulomb branch $\mathcal{U}$: r M5 branes on $\mathbb{R}^6$ separated in the transversal $\mathbb{R}^5$ space



$\mathcal{U}$ described by v.e.v.s of $\mathrm{Tr}\, Y^I Y_I$ and all other Casimirs of A_{r-1}

Y^I five real scalars param. the position of M5s in transv. $\mathbb{R}^5$

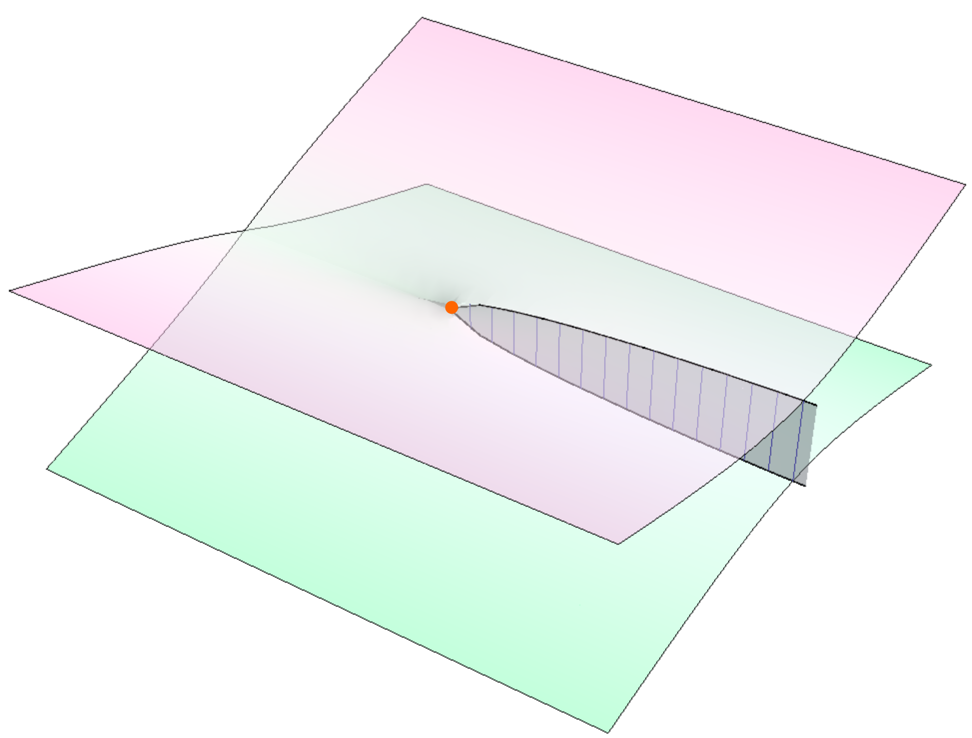
Toda Lattice equations from M-theory

twisted compactification on $\mathbb{R}^4 \times \mathcal{C}$: local geometry near M5s

$$\mathbb{R}^4 \times T^*\mathcal{C} \times \mathbb{R}^3$$

superconformal group reduces $\supset$ SO(4,2) x U(1) x SU(2)

Coulomb branch described by v.e.v.s of $\frac{1}{2}(Y^1 + iY^2) \equiv \varphi \in \Gamma(K)$ x End(E)



$$\mathcal{U} \equiv \oplus_{k=1}^{r} H^0(\mathcal{C}, K^{\otimes d_k})$$

r-covering of $\mathcal{C}$, sheets labeled by eigenvalues of φ

S^1 compactification gives rise to U(r) Super Yang-Mills theory in 5d on $\mathbb{R}^3 \times \mathcal{C}$. BPS vacua invariant under Super-Poincare' of $\mathbb{R}^3$ satisfy **Hitchin's equations**

$$F + R^2[\varphi, \bar{\varphi}] = 0,$$
$$\partial_{\bar{z}}\varphi + [A_{\bar{z}}, \varphi] = 0,$$
$$\partial_z\bar{\varphi} + [A_z, \bar{\varphi}] = 0,$$

these are equivalent to the flatness of the $SL(r, \mathbb{C})$ connection

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\overline{\varphi};$$

oper limit: $R \to 0, \zeta \to 0$ $\zeta / R = \hbar$

radial component of flat Hitchin connection on cylinder w. regular singularities

$$\mathcal{A} = \frac{\partial}{\partial r}\mathbf{q} + w^{-1}\left(e^{-\alpha_0 q}E_{-\alpha_0} + \sum_{\alpha \text{ simple}} e^{\alpha q}E_{\alpha}\right) + w\left(e^{-\alpha_0 q}E_{\alpha_0} + \sum_{\alpha \text{ simple}} e^{\alpha q}E_{-\alpha}\right)$$

$$\mathbf{q} = \mathrm{diag}(q_0, \ldots, q_{N-1})$$

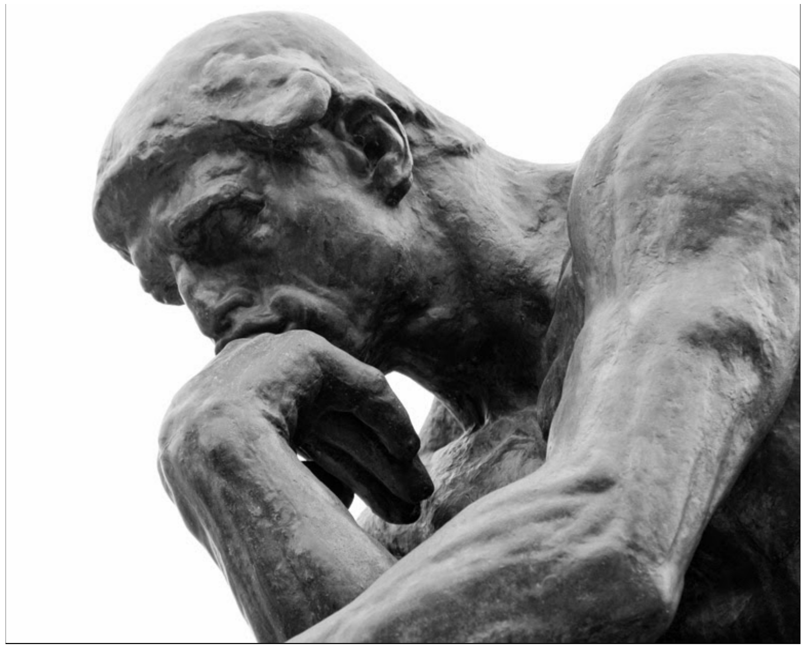
obey's ***Toda lattice equations*** with boundary conditions set by the surface operator.

Toda lattice equations from 4d/2d correspondence

The Toda system is the ***radial reduction*** of 2D Toda lattice equations on the cylinder $\mathbb{C}^*$.

These naturally arise as ***tt* equations*** for a *Landau-Ginzburg* model describing complex deformations of a $\mathcal{Z}(G)$ singularity

Why is this relevant for surface defects ?



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Recall that 1/2 BPS surface defects can also introduced by coupling the 4d theory to a (2,2) 2d GLSM describing maps

$$D \to G/\mathbb{L}$$

for full surface defects $\mathbb{L} = \mathbb{T}$ the target space is a ***complete flag*** variety whose Hori-Vafa mirror is precisely the above Landau-Ginzburg model !!

Seiberg-Witten theory viewpoint

Seiberg-Witten curve of N=2 SYM is the spectral curve of affine ***Toda chain*** of type $(\hat{G})^\vee$ ← Langlands dual

The RG equations for surface defects are the ***de-autonomization*** of Toda chain equations. Simplest example:

SU(2) $\xrightarrow{\text{SW curve}}$ 2 particle Toda chain $\xrightarrow{\text{de-autonomization}}$ Painleve' III degen.

The de-autonomization is the deformation of the integrable system describing susy gauge theory in a *self-dual Omega background* and thus *gravitational corrections to SW prepotential .*

The latter are equivalent to *topological string amplitudes* on a suitable local Calabi-Yau via geometric engineering.

Solutions

Tau-system

Toda system in tau form:

$$D^2(\tau_{\boldsymbol{\beta}}) = -\frac{\boldsymbol{\beta}^\vee \cdot \boldsymbol{\beta}^\vee}{2} t^{1/h^\vee} \prod_{\beta \in \hat{\Delta}, \boldsymbol{\beta} \neq \boldsymbol{\alpha}} [\tau_{\boldsymbol{\alpha}}]^{-\boldsymbol{\alpha} \cdot \boldsymbol{\beta}^\vee}$$

Kyiv-like ansatz

$$\tau_{\boldsymbol{\alpha}_{\rm aff}}(\boldsymbol{\sigma}, \boldsymbol{\eta} | \kappa_{\mathfrak{g}} t) = \sum_{\mathbf{n} \in Q^\vee_{\rm aff}} e^{2\pi\sqrt{-1}\boldsymbol{\eta}\cdot\mathbf{n}} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2} B(\boldsymbol{\sigma} + \mathbf{n} | t)$$

$$B(\boldsymbol{\sigma}|t) \;=\; B_0(\boldsymbol{\sigma}) \textstyle\sum_{i\geq 0} t^i Z_i(\boldsymbol{\sigma}) \qquad Z_0(\boldsymbol{\sigma}) \;\equiv\; 1$$

Asymptotic conditions

$$\log(B_0) \sim -\frac{1}{4} \sum_{\mathbf{r} \in R} (\mathbf{r} \cdot \boldsymbol{\sigma})^2 \log{(\mathbf{r} \cdot \boldsymbol{\sigma})^2}$$

$t \to 0$ and $\boldsymbol{\sigma} \to \infty$

We find that for the solutions satisfying the above asymptotic conditions

$t^{\frac{1}{2}\boldsymbol{\sigma}^2} B(\boldsymbol{\sigma}|t)$ is the full Nekrasov p.f. in a ***self-dual*** Omega background $(\epsilon_1, \epsilon_2) = (\epsilon, -\epsilon)$

with $t := (\Lambda/\epsilon)^{2h^\vee}$ RG scale $\boldsymbol{\sigma} = \mathbf{a}/\epsilon$, vev of N=2 scalar

We obtain new recurrence relations determining all instanton corrections in self-dual bckg for all simple groups from A_n to E .

One-loop

solution satisfying the asymptotic conditions

$$B_0(\boldsymbol{\sigma}) = \mathcal{Z}_{1-loop}(\boldsymbol{\sigma}) \equiv \prod_{\mathbf{r}\in R} \frac{1}{G(1+\mathbf{r}\cdot\boldsymbol{\sigma})}$$

adjoint representation Barnes G-function

by using

$$\log G(1+x) = \frac{1}{12} - \log(A) + \frac{1}{2}x\log(2\pi) - \frac{3}{4}x^2 + \left(\frac{1}{2}x^2 - \frac{1}{12}\right)\log(x) + \mathcal{O}(x^{-2})$$

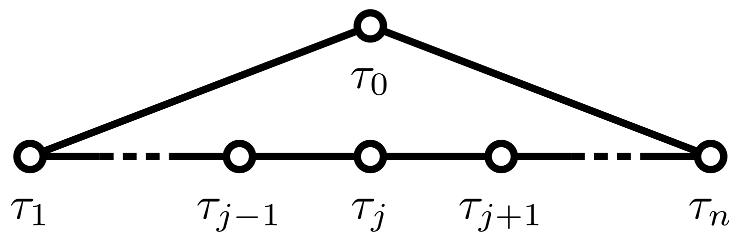
the asymptotic expansion matches the ***perturbative one-loop*** calculation of gauge theory upon a suitable choice of the branch for the log

$$\ln\left[\sqrt{-1}\mathbf{r}\cdot\mathbf{a}/\Lambda\right] \in \mathbb{R}$$

this fixes some directions in the complex plane which are the ***Stokes rays*** of the related non-linear equations - isomonodromic deformation problem on the Riemann sphere with two irregular singular point of Poincare’ rank 1.

Instantons

Start with A_n case



tau-system

$$D^2(\tau_j) = -t^{\frac{1}{n+1}}\tau_{j-1}\tau_{j+1}$$

$\mathbb{Z}_{n+1}$ symmetry

$$\tau_j(\boldsymbol{\sigma}|t) = \tau_0(\boldsymbol{\sigma} + \boldsymbol{\lambda}_j|t)$$

reduces to

$$D^2(\tau_0(\boldsymbol{\sigma})) = -\tau_0(\boldsymbol{\sigma} \pm e_1)$$

ansatz

$$\tau_0(\boldsymbol{\sigma}, \boldsymbol{\eta}|t) = \sum_{\mathbf{n}\in Q,\, i\geq 0} e^{2\pi\sqrt{-1}\mathbf{n}\cdot\boldsymbol{\eta}} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2+i} B_0(\boldsymbol{\sigma}+\mathbf{n}) Z_i(\boldsymbol{\sigma}+\mathbf{n})$$

Recursion relations

by plugging the ansatz into the tau-system one gets the recurrence relation:

$$k^2 Z_k(\boldsymbol{\sigma}) = - \sum_{\substack{\mathbf{n}^2+j_1+j_2=k \\ \mathbf{n}\in e_1+Q,\, j_{1,2}<k}} \frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2} Z_{j_2}(\boldsymbol{\sigma}-\mathbf{n}) Z_{j_1}(\boldsymbol{\sigma}+\mathbf{n})$$
$$+ \sum_{\mathbf{n}^2+i_1+i_2=k\mathbf{n}\in Q,\, i_{1,2}<k} (i_1 - i_2 + 2\mathbf{n}\cdot\boldsymbol{\sigma})^2 \frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2} Z_{i_1}(\boldsymbol{\sigma}+\mathbf{n}) Z_{i_2}(\boldsymbol{\sigma}-\mathbf{n}),$$

the first step provides ***one-instanton in terms of one-loop:***

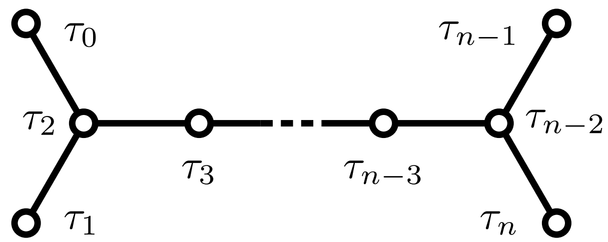
$$Z_1(\boldsymbol{\sigma}) = -\sum_{i=1}^{n+1} \frac{B_0(\boldsymbol{\sigma}\pm e_i)}{B_0(\boldsymbol{\sigma})^2} = (-1)^{n+1} \sum_{i=1}^{n+1} \frac{1}{\prod_{j\neq i}(\sigma_i-\sigma_j)^2}$$

going on with two-instantons:

$$Z_2(\boldsymbol{\sigma}) = -\frac{1}{4}\sum_{i=1}^{n+1} \frac{B_0(\boldsymbol{\sigma}\pm e_i)}{B_0(\boldsymbol{\sigma})^2} [Z_1(\boldsymbol{\sigma}+e_i) + Z_1(\boldsymbol{\sigma}-e_i)]$$
$$+ \sum_{i<j}^{n+1} (\sigma_i-\sigma_j)^2 \frac{B_0(\boldsymbol{\sigma}\pm(e_i-e_j))}{B_0(\boldsymbol{\sigma})^2}$$

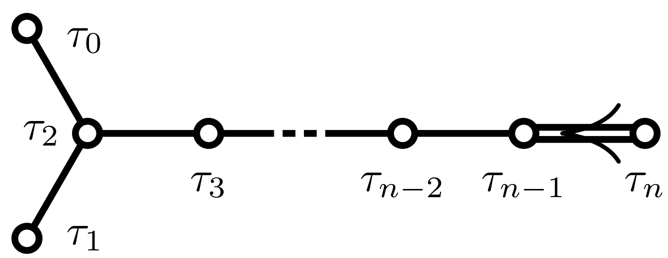
Other groups

D_n



$$D^2(\tau_0) = D^2(\tau_1), \quad D^2(\tau_{n-1}) = D^2(\tau_n)$$

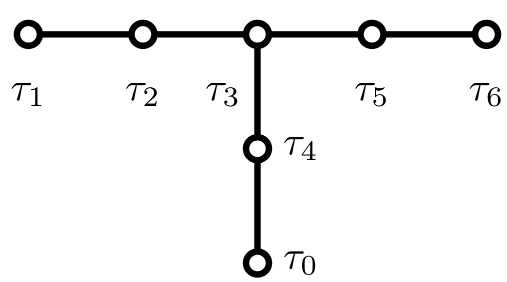
B_n



$$D^2(\tau_0) = D^2(\tau_1)$$

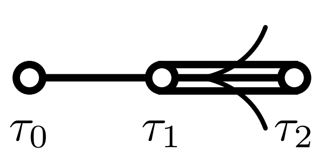
$$D^2(\tau_{n-1}) = -2t^{\frac{1}{2n-1}}\tau_{n-2}\tau_n, \quad D^2(\tau_n) = -t^{\frac{1}{2n-1}}\tau_{n-1}^2.$$

E_6



$$\tau_6 D^4(\tau_0) = \tau_0 D^4(\tau_6)$$

G_2



$$D^2(\tau_0^{-1}D^4(\tau_0)) = 3t(D^2(\tau_0))^3$$

G_2 case

G_2 ansatz

$$\tau_0(\boldsymbol{\sigma},\boldsymbol{\eta}|t) = \sum_{\mathbf{n}\in Q^\vee} e^{2\pi\sqrt{-1}\boldsymbol{\eta}\cdot\mathbf{n}} \left(-\tfrac{t}{3}\right)^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2} B\left(\boldsymbol{\sigma}+\mathbf{n}|-\tfrac{t}{3}\right)$$

reduced tau-system implies

$$\sum_{\substack{\{\mathbf{n}_k\}\in Q^\vee \\ \{i_k\}\in\mathbb{N}}} \prod_{k=1}^{4} e^{2\pi\sqrt{-1}\boldsymbol{\eta}\cdot\mathbf{n}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k} B_0(\boldsymbol{\sigma}+\mathbf{n}_k) Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)$$
$$\left(\frac{1}{4!}\prod_{k_1<k_2}(\frac{1}{2}\mathbf{n}_{k_1}^2+i_{k_1}-\frac{1}{2}\mathbf{n}_{k_2}^2-i_{k_2}+(\mathbf{n}_{k_1}-\mathbf{n}_{k_2})\cdot\boldsymbol{\sigma})^2\right.$$
$$+\frac{9}{4}(\frac{1}{2}\mathbf{n}_1^2+i_1-\frac{1}{2}\mathbf{n}_2^2-i_2+(\mathbf{n}_1-\mathbf{n}_2)\cdot\boldsymbol{\sigma})^2$$
$$\left.(\frac{1}{2}\mathbf{n}_3^2+i_3-\frac{1}{2}\mathbf{n}_4^2-i_4+(\mathbf{n}_3-\mathbf{n}_4)\cdot\boldsymbol{\sigma})^2\right) = 0\,.$$

one-instanton

$$Z_1(\boldsymbol{\sigma})^{[G_2]}|_{\sigma_3=-\sigma_1-\sigma_2} = -\frac{2}{3\sigma_1^2\sigma_2^2(\sigma_1+\sigma_2)^2}$$

two-instantons

$$Z_2(\boldsymbol{\sigma})^{[G_2]}|_{\sigma_3=-\sigma_1-\sigma_2} = \frac{3\left(9\sigma_1^4\left(6\sigma_2^2+1\right)+18\sigma_1^3\left(6\sigma_2^3+\sigma_2\right)+3\sigma_1^2\left(18\sigma_2^4+9\sigma_2^2-2\right)+6\sigma_1\sigma_2\left(3\sigma_2^2-1\right)+\left(1-3\sigma_2^2\right)^2\right)}{\sigma_1^2\left(1-3\sigma_1^2\right)^2\sigma_2^2\left(1-3\sigma_2^2\right)^2(\sigma_1+\sigma_2)^2(1-3(\sigma_1+\sigma_2)^2)^2}$$

Matrix models for the magnetic phase

[Bonelli, Grassi, A.T.]

Toda time $\longrightarrow$ $t = \left(\frac{\Lambda}{\epsilon}\right)^{2h^\vee}$ $\longleftarrow$ RG scale

Late time expansion of tau-functions $\Longleftrightarrow$ ***dual Seiberg Witten prepotential***

Spectral determinant

$$\tau_{A_1}(\sigma, \eta = 0, t) = \sum_{M \geq 0} \kappa^M Z_M(\Lambda/\epsilon)$$

$$Z_M(\Lambda) = \frac{1}{M!} \int \prod_{i=1}^{M} \frac{\mathrm{d}x_i}{4\pi} \mathrm{e}^{-\frac{2\Lambda}{\pi^2 \epsilon} \cosh x_i} \prod_{i<j} \tanh\left(\frac{x_i - x_j}{2}\right)^2$$

$$\log Z_M = \sum_{g>0} \epsilon^{2g-2} F_g^D(a_D)$$

matrix model for SU(2) SYM around the monopole point

non-perturbative completion of topological string on (a limit of) local Hirzebruch geometry

Matrix models for the magnetic phase

Generalisation to $\mathcal{T}A_{n-1}$

$$Z_{4\mathrm{d}}^{\mathrm{SYM}}(M_1, \cdots, M_{N-1}) = \frac{1}{M_1! \cdots M_{N-1}!} \int \frac{\mathrm{d}^M x}{(2\pi)^M} \prod_{j=1}^{N-1} \prod_{i_j \in I_j} \mathrm{e}^{-NT \sin\left(\frac{\pi j}{N}\right) \cosh(x_{i_j})}$$
$$\times \frac{\prod_{1 \le i < j \le M} 2\sinh\left(\frac{x_i - x_j}{2} + \frac{1}{2}(d_i - d_j)\right) 2\sinh\left(\frac{x_i - x_j}{2} + \frac{1}{2}(f_i - f_j)\right)}{\prod_{i,j=1}^{M} 2\cosh\left(\frac{x_i - x_j}{2} + \frac{1}{2}(d_i - f_j)\right)}$$

d_i, f_i are N-dependent phase shifts.

Computes dual prepotential around the massless monopole point $a_{D_i} = M_i T^{-1}$

$$F_0^D(T_1, \ldots, T_{N-1}) = \sum_{j=1}^{N-1} \left[\frac{T_j^2}{2} \left(\log\left(\frac{T_j}{c_j}\right) - \frac{3}{2} \right) - T_j \frac{\sin\left(\frac{\pi j}{N}\right)}{\sin\left(\frac{\pi}{N}\right)} \right]$$
$$+ \sum_{i<j} T_i T_j \log\left(a_{ij}\right) + \mathcal{O}(T_i^3),$$

$$a_{ij} = \frac{\cos\left(\frac{\pi(i-j)}{N}\right) - 1}{\cos\left(\frac{\pi(i+j)}{N}\right) - 1},$$

$$c_j = 4\sin\left(\frac{\pi j}{N}\right)^3 / \sin\left(\frac{\pi}{N}\right)$$

controls the mass spectrum when breaking to $\mathcal{N} = 1$ supersymmetry.

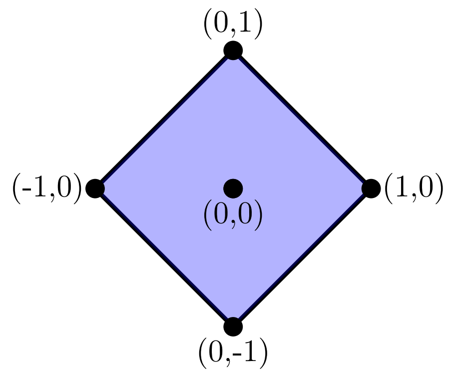
Surface defects in 5d

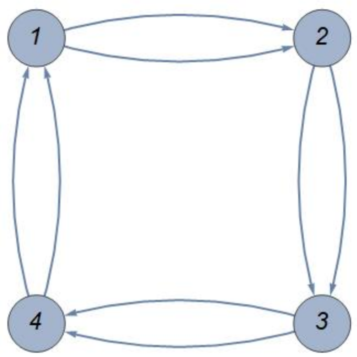
5d Gauge theories on a circle

Codim. 2 defects in 5d gauge theories obey a ***q-difference*** uplift of the tau-system. ***Discrete dynamical flow*** generated by ***automorphism group*** of the **5d *BPS quiver.***

5d *BPS quiver* from geometric engineering via Calabi-Yau compactification
nodes: Dp branes wrapping calibrated cycles - BPS states of 5d SCFT
arrows: Dirac pairings among BPS particles

Example: Pure SU(2) Super Yang-Mills





$$Dih_4 \ltimes W(A_1^{(1)})$$

local $\mathbb{F}_0$ Newton polygon | 5d BPS quiver | automorphism group

generators $\pi_1 = (1,3)\iota,$ $\pi_2 = (4,3,2,1),$ $T_{\mathbb{F}_0} = (1,2)(3,4)\mu_1\mu_3.$

Cluster integrable system

quiver mutation

$$\mu_k(B_{ij}) = \begin{cases} -B_{ij}, & i = k \text{ or } j = k, \\ B_{ij} + \frac{B_{ik}|B_{kj}| + B_{kj}|B_{ik}|}{2}, & \end{cases}$$

X - *cluster variables*

$$\mu_k(x_j) = \begin{cases} x_j^{-1}, & j = k, \\ x_j(1 + x_k^{\operatorname{sgn} B_{jk}})^{B_{jk}}, & j \neq k. \end{cases}$$

Poisson bracket

adjacency matrix of quiver

$$\{x_i, x_j\} = B_{ij} x_i x_j.$$

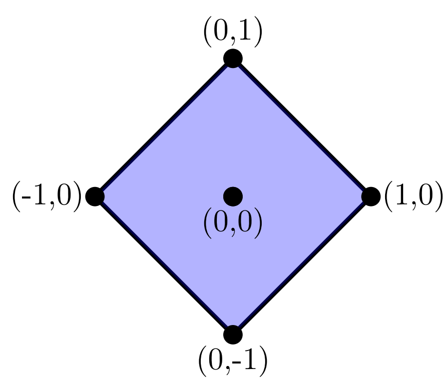
space of Casimirs $\ker(B)$, $q \equiv \prod_i x_i$ is always a Casimir by construction

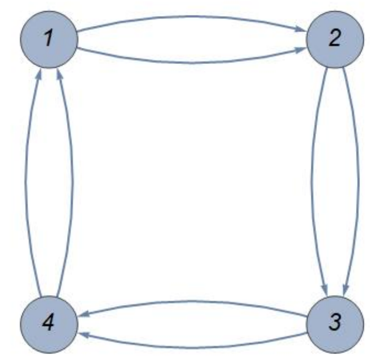
Integrable dynamics on the level surface $q = 1$ ***discrete flows*** generated by the group of ***quiver automorphisms***.

independent Hamiltonians ⟷ internal points of Newton polygon

Relativistic Toda

For the example of SU(2) SYM





$$Dih_4 \ltimes W(A_1^{(1)})$$

local $\mathbb{F}_0$ Newton polygon | 5d BPS quiver | automorphism group

In the variables $x \equiv x_1 \,,\, y \equiv x_2 \,,\, Z \equiv x_1 x_3$ one gets the Hamiltonian

$$H = \sqrt{xy} + \sqrt{\frac{x}{y}} + \frac{1}{\sqrt{xy}} + Z\sqrt{\frac{y}{x}}$$

relativistic two-particle periodic Toda chain with flow generated by the quiver automorphism

$$T_{\mathbb{F}_0} = (1,2)(3,4)\mu_1\mu_3 .$$

Tau functions and de-autonomization

Tau - *cluster variables*

$$\mu_k(\tau_j) = \begin{cases} \tau_j, & j \neq k, \\ \frac{y_k \prod_{i=1}^{n} \tau_i^{[B_{ik}]_+} + \prod_{i=1}^{|Q|} \tau_i^{-[B_{ik}]_+}}{\tau_k (1 \oplus y_k)}, & j = k, \end{cases}$$

de-autonomization $\quad q \neq 1$

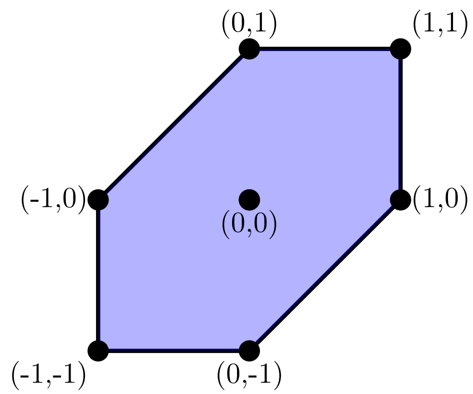
induces a discrete flow of the Hamiltonians and of the tau functions under quiver automorphism $\quad T_{\mathbb{F}_0} = (1,2)(3,4)\mu_1\mu_3.$

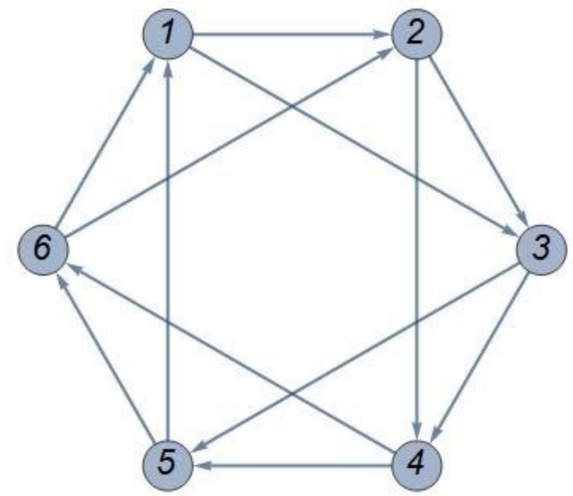
$$\begin{cases} T_{\mathbb{F}_0}(\tau_1) = \tau_2, \\ T_{\mathbb{F}_0}(\tau_2) = \frac{\tau_2^2 + (qt)^{1/2}\tau_4^2}{\tau_1}, \\ T_{\mathbb{F}_0}(\tau_3) = \tau_4, \\ T_{\mathbb{F}_0}(\tau_4) = \frac{\tau_4^2 + (qt)^{1/2}\tau_2^2}{\tau_3} \end{cases}, \qquad \begin{cases} T_{\mathbb{F}_0}^{-1}(\tau_1) = \frac{\tau_1^2 + t^{1/2}\tau_3^2}{\tau_2}, \\ T_{\mathbb{F}_0}^{-1}(\tau_2) = \tau_1, \\ T_{\mathbb{F}_0}^{-1}(\tau_3) = \frac{\tau_3^2 + t^{1/2}\tau_1^2}{\tau_4}, \\ T_{\mathbb{F}_0}^{-1}(\tau_4) = \tau_3, \end{cases}$$

$$\overline{\tau_1}\underline{\tau_1} = \tau_1^2 + t^{1/2}\tau_3^2, \qquad \overline{\tau_3}\underline{\tau_3} = \tau_3^2 + t^{1/2}\tau_1^2$$

q-difference uplift of $\hat{A}_1$ Toda tau-system of 4d SU(2) SYM
bilinear relations for tau functions of q-Painleve'.

E_3 **SCFT**





$\tilde{W}((A_2+A_1)^{(1)})$

Four commuting discrete flows:

$$T_1 = s_0 s_2 \pi, \qquad T_2 = s_1 s_0 \pi, \qquad T_3 = s_2 s_1 \pi \qquad T_1 T_2 T_3 = 1$$

and $$T_4 = r_0 \pi^3 = (4,6)\mu_2\mu_4\mu_6\mu_2(4,5,6,1,2,3)$$

from $$s_0 = (3,6)\mu_6\mu_3 \qquad s_1 = (1,4)\mu_4\mu_1, \qquad s_2 = (2,5)\mu_5\mu_2 \qquad \pi = (1,2,3,4,5,6)$$

$$r_0 = (4,6)\mu_2\mu_4\mu_6\mu_2 \qquad r_1 = (3,5)\mu_1\mu_3\mu_5\mu_1 \qquad \sigma = (1,4)(2,3)(5,6)\iota$$

generators of the extended Weyl group $\tilde{W}((A_2+A_1)^{(1)})$

5d SU(2) with two flavours

It is known that the massive deformation of E_3 5d SCFT by the Yang-Mills action produces 5d SU(2) with two flavours. All the first three flows give rise to the bilinear equations

$$\overline{\tau_3}\tau_2 = q^{1/4}t^{1/2}\overline{\tau_5}\tau_6 + \tau_3\overline{\tau_2}, \qquad \overline{\tau_6}\tau_5 = \overline{\tau_5}\tau_6 + q^{1/4}t^{1/2}\tau_3\overline{\tau_2},$$

$$\overline{\tau_2}\underline{\tau_3} = -Q_1 t^{1/2}\tau_5\tau_6 + \tau_2\tau_3, \qquad \overline{\tau_5}\underline{\tau_6} = -Q_2 t^{1/2}\tau_2\tau_3 + \tau_5\tau_6.$$

whose tau functions are 5d partition functions for that gauge theory:

$$\tau_1 = Z^D_{1/2}(Q_1, Q_2 q^{1/2}, tq^{1/2}), \qquad \tau_4 = Z^D_0(Q_1 q^{1/2}, Q_2, tq^{1/2})$$

$$\tau_2 = Z^D_0(Q_1 q^{1/2}, Q_2, tq^{-1/2}), \qquad \tau_3 = Z^D_0(Q_1 q^{-1/2}, Q_2, tq^{1/2}),$$

$$\tau_5 = Z^D_{1/2}(Q_1, Q_2 q^{1/2}, tq^{-1/2}), \qquad \tau_6 = Z^D_{1/2}(Q_1, Q_2 q^{-1/2}, tq^{1/2}).$$

$$Z^D_0 \equiv \sum_n s^n Z(Q_1, Q_2, uq^n, t), \quad Z^D_{1/2} = \sum_n s^n Z(Q_1, Q_2, uq^{n+1/2}, t) = Z^D_0(uq^{1/2})$$

full Nekrasov partition function - topological string on local dP3 at large volume

An AD surprise

The fourth flow give rise to *new bilinear relations* !

$$\begin{cases} \overline{\tau_6}\underline{\tau_2} - q^{1/4}t^{1/2}\overline{\tau_2}\underline{\tau_6} = -t^{1/2}\left(Q_2 + q^{1/2}Q_1\right)\tau_2\tau_6, \\ Q_+^{1/2}q^{1/4}\overline{\tau_6}\underline{\tau_4} + Q_2 t^{1/2}\overline{\tau_4}\underline{\tau_6} = t^{1/2}\left(Q_2 + q^{1/2}Q_1\right)\tau_4\tau_6, \\ \overline{\tau_4}\underline{\tau_2} - \overline{\tau_2}\underline{\tau_4} = t^{1/2}\left(Q_2 + q^{1/2}Q_1\right)\tau_2\tau_4. \end{cases} \quad , \quad \overline{\tau_i} = \tau_i(q^{1/2}Q_1, q^{-1/2}Q_2).$$

$$\overline{Q_-} = qQ_- \qquad Q_- = \frac{Q_1}{Q_2}$$

4d sub-quivers and Argyres-Douglas theory

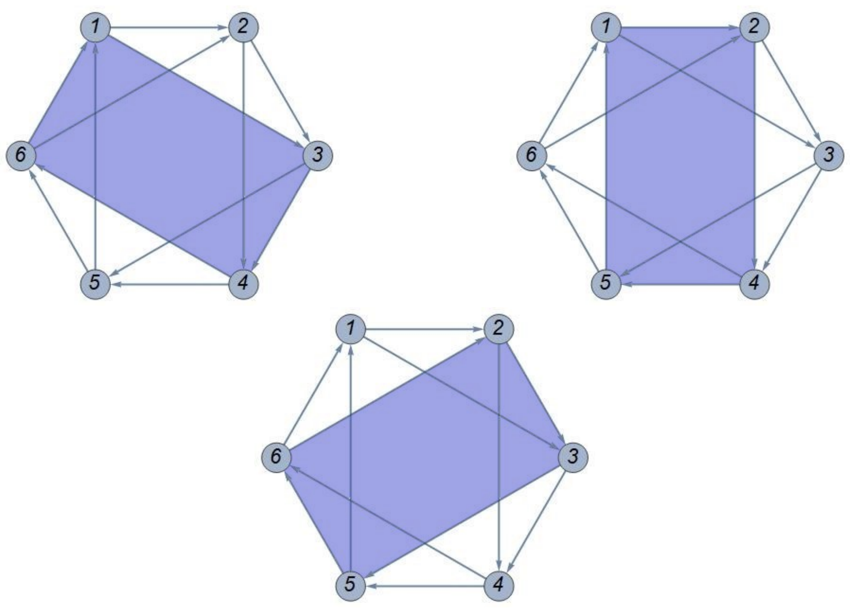
5d BPS quivers are obtained from 4d ones by adding ***two nodes***, corresponding to ***Kaluza-Klein modes*** on the circle and to ***5d instanton-particle***.

Conversely, one can recover 4d sub-quivers from the 5d one by sending to zero/infinity two X-cluster variables consistently with the flow.

$$T_4(Q_-) = qQ_-, \qquad x_2x_4x_6 = (qQ_-)^{1/2} \to 0, \qquad x_1x_3x_5 = q^{-1}Q_-^{-1/2} \to \infty$$

keeping other Casimirs finite.

For the new flow, the consistent choices are:

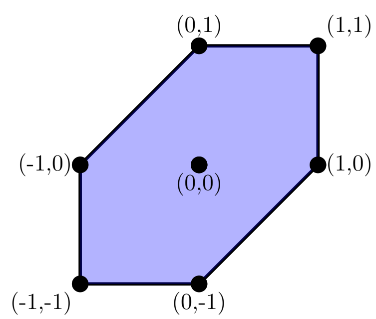


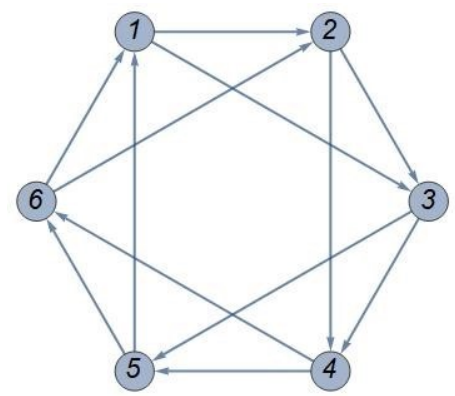
and bring to the 4d quiver of H_2 ***Argyres-Douglas theory.***

A puzzle

Recall that H_2 ***Argyres-Douglas theory*** is the IR SCF point of 4d SU(2) gauge theory with ***three flavours.***

On the other hand, the toric diagram of E_3 SCFT



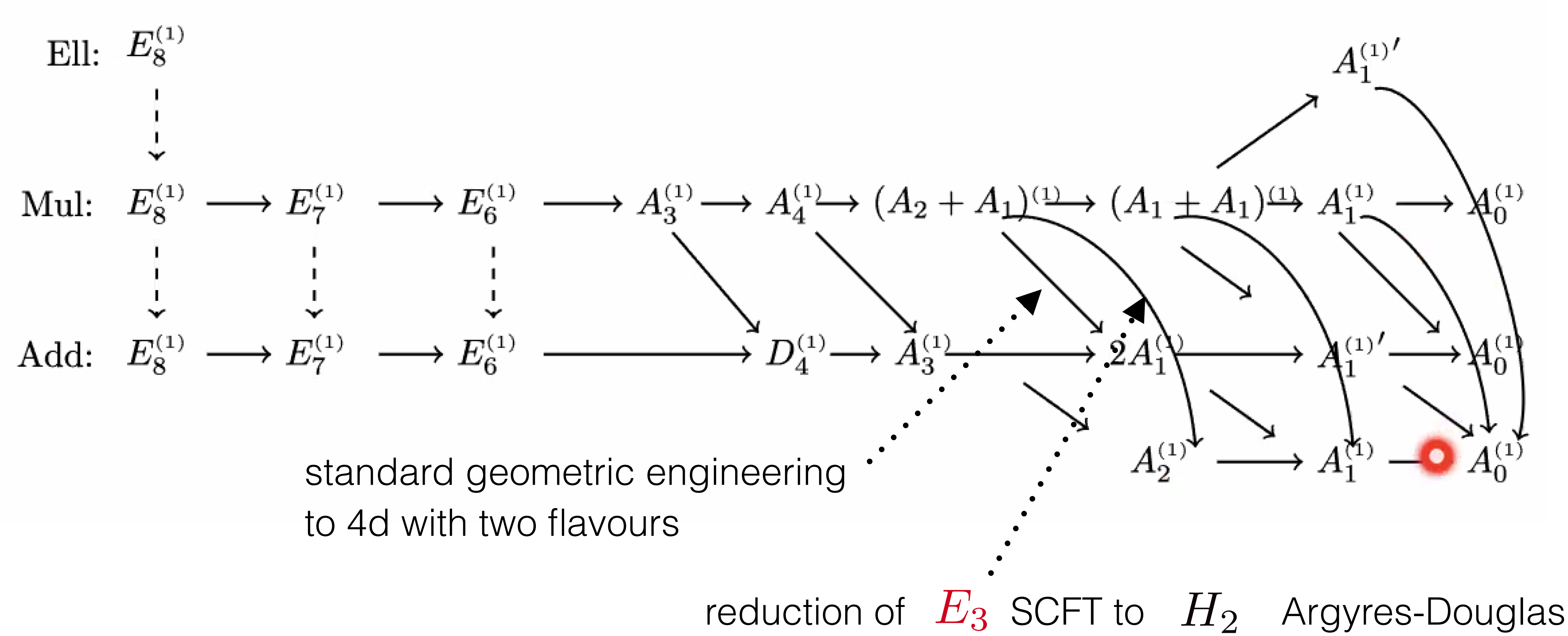


engineers 5d SU(2) gauge theory with ***two flavours.***

So, what is going on ?

Sakai’s classification and rank one 5d SCFTs

Sakai’s classification of q-Painleve’ equations based on symmetry lattice of eight-point blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$



different limit pointing to a ***Argyres-Douglas SCFT !***

Matrix models for the magnetic phase

Conjectural proposal for non-perturbative completion of topological strings

Quantum mirror curve $\widehat{\mathcal{O}} = \sum_{(m,n)\neq(0,0)} a_{m,n} e^{m\widehat{x}+n\widehat{p}}$ $[\widehat{x},\widehat{p}] = i\hbar$

(5d quantum SW curve)

Spectral determinant $\Xi(\kappa) = \det\left(1+\kappa\widehat{\mathcal{O}}^{-1}\right)$

$$\Xi(\kappa,\hbar) = \sum_{N\geq 0} \kappa^N Z_X^\rho(N,\hbar)$$

⟶ **matrix model for topological string partition function**

TS/ST conjecture

Problem: inverting quantum mirror curve is hard !

Matrix models for the magnetic phase

[Bonelli, Globlek, Kubo, Nosaka, A.T.]

Proposal for local dP_5 which engineers 5d SU(2) $N_F = 4$ gauge theory

$$Z_k^{\mathrm{VI}}(N; M_1, M_2, M, \zeta_1, \zeta_2) = \frac{1}{N!\,(N+M)!} \int \prod_{n=1}^{N} \frac{d\mu_n}{2\pi k} \prod_{n=1}^{N+M} \frac{d\nu_n}{2\pi k}$$

$$\times \prod_{n=1}^{N} e^{\left(-\frac{i\zeta_1}{k} + \frac{2k-M-M_2}{2k}\right)\mu_n} \frac{\Phi_b\left(\frac{\mu_n}{2\pi b} - \frac{iM_1}{2b} + \frac{i}{2}b\right)}{\Phi_b\left(\frac{\mu_n}{2\pi b} + \frac{iM_1}{2b} - \frac{i}{2}b\right)} \frac{\Phi_b\left(\frac{\mu_n}{2\pi b} - \frac{iM_2-2\zeta_2}{2b} + \frac{i}{2}b\right)}{\Phi_b\left(\frac{\mu_n}{2\pi b} + \frac{iM_2+2\zeta_2}{2b} - \frac{i}{2}b\right)}$$

$$\times \prod_{n=1}^{N+M} e^{\left(\frac{i\zeta_1}{k} + \frac{M_1+M_2}{2k}\right)\nu_n} \frac{\Phi_b\left(\frac{\nu_n}{2\pi b} + \frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\nu_n}{2\pi b} - \frac{iM_1}{2b}\right)} \frac{\Phi_b\left(\frac{\nu_n}{2\pi b} + \frac{iM_2+2\zeta_2}{2b}\right)}{\Phi_b\left(\frac{\nu_n}{2\pi b} - \frac{iM_2-2\zeta_2}{2b}\right)}$$

$$\times \left(\frac{\prod_{m<m'}^{N} 2\sinh\frac{\mu_m-\mu_{m'}}{2k} \prod_{n<n'}^{N+M} 2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{N}\prod_{n=1}^{N+M} 2\cosh\frac{\mu_m-\nu_n}{2k}} \right)^2$$

Check: its spectral determinant satisfies the expected q-difference equation in Sakai's list

Summary

Surface defects strongly determine non-perturbative dynamics of gauge theories and have many relations with ***integrability, random matrices*** and ***representation theory.*** Main results:

- new recurrence relations for instanton counting on self-dual Omega background for all simple groups from A_n to E .
- 4d/2d correspondence and t t* equations.
- new matrix models for the magnetic phase of a class of 4d and 5d theories.
- connection between 5d BPS quivers, cluster algebrae and q-difference eq.
- new viewpoint on Argyres-Douglas SCFT.

Outlook

Some natural lines of development:

- 5d uplift of tau-system for general gauge groups (de-autonomization of relativistic Toda chain).
- matrix model for the magnetic phase beyond A_n series.
- spectrum of quantum integrable system from zeroes of the tau-functions.
- solutions of the new bilinear equations and Argyres-Douglas SCFTs.
- general Omega background and quantum cluster algebrae.

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THANKS!