2201.12223
1811.02875 + Banerjee & Romo
+
2101.01681

<u>GOAL:</u>

- Study BPS states of 5d $\mathcal{N}=1$ QFT
  - related to 4d $\mathcal{N}=2$ — much progress to borrow / extend (GMN in particular)
  - closer to enumerative geometry of CY_3

<u>OUTCOMES</u>

- Framework for computations: Exponential Networks
- Exact results for some toric CY, both old + new
- Definition of counts of sLag A-branes
  in some cases, with CLEAR + SIMPLE math interpretation

General Setup

$5d\ \mathcal{N}=1$

M-theory : $X \times S^1 \times \mathbb{R}^4$ $\qquad$ $T_{5d}[X] : S^1 \times \mathbb{R}^4$

BPS states

M5 $\quad C_4 \times S^1 \times \mathbb{R}$ $\qquad$ monopole string $\quad S^1 \times \mathbb{R}$

M2 $\quad C_2 \times pt \times \mathbb{R}$ $\qquad$ instanton particle $\quad \mathbb{R}$

counting:

S^1-reduction $\longrightarrow$ D4, D2, D0 $\qquad$ "BPS index"

as obj in $D^b Coh(X)$ $\quad$ rank-0 DT $\longleftarrow$ will compute

—— o ——

For this purpose, we introduce

$3d\ \mathcal{N}=2$

M5 : $L_0 \times S^1 \times \mathbb{R}^2$ $\qquad$ $T_{3d}[L_0] : S^1 \times \mathbb{R}^2$

coupled to T_{5d}

$\rightarrow$ "<u>3d-5d system</u>"

If L_0 noncompact, $b_1 = 1$

its moduli space is ONE - DIM

$$x = \exp(r + i\theta)$$ LOCAL COORD.

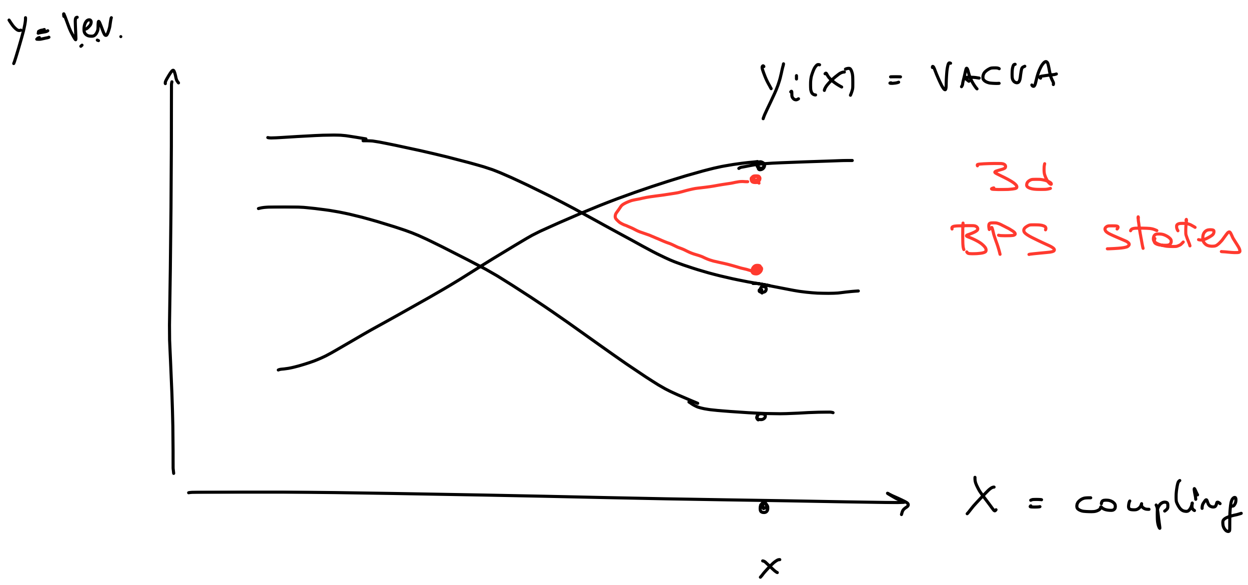
θ: U(1) holonomy

GLOBALLY the moduli space receives corrections from holom. string instantons / M2

$\rightsquigarrow$ g_s small : genus 0

$$\Sigma : \quad F(x, y) = 0 \quad \subset (\mathbb{C}^*)^2$$

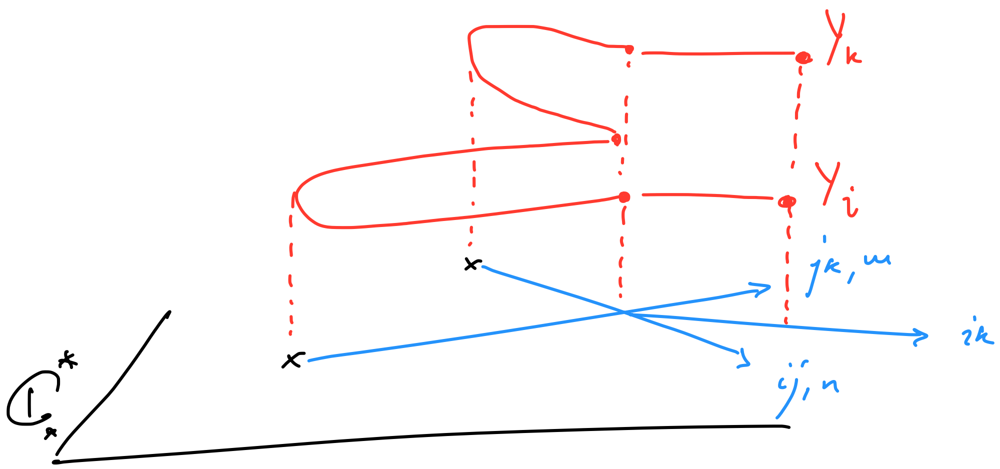
We view Σ from P.O.V. of $T_{3d}[L_0]$



- sheets = vacua
- open paths = BPS states

BPS equations for topological sector $y_i \to y_j$

$$\left(\log y_j - \log y_i + 2\pi i n\right) \cdot \frac{d \log x}{dt} \in e^{i\theta} \mathbb{R}_+$$

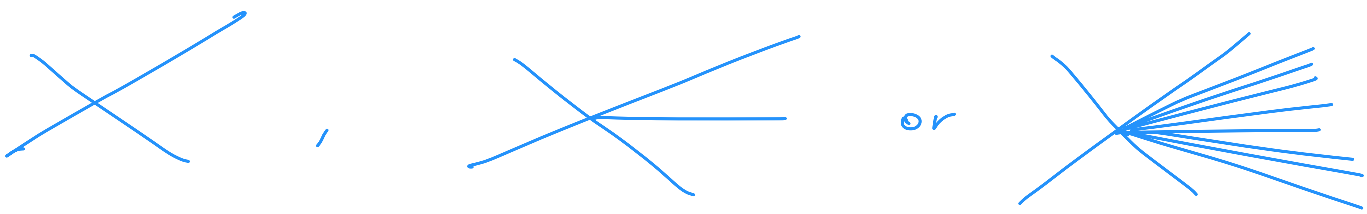


At given value of cplg x, BPS states of "phase ϑ" are 1:1 sol's to BPS eqs.

Exponential network: $W(\vartheta)$

web of trajectories $\rightarrow$ sol$^{\text{n}}$ to 3d BPS eqs.

- Start at each branch pt $y_i = y_j$
- New trajectories from intersections



Junctions : wall-crossing of 3d BPS states

But in fact there is another type of wall-crossing involving 3d AND 5d states

5d BPS states from 3d-5d wall-crossing :

A "simple" description is geometric :

Using $W(\theta)$, define nonabelianization map [GMN]

ABELIAN CONN. ON Σ $\longrightarrow$ NON-ABEL. conn on $\mathbb{C}^*_\times$

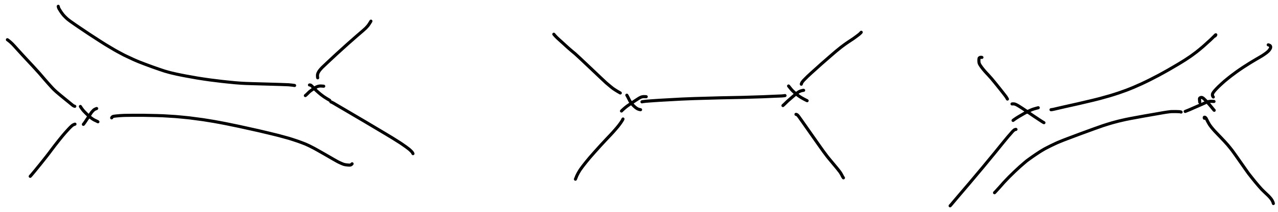
$$\mathrm{Hol}(\wp, \nabla^{n.a.}) = \sum_{\gamma} \underline{\overline{\Omega}}(\wp, \gamma, \theta)\ \mathrm{Hol}(\gamma, \nabla^{ab})$$

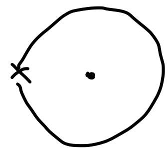
$\wp$ path on $\mathbb{C}^*_\times$ γ path on Σ

$\underline{\overline{\Omega}} \in \mathbb{Z}$ topological : locally constant in θ.

computed from $W(\theta)$ by counting "detours"

For certain $\theta = \theta_{cr}$, $\mathcal{W}(\theta)$ changes topology





etc. many more examples

Then

$$\underline{\overline{\Omega}}(\theta_{cr} + \epsilon) \rightarrow \underline{\overline{\Omega}}(\theta_{cr} - \epsilon)$$

must jump.

One can show that the universal form of these jumps is of Kontsevich-Soibelman type

$$\mathrm{Hol}(\gamma, \nabla^{ab}) = X_\gamma \longmapsto X_\gamma (1 \pm X_{\gamma'})^{\Omega(\gamma')\langle\gamma,\gamma'\rangle}$$

γ' Determined by the saddle.

$$\boxed{\text{Read off } \Omega(\gamma') \text{ from jumps}}$$

OVERALL - SIMILAR TO SPEC. NET OF GMN, BUT TECHNICAL DIFF. WHICH I SUPPRESSED MOSTLY

- There is a q-deformed version

$$\longrightarrow \quad \Omega(\gamma, q) \in \mathbb{Z}[q, q^{-1}]$$

protected spin character

worked out in 4d limit
but expected to generalize

- Jumps akin to Stokes automorphisms from exact WKB for ODEs, here Exp Net $\leftrightarrow$ qDE.

related work recently by
- Kashaev Garoufalidis,
- Hollands Alim Tulli
- Gaiotto Moore Neitzke

- BPS Graphs : a special class of networks

$R \subset B_\Sigma$ cplx moduli such that

$\vartheta_{cr} \in \{0, \pi\}$ only.

Then a single jump of $W(\vartheta)$

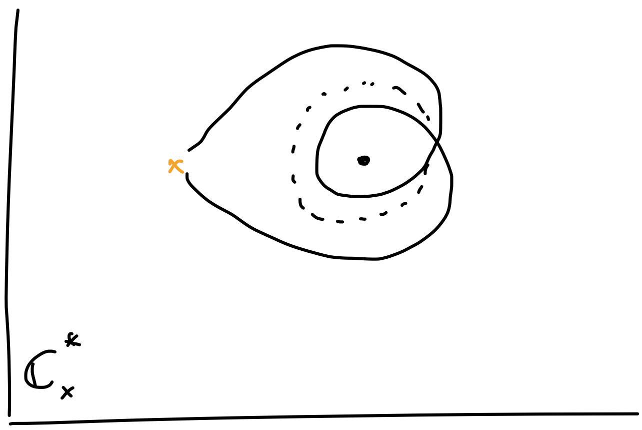
$$\overline{\Omega}(\vartheta = \epsilon) - \overline{\Omega}(\vartheta = -\epsilon) \longleftrightarrow$$

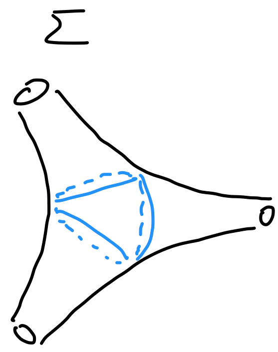
Wall-Crossing invariant of KS
("spectrum generator")

2. Some Results

Ex 1 $\quad X = \mathbb{C}^3 \qquad \Sigma : \; 1 + y + x y^2 = 0$

$\vartheta_{cr} = 0$ only





$$\Omega(k\gamma) = -1 \qquad k \in \mathbb{Z}^*$$

$$Z_{k\gamma} = k \cdot \frac{2\pi}{R} \longrightarrow k \text{ D0-branes}$$

indeed $\Omega \sim \chi(\mathbb{C}^3)$

Ex 2

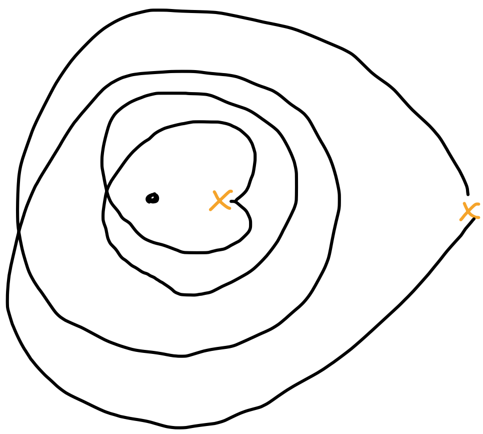
$X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$1 + y + xy + Qxy^2 = 0$

take $Q > 1$ say

then, many ϑ_{cr}

$\vartheta_{cr} = 0$



(2 copies of $\mathbb{C}^3$)

$\Omega(k\gamma) = -2$

$Z_{k\gamma} = k \frac{2\pi}{R}$

$\rightarrow \boxed{\text{D0's}}$

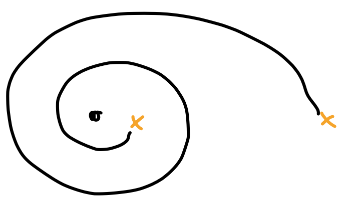
$\vartheta_{cr} = \pi/2$

$\Omega(k\gamma) = \delta_{k,1}$

$Z_\gamma = \frac{i}{R} \log Q$

$\boxed{\rightarrow \text{D2}}$

$\vartheta_{cr} = \arg\left(\frac{2\pi}{R} + \frac{i}{R} \log Q\right)$



$\Omega(\gamma) = 1$

D2-D0

etc. $\boxed{\Omega(\text{D2}-k\text{D0}) = 1}$

Ex 3

$$X = K_{\mathbb{F}_0} \qquad Q_b\,(y+y^{-1}) + Q_f\,(x+x^{-1}) - 1 = 0$$

- in 2012.09769 studied BPS states @ $Q_b = Q_f = 1$

  $\rightarrow$ checked via wall-crossing against results from Coulomb branch / Attractor formulae by [Beaujard - Manschot - Pioline]

- in 2101.01681 exact spectrum by combining networks + quiver rep. theory

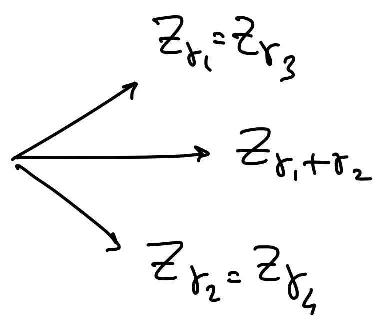
  $\rightarrow$ [Closset Del Zotto] + [Beaujard Mans. Piol.]

$\gamma_1 : D4$

$\gamma_2 : D2_f\ \overline{D4}$

$\gamma_3 : D0\ D2_b\ \overline{D2_f}\ \overline{D4}$

$\gamma_4 : \overline{D2_b}\ D4$



$$\Omega(\pm\gamma_1 + k(\gamma_1+\gamma_2)) = 1$$

$$\Omega(\pm\gamma_3 + k(\gamma_3+\gamma_4)) = 1$$

$$\Omega(\pm(\gamma_1+\gamma_2) + k\,\gamma_{D0}) = -2$$

$$\Omega(k\,\gamma_{D0}) = -4$$

checked against formula for WCI by [Mozgovoy Pioline]

Counting special Lagrangian A-branes

So far: • 5d BPS states counted by networks

• framework based on QFT of 3d-5d

but related to

• rank-0 DT on CY_3

counts of B-branes

• mirror symmetry $\rightarrow$ A-branes on $X^\vee$

• In fact, networks analyze geometry of the mirror CY.

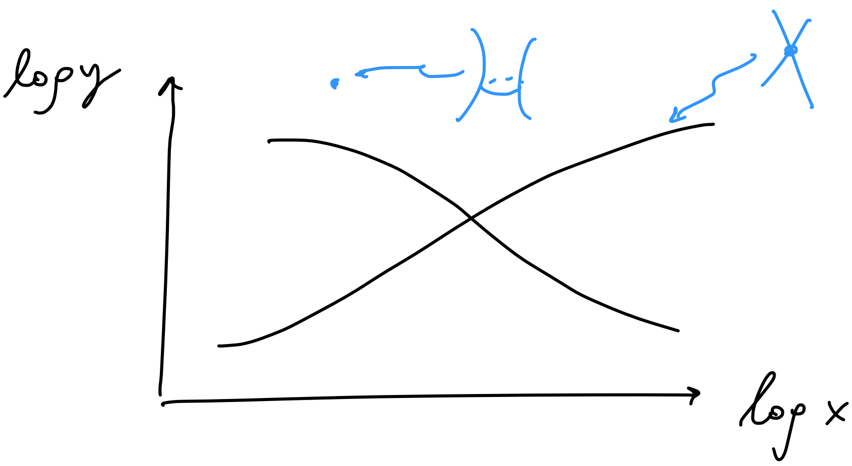
• Thus they define counts of A-branes from purely geometric data

• Goal now: explain what is $\Omega(\gamma)$ in terms of geometry of slags

Let $X^\vee$ be a conic bundle over $(\mathbb{C}^*)^2$

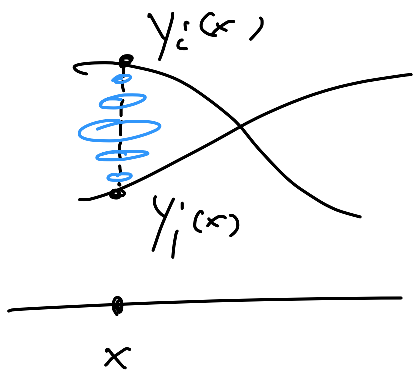
$$uv = F(x,y) \quad \subset \quad \mathbb{C}^2 \times (\mathbb{C}^*)^2$$

$$\Omega^{3,0} = i \, \frac{du}{u} \wedge \frac{dx}{x} \wedge \frac{dy}{y}$$



$\boxed{\text{Conic degenerates over } \Sigma}$

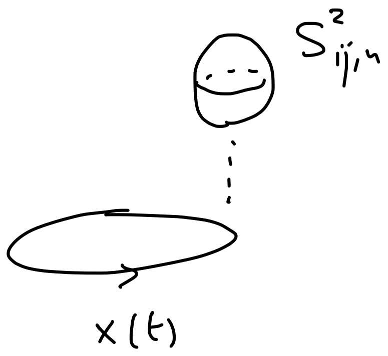
Consider L fibered by $S^2_{ij,n}$



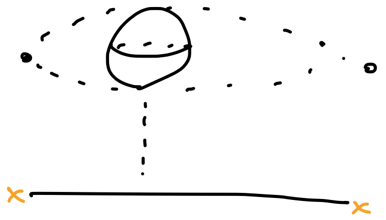
n = shift in log-branch for log y
= winding in y-plane

log y branch related to graded lift of slag

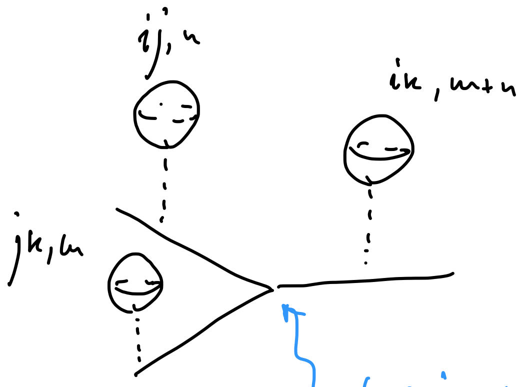
3-mfld L from paths $x(t)$



$\longrightarrow \quad L \simeq S^2 \times S^1$



$\longrightarrow \quad L \simeq S^3$



glue in with extra piece (S^3 w/ 3 holes)

Calibration

$$\Omega^{3,0}\big|_L \sim e^{i\vartheta}\ \mathrm{vol}_L$$

$$S^2_{ijn}(x) \longrightarrow L \longrightarrow x(t)$$

$\downarrow$

calibrated

by $\Omega^{2,0} = \frac{du}{u} \wedge \frac{dy}{y}$

But $S^2_{ij,n}(x)$ is rigid ($b_1 = 0$)

exact shape :

$$\begin{cases} u = u_0(s) \cdot e^{i\alpha} \\ \log y = (1-s)(\log y_i + 2\pi i M) + s(\log y_j + 2\pi i N) \end{cases}$$

$[N - M = n]$

So all calibration is enforced on $x(t)$

$$\Omega^{3,0}(\partial_t, \partial_s, \partial_\alpha) = \dots$$ quick computation

$$= \frac{d\log x}{dt}\left(\log y_j - \log y_i + 2\pi i n\right)$$

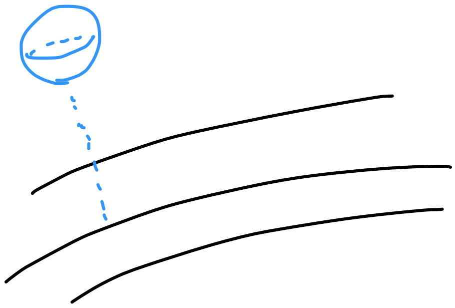
$$\in e^{i\vartheta}\,\mathbb{R}_+$$

like exp. networks – BUT will use diff'tly

"Foliations" $\phi_{ij,n}$ (only locally, really) due to self-inters.

OR, AWAY FROM CUTS

leaves = integral curves of the eqs.



$\longrightarrow$ $\boxed{L \ \text{slag}}$

• rigidity of S^2 $\Longrightarrow$ $\boxed{\mathfrak{M}_L \simeq \mathfrak{M}_\phi}$ "Leaf space"

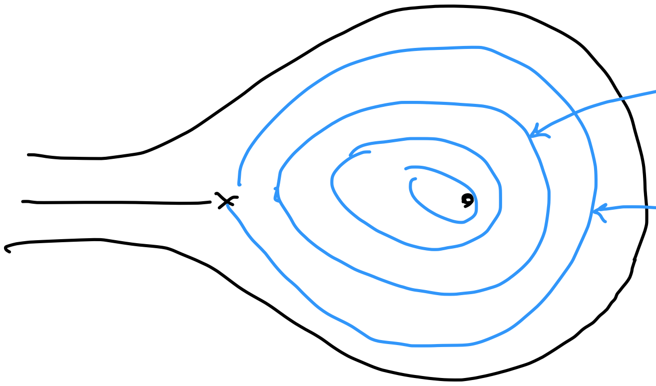
slag moduli space

• A-brane: $U(1)$ flat conn. on L

$$\boxed{T^{b_1(L)} \longrightarrow \mathcal{M}_L \longrightarrow \mathfrak{M}_L}$$

moduli of A-brane

Ex :



$$L \simeq S^2 \times S^1$$

$$L_* \simeq S^3/\sim$$

Count A-branes

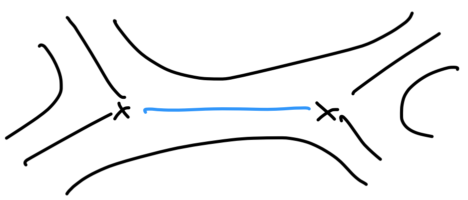
- Lift to D3 on $L \times \mathbb{R}$
- $\mathcal{N}=4$ SUSY QM : maps $\mathbb{R} \to \mathcal{M}_L$
- BPS states $\simeq$ $H^*_{dR,cpt}(\mathcal{M}_L)$
- Witten index : our definition for COUNTING

(F = cohom degree + shift)

$$\boxed{\Omega(L) :} = \boxed{\mathrm{Tr}_{\mathcal{H}}(-1)^F = (-1)^{\dim \mathcal{M}} \chi(\mathcal{M})}$$

IMPORTANT : L primitive

Ex

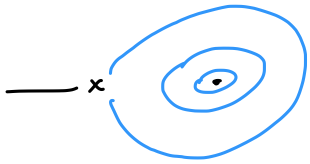


$$L \simeq S^3$$

$$\mathcal{M} = pt$$

$$\Omega = 1$$

ex : D2 in res. conifold



$\mathcal{M} = \mathbb{R}_+$

$\mathcal{M} = \mathbb{C}$

$\Omega = -1$

D2 in $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$

Claim:

$$\boxed{\Omega(L) = \Omega(\gamma)}$$

(L primitive)

As defined by $\chi(\mathcal{M})$

As defined by networks

why:

- $\mathcal{M}$ admits $G = T^{b_1(L)}$ action
- $\chi(\mathcal{M}) = \int e_G(T\mathcal{M}) = \sum_F 1 = \boxed{\#\text{ Fixed pts}}$

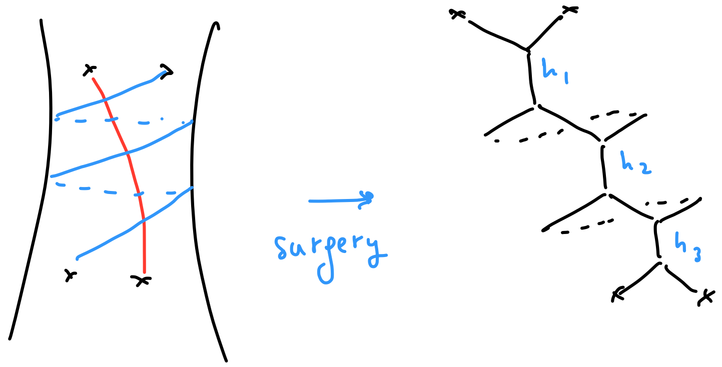
  by equivariant localization
- $\boxed{\text{Fixed points on } \mathcal{M}} \longleftrightarrow \boxed{\text{critical leaves}} \equiv \boxed{\text{networks}}$

- Then we show that the $\boxed{\text{def'n of } \Omega(\gamma)}$ from $\boxed{\text{nonabelianization}}$ agrees w/ $\boxed{\text{count of}}$ $\boxed{\text{degenerate slag}}$.

$\longrightarrow$

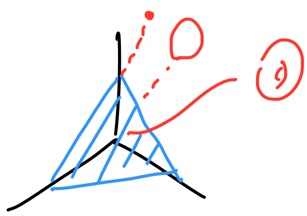
- $\boxed{\text{More than } \Omega(\gamma)}$: foliations capture the $\boxed{\text{moduli space}}$ of A-branes, and in examples this matches the mirror moduli space of B-branes

Ex



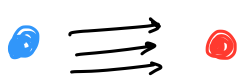
$$L = [S^2 \times S^1] \# [S^2 \times S^1]$$

$$\boxed{\begin{aligned} \mathcal{M}_L : \; & h_1 + h_2 + h_3 = 1 \\ & h_i \geq 0 \end{aligned}}$$



$$\left.\begin{aligned} \mathcal{M} &= \Delta^2 \\ \mathcal{M} &= \mathbb{P}^2 \\ \Omega &= 3 \end{aligned}\right\}$$

B-branes:



$\mathbb{C} \equiv \mathbb{C}$ $\rightarrow \mathcal{M}_{1,1} = \mathbb{P}^2$