CY^3 cones with T^2-action

Cone singularities in dimension 3:

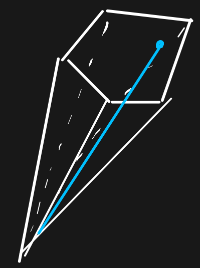
with $\mathbb{C}^*$-action:

$\mathrm{AffineCone}_{-K_X} X$

X Fano surface with KE metric

$$\boxed{\text{with } (\mathbb{C}^*)^2\text{-action} \quad ?}$$

with $(\mathbb{C}^*)^3$-action (toric):



In general: Cone singularities with $(\mathbb{C}^*)^{\dim -1}$-action

Polyhedral divisors:

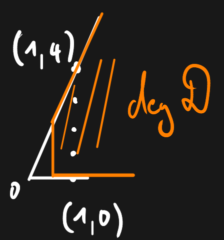
- $\sigma \subset \mathbb{R}^n$ — rational polyhedral cone
- $\mathcal{D} = \sum_i \Delta_i y_i$ — $y_i \in \mathbb{P}^1$, $\Delta_i \subset \mathbb{R}^2$ polyhedra

- $\deg \mathcal{D} = \sum_i \Delta_i \subsetneq \sigma$ — $\sigma = \mathrm{tail}(\Delta_i) = \{v \mid \forall_{w \in \Delta_i} : v + w \in \Delta_i\}$

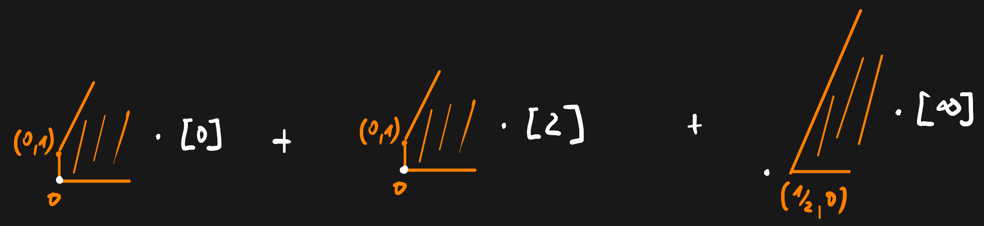
$$\deg \mathcal{D} = \left\{ \sum_i v_i \;\middle|\; v_i \in \Delta_i \right\}$$

Example

$\sigma =$

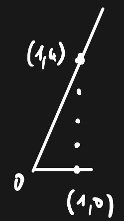


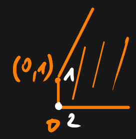
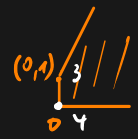
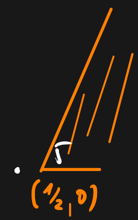
$\mathcal{D} =$

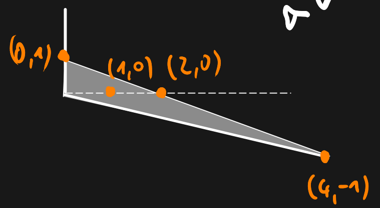


Polyhedral divisor $\longrightarrow$ affine variety: $X = \operatorname{Spec} A(\mathcal{D})$, $\mathcal{D}(u) = \sum_i \min \langle \Delta_i, u \rangle y_i$

$$A(\mathcal{D}) = \bigoplus_{u \in \sigma^\vee \cap \mathbb{Z}^2} H^0(\mathbb{P}^1, \mathcal{O}(\lfloor \mathcal{D}(u) \rfloor)) x_1^{u_1} x_2^{u_2} \subset \mathbb{C}(y)[x_1^{\pm 1}, x_2^{\pm 1}]$$

Example: $\sigma =$ 

$\mathcal{D} =$  $\cdot [0] +$  $\cdot [2] +$  $\cdot [\infty]$

$\sigma^\vee$ 

u	$\lfloor \mathcal{D}(u) \rfloor$	generators of $H^0(\mathbb{P}^1, \mathcal{O}(\mathcal{D}(u)))\, x^u$	
$(0,1)$	0	x_2	$=: z_1$
$(1,0)$	$\lfloor \frac{1}{2}[\infty] \rfloor = 0$	x_1	$=: z_2$
$(2,0)$	$[\infty]$	$\cancel{x_1^2}\,,\; x_1^2(y-1)$	$=: z_3$
$(4,-1)$	$-[0] - [2] + 2[\infty]$	$y(y-2)x_1^4 x_2^{-1}$	$=: z_4$

$$z_1 z_4 - z_3^2 + z_2^4$$

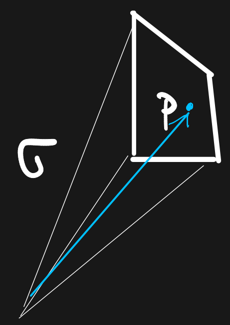
Remark: In general not a complete intersection! But always the quotient of a complete intersection

Alternative description of $A(\mathcal{D})$: $R(\mathcal{D}) = \mathbb{C}[\underset{1}{z_1}, \underset{-1}{z_2}, \underset{1}{z_3}, \underset{-1}{z_4}, z_5] / \langle z_1 z_2 + z_3 z_4 + z_5^2 \rangle$

$$A(\mathcal{D}) = R(\mathcal{D})^{\mathbb{C}^*} \qquad X = \hat{X} \setminus \{0\} / \mathbb{C}^*$$

CY condition

Toric case: $\sigma = \mathrm{cone}(P)$
$= \mathbb{R}_{\geq 0} \cdot (P \times \{1\})$

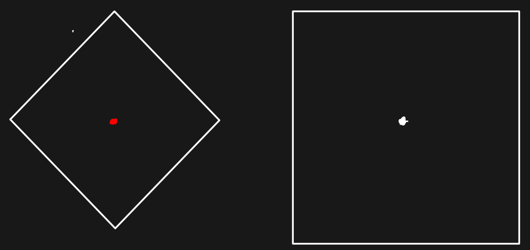


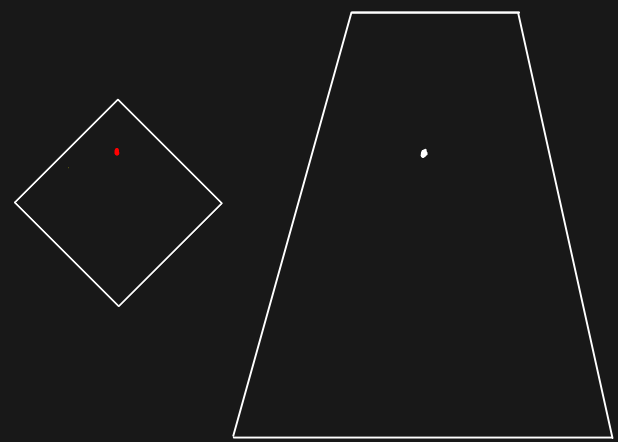
- Reeb Vector field: $v \in P$

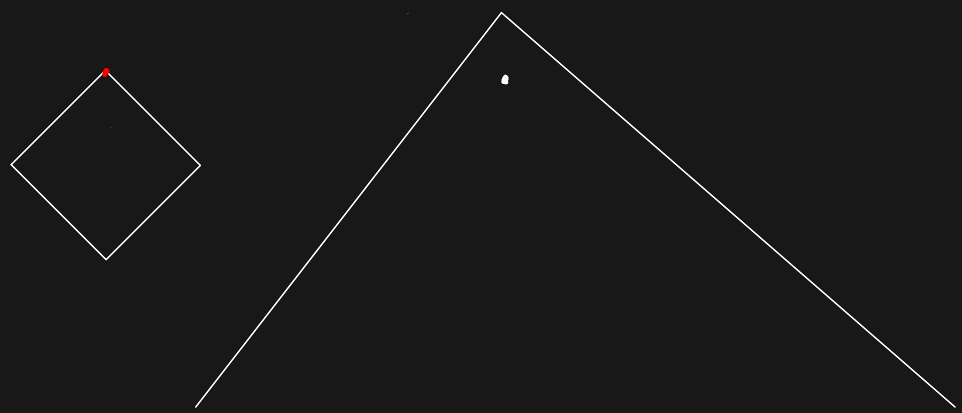
- volume: $\mathrm{vol}\,(P - v)^*$

- minimiser: $s = s(P)$ s.t. $\mathrm{vol}\,(P - s)^* \overset{!}{=} \min$

Santaló point

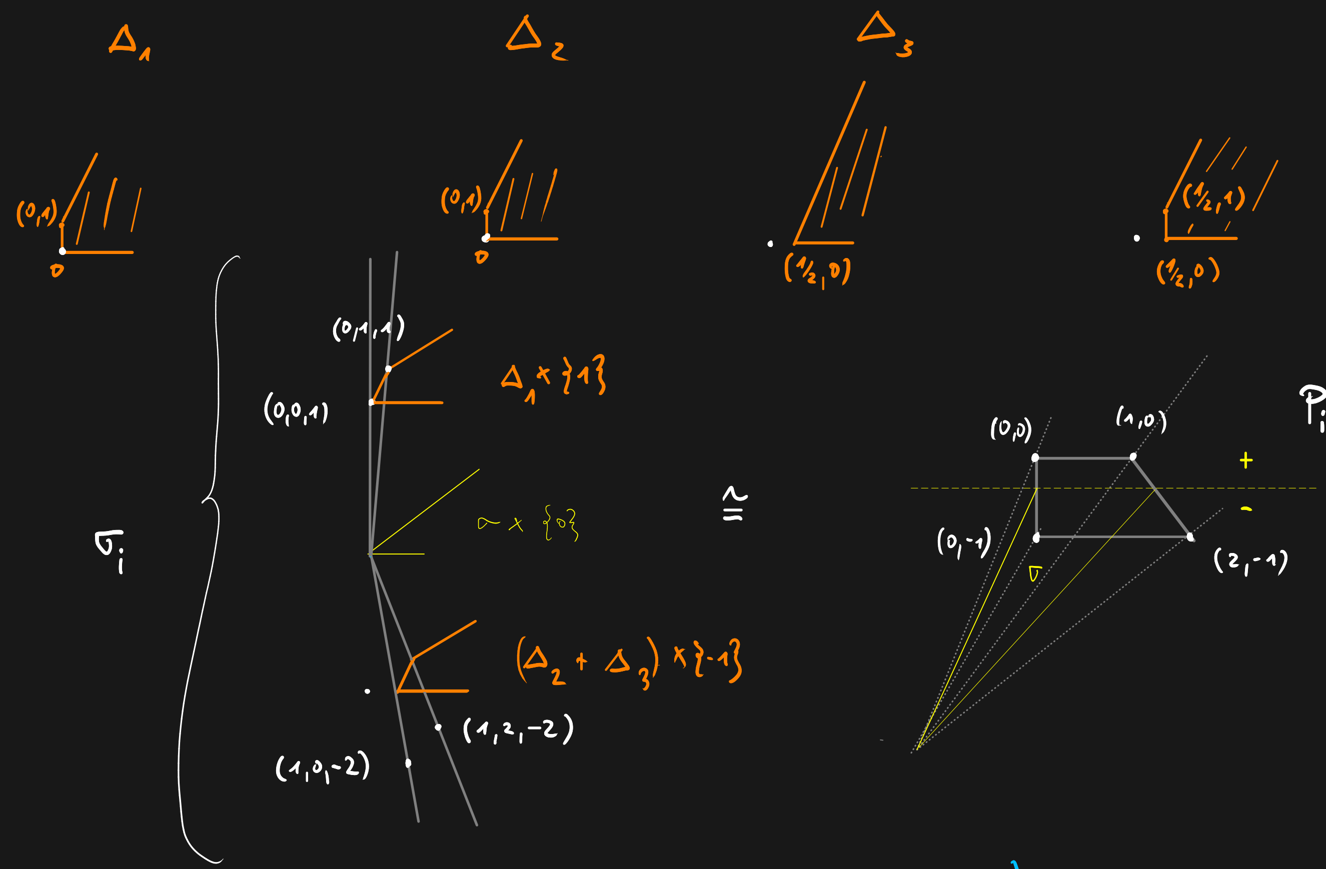






Complexity-1 case

degeneration: $p_i \in \mathbb{P}^1 \rightsquigarrow \sigma_i := \mathbb{R}_{\geq 0}\left(\Delta_i \times \{1\} \cup \sigma \times \{0\} \cup \left(\sum_{j \neq i} \Delta_j\right) \times \{-1\} \right)$



Criterion: X admits a flat Kähler cone metric (with Reeb field $\xi \in \sigma$)

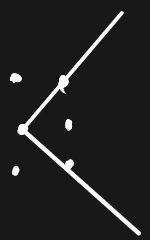
$\Leftrightarrow$

$\forall_i \quad s(P_i) \in P_i^-$ (and $\mathrm{vol}(P_i - \xi)^* = \min_{v \in P \cap \sigma} \mathrm{vol}(P_i - v)$)

In fact: • Need to check this condition only for **normal** degenerations!

- normal degenerations do not exist if 3 of the Δ_i are non-integral
- CY condition is automatically fulfilled in this case.

Example:



σ

σ translated by $(-1/2, 0)$

$$\mathcal{D} = \left((-1/2, 0) + \sigma\right) \cdot [0] + \left((1/3, 0) + \sigma\right) \cdot [1] + \left((1/5, 0) + \sigma\right) \cdot [\infty]$$

$$X(\mathcal{D}) = Y^{2,0} / I$$

I = icosahedral group.

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