Generalized Symmetries in String Compactifications

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Geometric Engineering of QFTs

Associates to a D-dimensional – special holonomy, non-compact, usually singular – space **X** a theory in $10 - D$ or $11 - D$ dimensions $\mathcal{T}(\mathbf{X})$.

Geometric engineering is the dictionary between geometric (topological) data of **X** and the physics of the QFT $\mathcal{T}(\mathbf{X})$.

Most of this week can be thought of us studying the particle like spectrum (BPS states) of such a theory.

In this talk: consider extended operators, i.e. defects.

5d SCFTs and Canonical Singularities

The connection to CY3:

M-theory on canonical Calabi-Yau 3-fold singularities realizes 5d SCFTs.

- We decouple gravity, so the CY3 is non-compact (relative).
- Canonical singularities $\mathbf{X}$ admit a resolution $\widetilde{\mathbf{X}}$ such that $K_{\widetilde{\mathbf{X}}} = \pi^* K_{\mathbf{X}} + \sum a_i \mathcal{S}_i$, $a_i \geq 0$.
- $a_i = 0$ results in a smooth Calabi-Yau resolution.
  Some non-zero a_i: remnant terminal singularities.

M-theory provides a dictionary between the geometry of these canonical singularities and the physics of 5d SCFTs.

1. Generalized Symmetries

Modern perspective on global symmetries in QFT:
topological operators generate symmetries.

- Codim 1 topological operators generate a 0-form symmetry (Noether).

  E.g.: for a continuous symmetry with a conserved current $d * j(\mathbf{x}) = 0$ and a codim 1 surface S^{d-1}, then the Noether charge is

$$\Omega_\lambda[S^{d-1}] = e^{i\lambda \int_{S^{d-1}} *j(\mathbf{x})}$$

- Topological operator, measuring the charge of a local – 0-dimensional – operator inside S^{d-1}
- Forms a group, which is the flavor symmetry group (0-form symmetry)

Generalized, Categorical Symmetries

1. Higher-form symmetries $\Gamma^{(p)}$:
   charged objects are p-dimensional defects, charge measured by topological operators $D^g_{d-(p+1)}$.
2. Higher-group symmetries:
   p-form symmetries might not form product groups
3. Non-invertible symmetries:
   relax group law $\Rightarrow$ fusion algebra
4. Higher-categorical symmetries:
   topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion

1.1 Physics: Higher-Form Symmetries $\Gamma^{(p)}$

$\Gamma^{(p)}$ is the p-form symmetry group. Its Pontryagin dual group

$$\widehat{\Gamma^{(p)}} = \mathrm{Hom}(\Gamma^{(p)}, U(1))$$

is defined as

$$\widehat{\Gamma^{(p)}} := \mathcal{L}_p / \sim$$

$\mathcal{L}_p$ are the defects of dim p, with equivalence relation

$$L_p^{(1)} \sim L_p^{(2)} \quad \Leftrightarrow \quad \exists L_{p-1} \text{ at the junction between } L_p^{(1)} \text{ and } L_p^{(2)}$$

Line Operators and 1-form symmetry

Example:

$p = 1$: line operators, with junctions formed by local operators.

$L_1^{(1)} \sim L_1^{(2)} \quad \Longleftrightarrow \quad$ there exists $L_1^{(1)}$  $L_1^{(2)}$

$O_0^{(12)} \neq 0$

E.g. in a pure G (simply-connected) Yang Mills theory, we have fundamental Wilson lines. The only local operators are in the adjoint, so

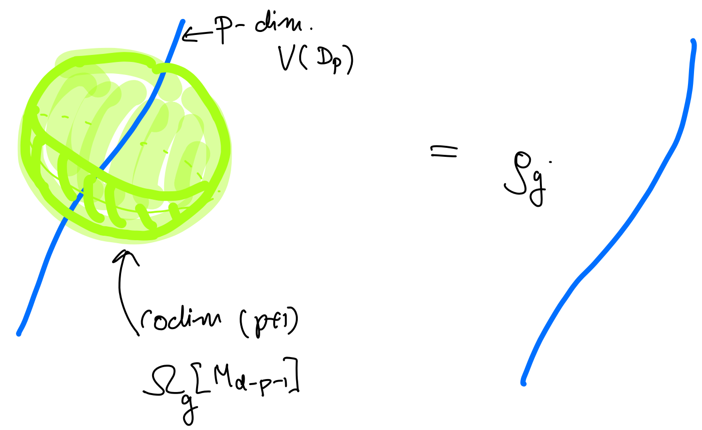
$$\Gamma^{(1)} = Z_G = \text{Center}(G)$$

Def: Higher-Form Symmetries $\Gamma^{(p)}$

A $\Gamma^{(p)}$-form symmetry is generated by topological operators of codimension $p+1$, $D^g_{d-(p+1)}$, $g \in \Gamma^{(p)}$ satisfying an (abelian) group law

$$D^g_{d-(p+1)} \otimes D^h_{d-(p+1)} = D^{gh}_{d-(p+1)}\,, \qquad g, h \in \Gamma^{(p)}$$

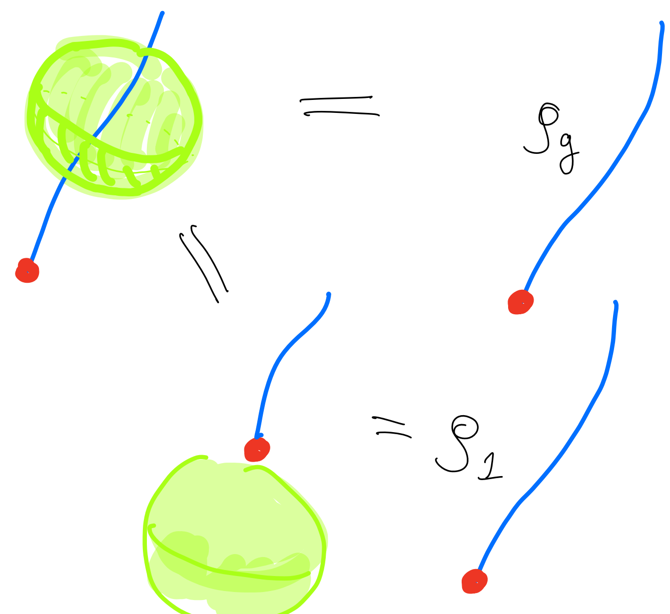
p-dim extended operators are charged under $\Gamma^{(p)}$



- Background field: $B_{p+1} \in H^{p+1}(M, \Gamma^{(p)})$ (flat)
- Gauging: summing over all such B.

Screening of Higher-Form Symmetries

Adding matter: lines can end on local operators and (part of the) 1-form symmetry gets screened

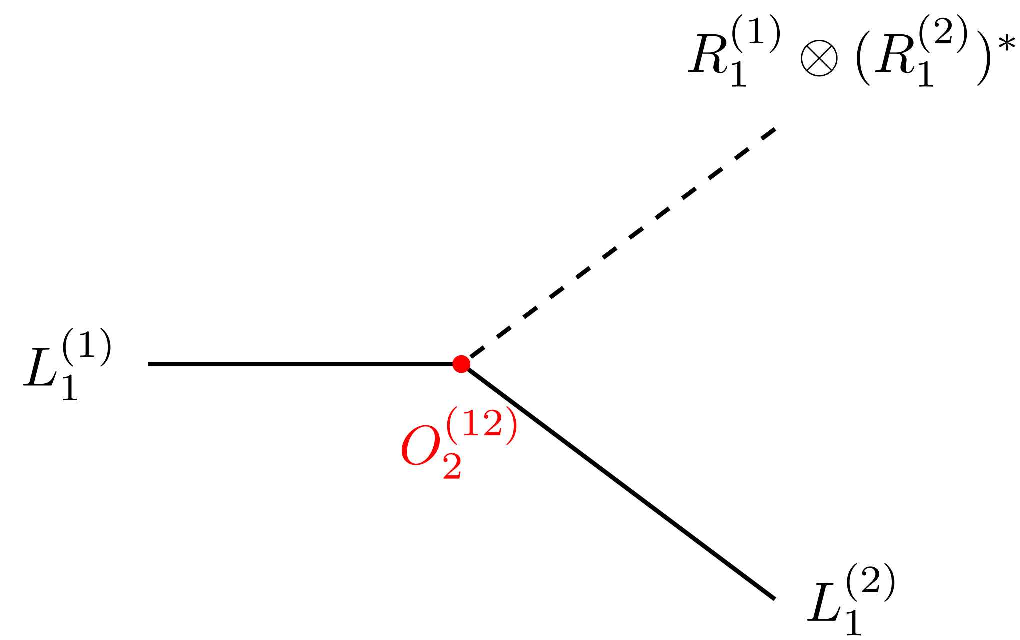


1.2 Higher-Group Symmetries

2-group symmetries:
in this screening picture, the local operators could be charged under a 0-form (flavor) symmetry $\mathcal{F}^{(0)} = F/C$ (F simply connected)

Refine the equivalence relation on the set of genuine line operators and flavor Wilson lines (L, R):

$$(L_1^{(1)}, R_1^{(1)}) \sim (L_1^{(2)}, R_1^{(2)}) \quad \Longleftrightarrow \quad \exists$$



Define: $\qquad \widehat{\mathcal{E}} = \{(L, R)\}/\sim$

$$\widehat{\mathcal{E}} = \{(L, R)\}/\sim$$

- $\widehat{\mathcal{E}} \twoheadrightarrow \widehat{\Gamma^{(1)}}$ by forgetting the flavor Wilson lines.
- $\widehat{C} \hookrightarrow \widehat{\mathcal{E}}$ by taking (id, R).
  $\widehat{C}$ which is the "center-symmetry" of the flavor symmetry F.

$$0 \to \Gamma^{(1)} \to \mathcal{E} \to C \to 0$$

If this sequence does not split then there is a non-trivial Bockstein homomorphism

$$\text{Bock} : H^2(-, C) \to H^3(-, \Gamma^{(1)})$$

and there is a 2-group is

$$\delta B_2 = \text{Bock}(w_2)$$

with $w_2 \in H^2(B\mathcal{F}, C)$ is the obstruction to lifting $\mathcal{F}$ to F bundles.

Def: 2-Group Symmetry

$\mathcal{F}$, $\Gamma^{(1)}$ satisfy group law and have background fields $B_{p+1} \in H^{p+1}(M, \Gamma^{(p)})$.

They form a 2-group given by the data

$$\left(\Gamma^{(1)}, \mathcal{F}, \Theta\right),$$

if there is a non-trivial Postnikov class

$$\Theta \in H^3(B\mathcal{F}, \Gamma^{(1)})$$

which specifies the relationship between the background fields

$$\delta B_2 = B_1^* \Theta \,, \qquad B_1 : \; M \to B\mathcal{F}$$

The 1-form symmetry background is not closed, and depends on the 0-form symmetry background.

2-Groups from Bockstein Homomorphisms

$$\delta B_2 = B_1^* \Theta \,, \qquad B_1 : \; M \to B\mathcal{F}$$

Postnikov class can be constructed in gauge theories as follows:

$$\Theta = \text{Bock}(w_2) \,,$$

where

$$\text{Bock} : H^2(B\mathcal{F}, C) \to H^3(B\mathcal{F}, \Gamma^{(1)})$$

associated to

$$0 \to \Gamma^{(1)} \to \mathcal{E} \to C \to 0$$

where $\mathcal{F} = F/C$ and w_2 is the obstruction to lifting $\mathcal{F}$-bundles to F-bundles.

Example:
$F = SU(2)$, $\mathcal{F} = SO(3)$, $C = \mathbb{Z}_2$, $\mathcal{E} = \mathbb{Z}_4$, $\Gamma^{(1)} = \mathbb{Z}_2$.

Ubiquity of Higher-Group Symmetries

Higher-groups are at least as ubiquitous as (non-anomalous) $\Gamma^{(p)}$ with mixed anomalies. Gauging $\delta B_2 = \text{Bock}(w_2)$, with a non-anomalous 1-form symmetry yields a mixed anomaly

$$\int B_{d-2} \cup \text{Bock}(w_2)$$

between the $\Gamma^{(d-3)}$ and 0-form symmetry. Other mixed anomalies yielding 2-groups

$$\int A_1 \cup B_2 \cup C_2$$

which are dual to 2-groups after gauging C_2 or B_2.

[Sharpe][Tachikawa][Benini, Cordova, Hsin][Cordova, Dumitrescu, Intriligator]

6d SCFT (full classification) [Apruzzi, Bhardwaj, Gould, SSN] 5d SCFTs [Apruzzi, Bhardwaj, Oh, SSN][Del Zotto, Heckman, Meynet, Moscrop, Zhang],

4d class S [Bhardwaj] 3d/4d: [Hsin, Lam][Lee, Ohmori, Tachikawa][Apruzzi, Bhardwaj, Gould, SSN].

1.3 Non-Invertible and Higher-categorical Symmetries

So far we assumed that the topological operators obey a group like fusion:

$$D^{g}_{d-(p+1)} \otimes D^{h}_{d-(p+1)} = D^{gh}_{d-(p+1)}\,, \qquad g, h \in \Gamma^{(p)}$$

Well-known that e.g. in $d = 2, 3$ topological lines can obey non-trivial fusion algebra relations:

$$D_1^{\alpha} \otimes D_1^{\beta} = \bigoplus_{\gamma} N_{\gamma}^{\alpha,\beta} D_1^{\gamma}$$

with $N \in \mathbb{Z}_{>0}$.

In $d = 2, 3$ these form structures like modular tensor categories (lines can fuse and braid).

Until last year there were no examples known of higher dimensional $d \geq 4$ such non-invertible symmetries: [Heidenreich, McNamara, Montero,Reece, Rudelius, Valenzuela] [Kaidi, Ohmori, Zheng][Choi,Cordova, Hsin, Lam, Shao][Bhardwaj, Bottini, SSN, Tiwari]

Non-Invertible Higher-Categorical Symmetries

No string theoretic realization **yet**, but there are interesting higher-categorical structures, which mathematicians should really study. We will define a symmetry higher-category in terms of the topological operators.

Simple example: 4d Spin$(4N)$ Yang-Mills [Bhardwaj, Bottini, SSN, Tiwari – today]

$$\Gamma^{(1)} = \mathbb{Z}_2^S \times \mathbb{Z}_2^C \,.$$

The diagonal $\mathbb{Z}_2^V$. The topological operators generating $\Gamma^{(1)}$ are

$$\mathcal{C}_{\text{Spin}(4N)}^{\text{objects}} = \left\{ D_2^{(\text{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)} \right\}$$

There is on each topological surface defect $D_2^{(g)}$ an endomorphism

$$\mathcal{C}_{\text{Spin}(4N)}^{\text{1-endo}} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

All these satisfy the group law

$$D_i^{(g)} \otimes D_i^{(h)} = D_i^{(gh)}$$

There is an outer automorphism

$$\Gamma^{(0)} = \mathbb{Z}_2$$

which acts

$$\Gamma^{(0)}: \qquad D_i^{(S)} \leftrightarrow D_i^{(C)}$$

Gauging $\Gamma^{(0)}$ results in the $\mathrm{Pin}^+(4N)$ gauge theory.

The invariant topological operators, i.e. objects

$$\mathcal{C}^{\text{objects}}_{\mathrm{Pin}^+(4N)} = \left\{ D_2^{(\mathrm{id})}, D_2^{(SC)}, D_2^{(V)} \right\}, \qquad D_2^{(SC)} = \left(D_2^{(S)} \oplus D_2^{(C)} \right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}$$

Trivial fusion of invariant objects:

$$D_2^{(\mathrm{id})} \otimes D_2^{(V)} = D_2^{(V)}, \qquad D_2^{(V)} \otimes D_2^{(V)} = D_2^{(\mathrm{id})}$$

But the non-invertible fusion of objects:

$$D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\mathrm{id})} \oplus D_2^{(V)}$$

1-morphisms are the invariant ones but also the dual to the 0-form symmetry: 2-form symmetry, generated by a topological line

$$\Gamma^{(0)} = \mathbb{Z}_2 \text{ gauged} \quad \Rightarrow \quad \Gamma^{(2)} = \mathbb{Z}_2 : \quad \text{generated by } D_1^{(-)}$$

Similarly: 1-morphism on the invariant $D_2^{(V)}$, in addition to $D_1^{(V)}$: $D_1^{(V_-)}$.

The morphisms of the new category are:

$$\mathcal{C}^{\text{1-endo}}_{\text{Pin}^+(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$

with the non-invertible fusion

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

There are also 2-morphisms, which are point-like local operators on the topological defects.

⇒ Generically the symmetries of 4d QFTs form 2-categories, with non-invertible fusion algebras

Connections to Geometry

Physically there are many motivations for studying these generalized symmetries (strong coupling, anomalies, vacuum structure etc.). In this mixed audience, the more interesting question is:

Given a geometric engineering framework: $\mathcal{T}(\mathbf{X})$.

How do we determine the generalized global symmetries
(and anomalies) of this QFT?

To be precise: lets focus on M-theory on canonical Calabi-Yau three-fold singularities, constructing 5d $\mathcal{N} = 1$ SCFTs.

5d $\mathcal{N} = 1$ SCFTs – from Geometry

A 5d superconformal field theory is

$$\mathcal{T}^{5\mathrm{d}}(\mathbf{X}) = 11\mathrm{d}\ \text{M-theory on}\ \mathbf{X} \times \mathbb{R}^{1,4}\,,$$

where $\mathbf{X}$ = canonical three-fold singularity (isolated or not), i.e. it admits a resolutions

$$\pi : \quad \widetilde{\mathbf{X}} \longrightarrow \mathbf{X}$$

such that the canonical class is

$$K_{\widetilde{\mathbf{X}}} = \pi^* K_{\mathbf{X}} + \sum_i a_i \mathcal{S}_i\,, \qquad a_i \geq 0$$

We will identify:

Canonical singularity $\longleftrightarrow$ SCFT

Kähler cone $\longleftrightarrow$ Coulomb Branch: vev of vector-multiplet scalars

Complex deformations $\longleftrightarrow$ Higgs Branch: vev of hyper-multiplet scalars

Coulomb Branch/Kähler Cone

X admits resolutions, crepant ($a_i = 0$) or with remnant terminal singularities ($a_i > 0$),

$$\pi : \quad \widetilde{\mathbf{X}} \longrightarrow \mathbf{X}$$

- Gauge Symmetry: M-theory C_3-form field expanded along $(1,1)$-forms PD to (compact) exceptional divisors

  $$\mathcal{S}_a\,, \qquad a = 1, \cdots, r = b_4(\widetilde{\mathbf{X}}) = \text{rank of the SCFT}$$

  Gauge coupling $g_a^{-2} = \text{vol}(\mathcal{S})$

- Global (flavor) symmetry:
  non-compact divisors D_α, $\alpha = 1 \cdots, f$ = flavor rank,

  $$b_2(\widetilde{\mathbf{X}}) = r + f\,.$$

- Effective description: $U(1)^r$ theory with matter and CS couplings.
  [Intriligator, Morrison, Seiberg]

- Along subloci, non-abelian gauge symmetry:
  M2-branes on rational curves:
  1. normal bundle degree $(-2,0)$: W-bosons
  2. normal bundle degree $(-1,-1)$: matter hypermultiplets

  Non-abelian gauge theory description if there exists a ruling of $\{\mathcal{S}_a\}$

$$\mathbb{P}^1 \hookrightarrow \mathcal{S}_a \to \Sigma_a$$

- SCFT:

$$\frac{1}{g_a^2} \sim \text{Volume}(\mathcal{S}_a) \to 0$$

Flavors of 5d

Amazing progress in the past few years has been the systematic derivation of global forms of $\mathcal{F}$. A peculiar and non-trivial feature of 5d SCFTs: $\mathcal{F}$ of the SCFT strictly larger than the IR Coulomb branch flavor symmetry.

Example: Rank 1 [Seiberg] theories: $SU(2) + N_F\mathbf{F}$, $N_F = 0, \cdots, 7$

- IR flavor symmetry: $SO(2N_F)$ and $U(1)_T$

$$j_T = \frac{1}{8\pi^2} \star \mathrm{Tr} F \wedge F$$

- At $g \to \infty$:

$$\mathfrak{f} \times U(1)_T \hookrightarrow \mathfrak{f}^{\mathrm{SCFT}} = E_{N_F+1}$$

- global flavor symmetry group: F simply connected Lie group associated to $\mathfrak{f}^{\mathrm{SCFT}}$

$$\mathcal{F} = F/C$$

[Series of papers: Apruzzi, Lawrie, Lin, Yi-Nan Wang, Eckhard, SSN]

Flop-Invariants: Combined Fiber Diagrams

[Series of papers with: Apruzzi, Lawrie, Lin, Yi-Nan Wang, Eckhard, SSN]

$\mathcal{F}^{\text{SCFT}}$: Encoded in the Combined Fiber Diagram (CFD):

- Vertices are curves $C_i = D_i \cdot (\sum_\alpha \mathcal{S}_\alpha)$
- (-2) (marked) vertices: $(-2, 0)$ curves
- (-1) (unmarked) vertices: $(-1, -1)$ curves
- Intersections of curves: edges

The subgraph of marked vertices is the Dynkin diagram of $\mathcal{F}^{\text{SCFT}}$. This is a flop-invariant (i.e. same accross the (not extended) CB).

Applicable to the most general 5d SCFT:
compactify M-theory on elliptically fibered CY3 (6d SCFTs on S^1, i.e. 5d KK-theories). Mass deforming (non-flat resolutions of the elliptic model) results in all known 5d SCFTs.

Relative Homology Cycles

As we are considering non-compact CY3 singularities, it is natural to ask what is the physics of relative cycles

$$H_q(\mathbf{X}, \partial\mathbf{X})$$

These non-compact q-cycles can be wrapped by M2-branes or M5-branes and give rise to infinitely massive excitations $\Rightarrow$ Defects

Example:
$H_2(\mathbf{X}, \partial\mathbf{X})$ wrapped by M2s: world-lines of infinitely massively particles – line operators.

On the set of lines we want to define an equivalence relation: physics-terms, line operators can be screened by local operators. The Pontryagin dual of the 1-form symmetry is the lines modulo screening by local operators:

$$\widehat{\Gamma}^{(1)} = \mathcal{L}/\sim$$

$$L_1 \sim L_2 \Leftrightarrow \exists \text{ local operator } \mathcal{O}_{1,2} \text{ at junction between } L_1 \text{ and } L_2\,.$$

Relative Homology cycles to Higher-form Symmetry

Translated into the geometry:

[Morrison, SSN, Willett][Albertini, del Zotto, Garcia Etxebarria, Hosseini]

- # M2-branes on compact 2-cycles: $H_2(\mathbf{X})$
  mass $m < \infty$ particles in 5d
- # M2-brane on non-compact 2-cycle: $H_2(\mathbf{X}, \partial\mathbf{X})$
  infinite mass particle, worldline defines line operator.

Equivalence relation is then

$$\widehat{\Gamma}^{(1)} = H_2(\mathbf{X}, \partial\mathbf{X})/H_2(\mathbf{X})$$

For most purposes in this talk the $\widehat{\Gamma} = \Gamma$ (abelian) will be dropped.

For q-form symmetry: e = M2, m = M5-branes wrapped

$$\Gamma_e^{(q)} = \mathfrak{h}_{(k=3-q)} \qquad \mathfrak{h}_{(k)} = (H_k(\mathbf{X}, \partial\mathbf{X})/H_k(\mathbf{X}))$$
$$\Gamma_m^{(q)} = \mathfrak{h}_{(k=6-q)}$$

$\Gamma^{(1)}$ from Intersection Theory and Link

$$\Gamma^{(1)} = H_2(\mathbf{X}, \partial\mathbf{X})/H_2(\mathbf{X}) = \mathbb{Z}^{b_4}/\mathcal{M}_{4,2}\mathbb{Z}^{b_2}$$

$\mathcal{M}_{4,2}$ is the intersection matrix between compact curves C and compact divisors $\mathcal{S}$:

$$\mathcal{M}_{4,2} = (\mathcal{S} \cdot C)_{r\times(r+f)}$$

Les relative homo:

$$\cdots H_2(\mathbf{X}) \xrightarrow{f_2} H_2(\mathbf{X}, \partial\mathbf{X}) \xrightarrow{g_2} H_1(\partial\mathbf{X}) \xrightarrow{h_1} H_1(\mathbf{X}) \cdots$$

$$\Rightarrow \ \Gamma = \mathrm{im}(g_2) = \ker(h_1) \subseteq H_1(\partial\mathbf{X})$$

='1-cycles of boundary that become trivial in the bulk'

1-Form Symmetry: Examples

1. Link $\partial\mathbf{X}$ particularly simple for $\mathbf{X}$ toric: Sasaki-Einstein manifolds. E.g.
$$\partial\mathbf{X} = Y^{p,q}\,, \qquad H_1(\partial\mathbf{X}) = \mathbb{Z}_{\text{gcd}(p,q)}\,.$$
UV-fixed points with IR description $SU(p)_q$.
2. Toric CY3:
SNF($\{\mathbf{v}_i\}$) $=$ diag(n_1, n_2, n_3) then $\Gamma^{(1)} = \oplus_i \mathbb{Z}_{n_i}$
3. Non-Lagrangian theories, which are numerous [Eckhard, SSN, Wang]
E.g. rank 1 $\mathbb{P}^2$-Seiberg theory:
$$\mathbb{P}^2: \qquad \Gamma_e^{(1)} = \mathbb{Z}_3\,.$$
4. Isolated hypersurface singularities [Clossed, SSN, Wang] have 3-form symmetries. In IIB these correspond to 1-form symmetries in 4d $\mathcal{N}=2$ theories.

2-group Symmetries

To compute the higher-form symmetries we need to determine the 0-form symmetry group, as well as $\Gamma^{(1)}$.

There are two approaches we have considered for the 2-groups:

1. $\mathcal{E}$ from the spectrum of local operators, i.e. curves in $\mathbf{X}$, including flavor group $\mathcal{F}$

   [Apruzzi, Bhardwaj, Oh, SSN]

2. From the boundary $\partial\mathbf{X}$ of the space

   [del Zotto, Garcia Etxebarria, SSN]

Two-groups from Intersections

[Apruzzi, Bhardwaj, Oh, SSN][Apruzzi, Bhardwaj, Gould, SSN]

The following applies in all geometric engineering frameworks where the full flavor symmetry that participates in the 2-group is manifest.

Let

$$\mathcal{M} = \left(\frac{\mathcal{M}_{4,2}}{\mathcal{M}_{4,2}^F} \right) ,$$

the intersections of compact and non-compact divisors with curves.

$$\mathcal{M}_{4,2} = (\mathcal{S} \cdot C) \,, \qquad \mathcal{M}_{4,2}^F = (D \cdot C) \,.$$

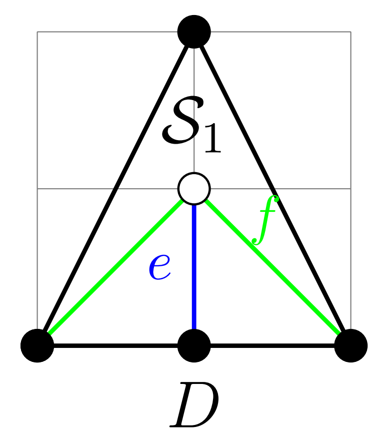
Then

$$\widehat{\mathcal{E}} = \mathbb{Z}^{r+f} / \mathcal{M} \mathbb{Z}^{r+f} \,.$$

$SU(2)_0$

$SU(2)_0$ in 5d is realized torically with one compact divisor $\mathcal{S}_1 = \mathbb{F}_2$

$SU(2)_0$:



On the Coulomb branch (of the SCFT, finite volume for f, zero volume for e) we find the charges

Curve in $\mathcal{S}_1$	$\mathcal{S}_1$	D_1	$\mathbb{Z}_2^D$
e	0	2	0 (mod 2)
f	2	-1	1 (mod 2)

The simply-connected form of the flavor group is $F = SU(2)$.

$SU(2)_0$ 5d SCFT

From the charges of the e and f curves we can compute:

$$\text{SNF}\begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

So that $\mathcal{E} = \mathbb{Z}_4$ generated by $\left(\frac{1}{4}, 1\right) \in U(1) \times \mathbb{Z}_2$. Projecting to the flavor center: $C =< 1 \in \mathbb{Z}_2 >= \mathbb{Z}_2$, with quotient $\Gamma^{(1)} = \langle (1/2, 0) \rangle = \mathbb{Z}_2$. These fit into the extension sequence:

$$0 \to \mathbb{Z}_2^{(1)} \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0\,.$$

The flavor symmetry group is from these considerations

$$\mathcal{F} = \frac{F}{C} = \frac{SU(2)}{\mathbb{Z}_2} = SO(3)\,.$$

Consistent with predictions from SCI [Kim, Kim, Lee], and anomalies [Benetti-Genolini, Tizzano].

0-Form Backgrounds

$$0 \to \mathbb{Z}_2^{(1)} \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0\,.$$

Consider a background for the 0-form symmetry: since the flavor is $SO(3)$, let

$$w_2 \in H^2(M, \mathbb{Z}_2): \qquad SO(3) \to SU(2)$$

be the obstruction to lifting $SO(3)$ bundles to $SU(2)$. In the presence of w_2 the gauge bundles are

$$\frac{G}{\mathcal{E} = \mathbb{Z}_4} \text{ bundles with obstruction to lifting to } \frac{G}{\mathbb{Z}_2}$$

Then $w_2 \in H^2(M, \mathbb{Z}_2)$ measure this obstruction.

2-group for the $SU(2)_0$

Again we are on the CB of the SCFT:

$$0 \to \mathbb{Z}_2^{(1)} \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0\,.$$

Then

$$\mathbf{w}_2 \in H^2(BSO(3), \mathbb{Z}_2)$$

is the second Stiefel-Whitney class, and

$$\mathbf{w}_3 = \delta B_2 = \mathrm{Bock}(\mathbf{w}_2)$$

is non-trivial, and the theory has a 2-group.

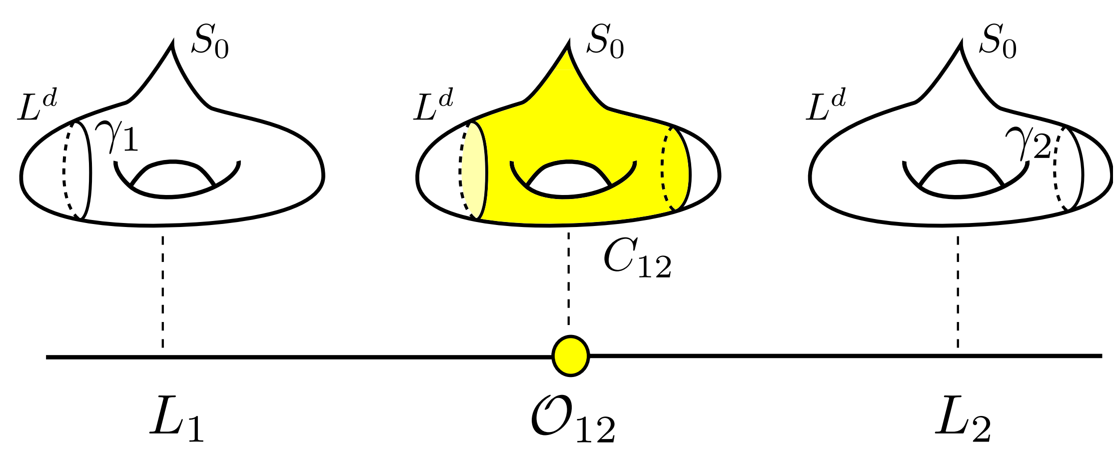
Two-groups from the Link

[del Zotto, Garcia Etxebarria, SSN]

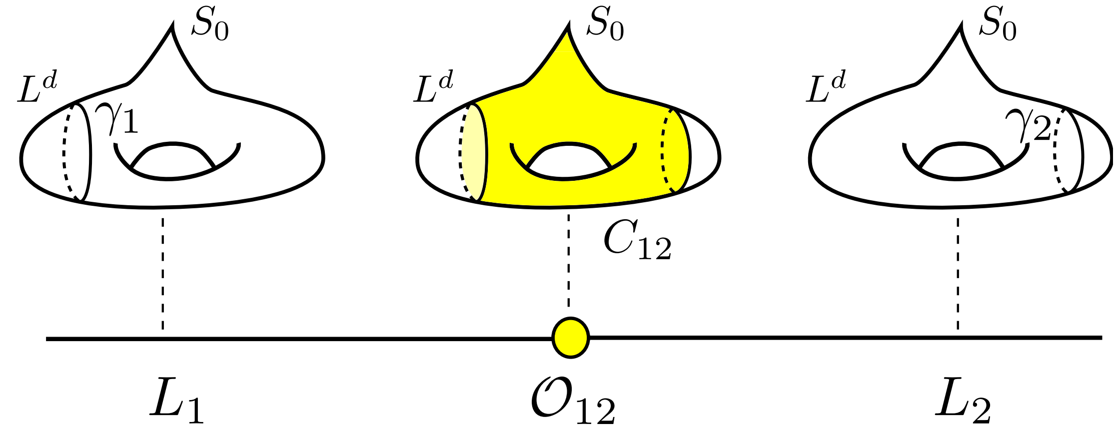
Assume also that **X** manifestly realizes the non-abelian flavor symmetry. Let $L_5 = \partial\mathbf{X}$. Let $\mathcal{S}_0 =$ singular locus modeling the flavor symmetry intersected with L_5.

On the boundary the relation on line operators $\sim$ has the following depiction:

let $\gamma_i = \partial C_i$, $C_i \in H_2(\mathbf{X}, \partial\mathbf{X})$. If there is a 2-chain C_{12} with $\partial C_{12} = \gamma_1 \cup \gamma_2$ then M2s on C_12 give rise to $\mathcal{O}_{12}$:



$\mathcal{E}$ from Link Geometry



The relation $\sim'$, which does not include line-changing operators charged under C:

Consider $L_5 - \mathcal{S}$ (neighborhood of $\mathcal{S}_0$). The chains that pass through $\mathcal{S}$ acquire an extra boundary and do not imply that $\gamma_1 = \gamma_2$.

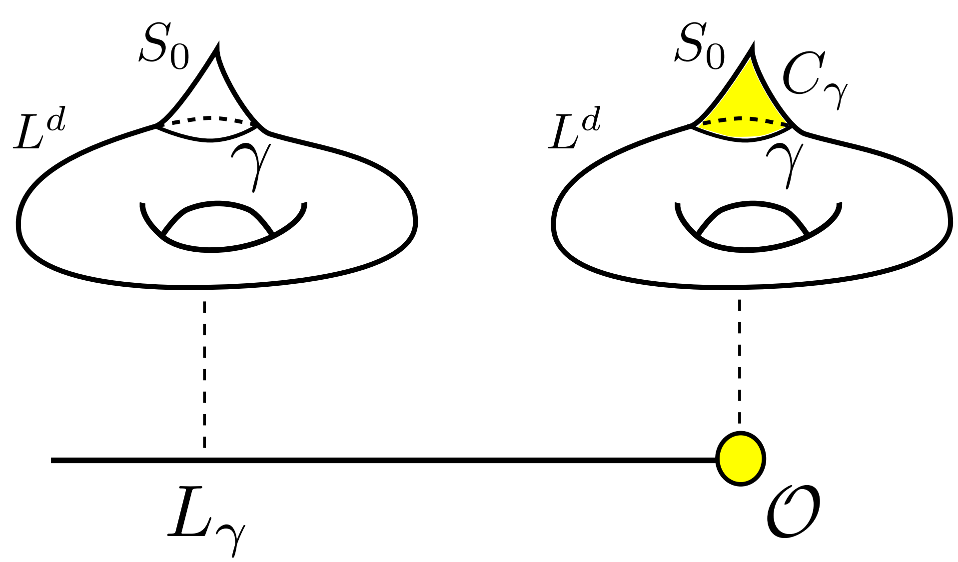
The homology relations now come from chains, that are in $L_5 - \mathcal{S}$, which are precisely those uncharged under C and

$$\widehat{\mathcal{E}} = H_1(L_5 - \mathcal{S}).$$

$\mathcal{E}$ from Link Geometry

To see that these are the line operators uncharged under C consider a line that is trivial in $\widehat{\Gamma}^{(1)}$: γ is trivial in $H_1(L_5)$.

This is charged under the flavor center symmetry C, if $\gamma \in \text{Tor}\, H_1(\partial \mathcal{S})$ is nontrivial.



Overall we find the geometrized version of the extension sequence:

$$0 \to \text{Tor}\, H_1(\partial \mathcal{S}) \to H_1(\mathbf{L}_5 - \mathcal{S}) \to H_1(\mathbf{L}_5) \to 0$$

Toric CY3

For toric models this can be readily applied to compute the 2-group symmetries.

The non-compact divisors (and the singularity $\mathcal{S}$) are encoded in vertices along the edges (minus corners).

The homology of the link can be computed by noting that it is glued together by Lens spaces [Garcia Etxebarria, Heidenreich]

$$H_n(L_5) = H_n(\mathcal{B}^3_L) \qquad \text{for } n \leq 2\,,$$

where $\mathcal{B}^3_L$ is a 3-chain of lens spaces

$$\mathcal{B}^3_L \simeq \mathbf{L}_{n_1} \veebar \mathbf{L}_{n_2} \veebar \ldots \veebar \mathbf{L}_{n_\nu}$$

where $\veebar$ denotes that the lens spaces are joined along their torsion cycle.

Toric CY3

The homology is then

$$\Gamma^{(1)} \cong H_1(\mathcal{B}^3_L) = \mathbb{Z}_{\gcd(n_1,\ldots,n_\nu)}\,,$$

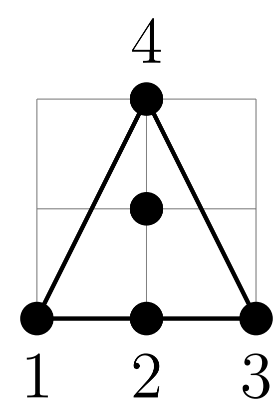
For each external vertex $\mathbf{v}_i$, $i \in \{1,\ldots,\nu\}$, construct the triangle T_i: $\{\mathbf{v}_{i-1},\mathbf{v}_i,\mathbf{v}_{i+1}\}$. Then

$$n_i = 2\text{Area}(T_i)\,.$$

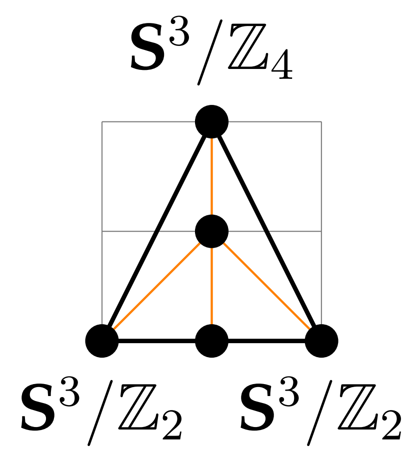
For $\widehat{\mathcal{E}}$ we excise the singularity, i.e. the vertices associated to the flavor symmetry and compute likewise

$$\widehat{\mathcal{E}} = H_1(\mathcal{B}^3_{L-S})$$

$SU(2)_0$



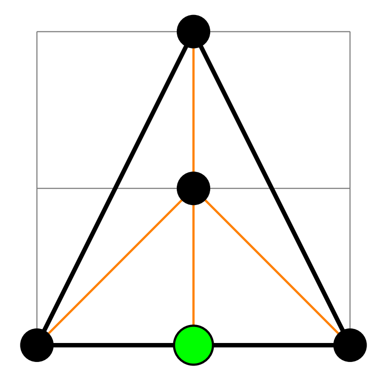
$\Gamma^{(1)}$ from triangles: $T_i = \Delta(i-1, i, i+1)$:



$\Rightarrow \qquad \Gamma^{(1)} = \mathbb{Z}_{\gcd(2,2,4)} = \mathbb{Z}_2.$

To determine $\widehat{\mathcal{E}}$, excise the vertex 2:

$\mathbf{S}^3/\mathbb{Z}_4$



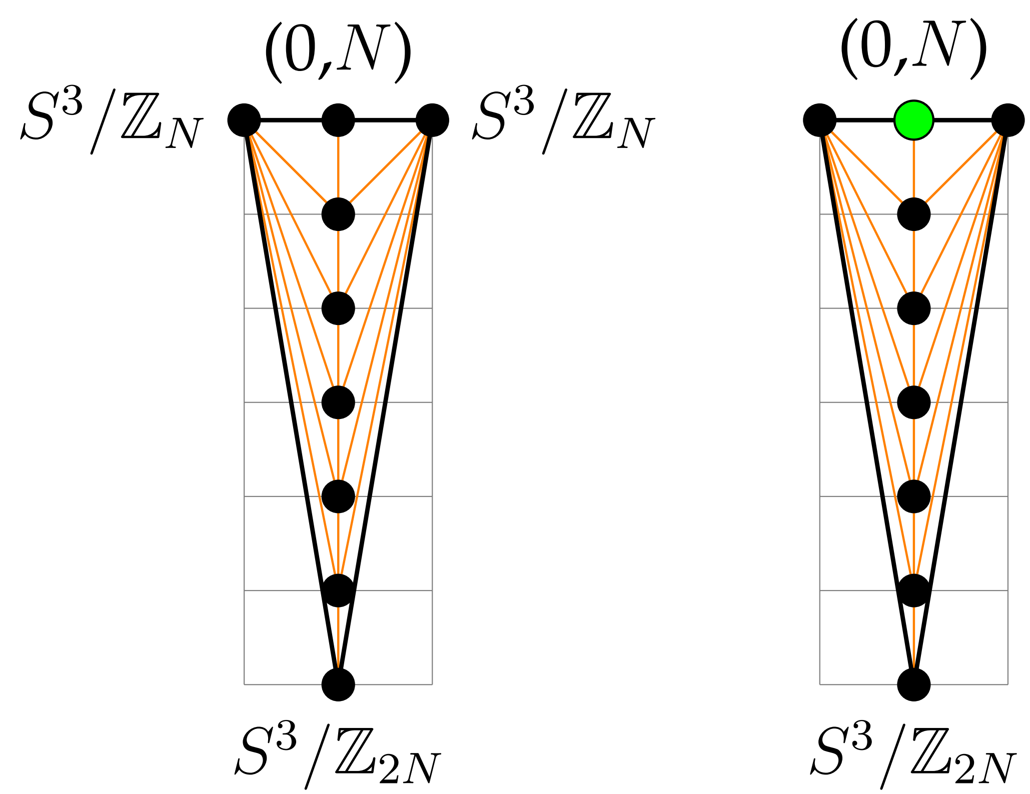
Then

$$\widehat{\mathcal{E}} = H_1(L - \mathcal{S}) = \mathbb{Z}_4$$

Therefore $C = \mathbb{Z}_2$ and we recover the extension sequence from the intersection computation.

$SU(2n)_{2n}$

$SU(2n)_2n$ has $\Gamma^{(1)} = \mathbb{Z}_{2n}$ and $\mathcal{F} = SO(3)$, and by intersection computations in [Apruzzi, Bhardwaj, Oh, SSN] a 2-group:



$$0 \to \mathbb{Z}_{2n} \to \mathbb{Z}_{4n} \to \mathbb{Z}_2 \to 0$$

does not split (for odd N it does and there is no 2-group).

Non-Lagrangian theories with 2-Groups

Generalizing the $\mathbb{P}^2$ rank 1 Seiberg theory, which has no non-abelian gauge theory description on the CB, there is a series of non-Lagrangian theories [Eckhard, SSN, Wang]

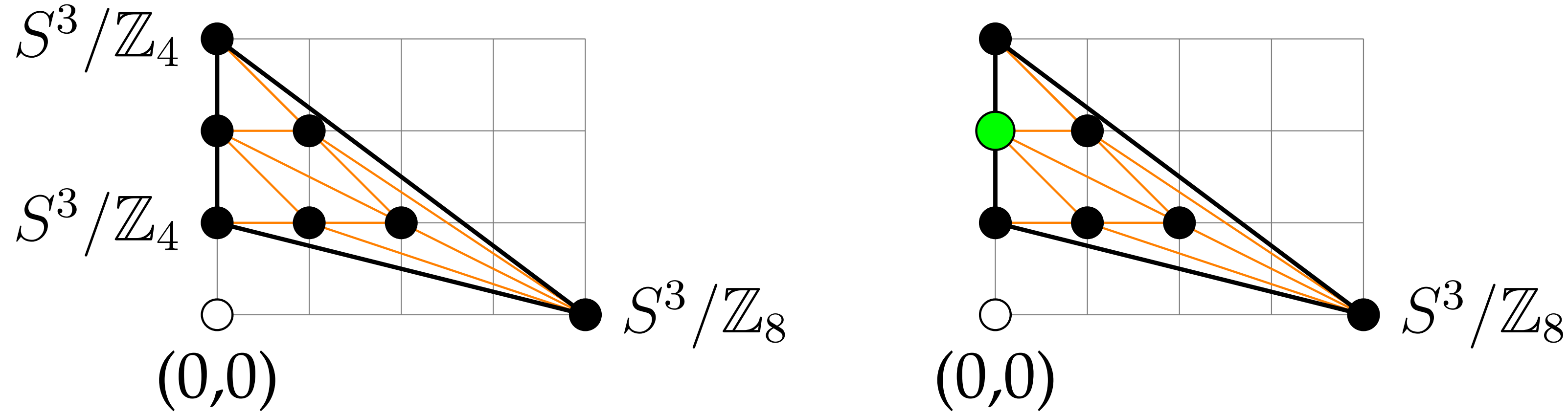
$$B_N^{(2)}: \qquad ((N,0,1),(0,N-1-k,1)),\, k=0,\cdots,N-2\,,$$

with

$$\Gamma^{(1)} = \mathbb{Z}_N$$

and flavor symmetry group

$$\mathcal{F} = SU(N-2)/\mathbb{Z}_{N-2}$$



$$B_4^{(2)}: \qquad \Gamma^{(1)} = \mathbb{Z}_{\text{gcd}(4,4,8)} = \mathbb{Z}_4$$

$$B_N^{(2)}: \qquad \Gamma^{(1)} = \mathbb{Z}_{\text{gcd}(N,N,(N-2)N)} = \mathbb{Z}_N \,.$$

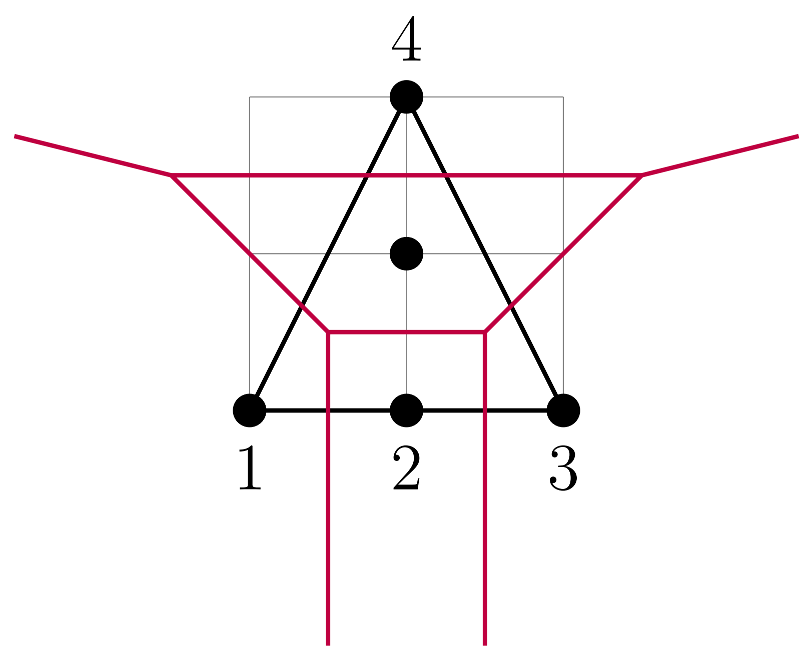
Excising the flavor vertices associated to $\mathcal{F}$ we find

$$\mathcal{E} = \mathbb{Z}_{N(N-2)} \,,$$

and thus there is a non-trivial extension only for $N = 2n$.

Generalizations to Generalized Torics

All toric CY3 are dual to so-called 5-brane-webs: (in a loose sense, this is the dual tropical geometry)



Given a toric polygon, we compute the charges of (p, q) 5-branes:

$$\mathbf{v}_i - \mathbf{v}_{i+1} = (a, b, 0) \qquad \Leftrightarrow \qquad (p, q) = (b, -a)$$

D5 horizontal: $(1, 0)$), NS5 vertical: $(0, 1)$.

This map from toric geometry to $W = \{(p_i, q_i)\}$ derives via string duality. As in the geometry, parallel 5-branes indicate flavor symmetries, or asymptotic $\mathbb{C}^2/\mathbb{Z}_n$-singularities.

The 1-form symmetry is simply computed from [Bhardwaj, SSN]

$$\Gamma^{(1)} = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}\,, \qquad \text{diag}(n_1, n_2) = \text{SNF}(W)\,.$$

Denoting the multiplicity of a (p, q) brane charges in a given web by $m_{(p,q)}$ the geometric derivation of $\mathcal{E}$ can be rephrased as

$$\mathcal{E} = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}\,, \qquad \text{diag}(n_1, n_2) = \text{SNF}(W^{\text{red}}_{(p,q)})\,.$$

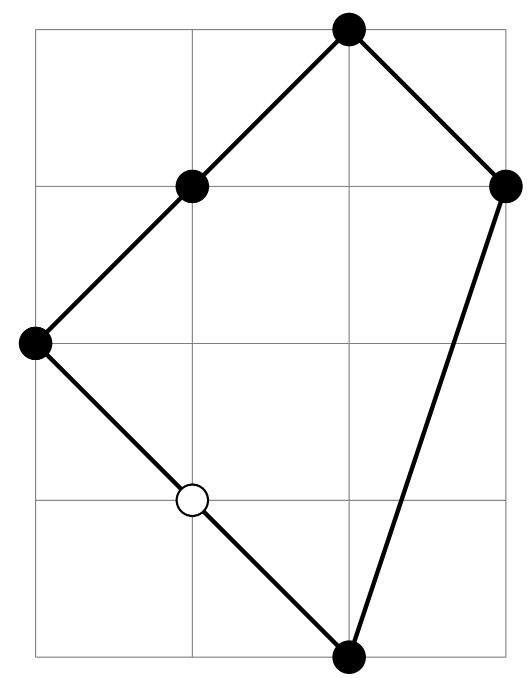
where

$$W^{\text{red}}_{(p,q)} = \text{Matrix obtained by removing } (p,q)^{m_{(p,q)}} \text{ from } W\,.$$

So far this is an equivalent analysis to the toric one. However brane-webs go beyond toric geometry.

Generalized Toric Polygons

Parallel 5-branes can end on a single 7-brane – in the geometry this should correpond to a kind of deformation. Such brane-webs are formally dual to generalized toric polygons [Benini, Benvenuti, Tachikawa][Cabrera, Hanany, Yagi][van Beest, Bourget, Eckhard, SSN][2]: introduce ”white” (empty) edge vertices, which are dual to 5-branes ending on the same 7-brane. E.g.



This describes $SU(4)_2 + 1\mathbf{AS}$, which has $\Gamma^{(1)} = \mathbb{Z}_2$.

Applying the web-based analysis we find

$$W^{\text{GTP}} = \begin{pmatrix} 2 & 2 \\ -3 & 1 \\ -1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -3 & 1 \\ -1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$$

and

$$\text{SNF}(W^{\text{red}}_{(1,1)}) = \text{diag}(1,4)$$

consistent with this theory $SU(4)_2 + 1\mathbf{AS}$ having a 2-group symmetry (shown in the geometric intersection analysis in [ABOS]).

Clearly: it would be very interesting to understand the dual generalized toric geometry.

Comments

Generalizations to any (singular) geometries in M-theory are straight-forward (in fact our analysis is agnostic to the dimension of the compactification space). The only limitation at this point in the link approach is that the flavor symmetry needs to be manifest in the geometric singularity. This is not the case in the intersection theory though.

E.g. for G_2 and F-theory elliptic fibrations, see recent works by [Tian, Wang][del Zotto, Heckman, Meynet, Moscrop, Zhang] for orbifolds, and for F-theory elliptic Calabi-Yaus [Morrison, Hubner, SSN, Wang][Cvetic, Heckman, Hubner, Torres].

View from the Edge

One message to take home is: much of the symmetries (and other properties like anomalies) are encoded in the boundary of the compactification space. This philosophy ties in with the Symmetry TFT [Freed] and in string theory [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, SSN], which encodes essentially the anomaly* theory of the QFT in terms of background fields. In string theory this can be derived from first principles in the supergravity reduced on the link L_5. E.g. we found 't Hooft anomalies for the 5d SCFTs using this approach.

<u>5d:</u> reducing the CS- and $C_3 X^8$ coupling in M-theory on the link:
E.g. for $SU(p)_q$ where the 5d link geometry is $Y^{p,q}$:

$$\mathcal{A}_{6d}^{(B^3)} = \frac{qp(p-1)(p-2)}{6\,\mathrm{gcd}(p,q)} B_2^3$$

$$\mathcal{A}_{6d}^{(FB^2)} = \frac{p(p-1)}{2\,\mathrm{gcd}(p,q)^2} F_I B_2^2$$

View from the Edge

Note, the SymTFT should also encode the 2-group more explicitly in terms of its dual mixed anomaly, obtained after gauging the 1-form symmetry (if this has no B^3 anomaly)

$$\mathcal{A}_{6d} = B_3 \text{Bock}(w_2)$$

where B_3 is the background for the dual 2-form symmetry.

Questions for Math

1. Develop the theory of higher-categorical structures that are relevant for QFTs
2. Interconnection between the symmetry higher-category and geometry (mostly topological data, but these defect operators, like Wilson lines can also be BPS)
3. Classification problem of 5d SCFTs: explicit form of Mori program for canonical non-compact CY3

Outlook: Physics

1. Geometric realization of non-invertible symmetries: probably using mixed anomalies; see [Kaidi, Ohmori, Zhen] and 4d, 5d and 6d implementations in field theory in [Bhardwaj, Bottini, SSN, Tiwari].
2. In 4d 1-form symmetries have a clear physical imprint, as providing a diagnostic for confinement.

   What is the role of higher-form symmetries in other dimensions, for SQFTs, SCFTs?
3. Generally: what physical implications do 2-group symmetries have in QFTs?
4. Determine the anomaly theories from a $d+1$ bulk for discrete generalized symmetries from first principles in string/M-theory. See Requires differential cohomology description of supergravity reductions:

   Symmetry TFT [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, SSN]