From BPS crystals to BPS algebras:

constructions, representations, and applications

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Quivers, CY_3 and DT invariants

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Setup and main question

Type IIA string on a toric CY_3 X

- $\frac{1}{2}$-BPS sector:

  D6/D4/D2/D0 branes wrapping holomorphic 6/4/2/0 cycles of X

- Consider rank-1 DT theory with

$$\#(\mathrm{D6},\ \mathrm{D4},\ \mathrm{D2},\ \mathrm{D0}) = (1, 0, m_i, n)$$

$$\sum_{\mathcal{H}_{\mathrm{BPS}}} N(m_i, n)(Q_i)^{m_i} q^n = Z^X_{\mathrm{DT}}(Q_i, q)$$

Question: Any algebraic structure underlying $\mathcal{H}_{\mathrm{BPS}}$?

Plan and reference

1. BPS states → BPS crystals → BPS algebras
   (NCDT) (vacuum rep) (Quiver Yangians)

   [2003.08909] with *Masahito Yamazaki*

2. Shifted quiver Yangians and non-vacuum representations
   (other chambers, open BPS)

   [2106.01230] with *Dmitrii Galakhov and Masahito Yamazaki*

3. Trig and elliptic generalizations

   [2108.10286] with *Dmitrii Galakhov and Masahito Yamazaki*

Definition of BPS algebra

Definition of BPS algebra

Harvey-Moore '96

Kontsevich-Soibelman '10

- elements: BPS states $|\mathrm{K}_i\rangle \in \mathcal{H}_{\mathrm{BPS}}$
- multiplication:

$$\begin{aligned} \mathcal{H}_{\mathrm{BPS}} \otimes \mathcal{H}_{\mathrm{BPS}} &\rightarrow \mathcal{H}_{\mathrm{BPS}} \\ |\mathrm{K}_1\rangle \times |\mathrm{K}_2\rangle &\rightarrow |\mathrm{K}_3\rangle \end{aligned}$$

Today: consider

$$\begin{aligned} \text{BPS algebra} \cdot \mathcal{H}_{\mathrm{BPS}} &\rightarrow \mathcal{H}_{\mathrm{BPS}} \\ \alpha \quad |\mathrm{K}_1\rangle &= |\mathrm{K}_2\rangle \end{aligned}$$

- BPS algebra: $\mathcal{H}_{\mathrm{BPS}} \rightarrow \mathcal{H}_{\mathrm{BPS}}$
- $\mathcal{H}_{\mathrm{BPS}}$: irrep of BPS algebra

Deriving BPS algebra

$$\text{BPS algebra} \cdot \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}$$
$$\alpha \;\; |\text{K}_1\rangle \;\; = |\text{K}_2\rangle$$

generators: consider "primative BPS states" Δ

$$\begin{cases} \text{raising} \quad e: & |\text{K}\rangle \to |\text{K}+\Delta\rangle \\ \text{Cartan} \quad \psi: & |\text{K}\rangle \to |\text{K}\rangle \\ \text{lowering} \quad f: & |\text{K}\rangle \to |\text{K}+\Delta\rangle \end{cases}$$

$\mathcal{H}_{\text{BPS}}$: $\infty-$dim $\quad \Longrightarrow \quad (e_i, \psi_i, f_i)$ $\infty-$dim

Q1: What is Δ? (How to determine e_i, ψ_i, f_i)?

- Condition: free algebra $< e_i, \psi_i, f_i >$: $\quad \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}$

Q2: What is action of (e_i, ψ_i, f_i) on $|\text{K}\rangle \in \mathcal{H}_{\text{BPS}}$?

Q3: Find all $F(e_i, \psi_i, f_i)$ s.t. $F(e_i, \psi_i, f_i)|\text{K}\rangle = 0$, $\forall |\text{K}\rangle \in \mathcal{H}_{\text{BPS}}$

$$\text{BPS algebra} = \frac{< e_i, \psi_i, f_i >}{\{F(e_i, \psi_i, f_i) = 0\}}$$

Deriving BPS algebra

1. Determine Δ $(\sim (e_i, \psi_i, f_i))$
2. Fix action of (e_i, ψ_i, f_i) on $|\mathrm{K}\rangle \in \mathcal{H}_{\mathrm{BPS}}$
3. Find $\{F(e_i, \psi_i, f_i) | F(e_i, \psi_i, f_i)|\mathrm{K}\rangle = 0\}$

$$\text{BPS algebra} = \frac{< e_i, \psi_i, f_i >}{\{F(e_i, \psi_i, f_i) = 0\}}$$

Deriving BPS algebra

1. Determine Δ ($\sim (e_i, \psi_i, f_i)$)

$|\mathrm{K}\rangle$: BPS crystal
Δ: atoms in BPS crystal

2. Fix action of (e_i, ψ_i, f_i) on $|\mathrm{K}\rangle \in \mathcal{H}_{\mathrm{BPS}}$

3. Find $\{F(e_i, \psi_i, f_i) | F(e_i, \psi_i, f_i)|\mathrm{K}\rangle = 0\}$

$$\text{BPS algebra} = \frac{< e_i, \psi_i, f_i >}{\{F(e_i, \psi_i, f_i) = 0\}}$$

BPS quiver Yangians from colored crystals

IIA string in generic toric CY_3 X

1. $\frac{1}{2}$−BPS sector: $\mathcal{N}=4$ quiver QM (Q,W)

   $\Downarrow$ define

2. { BPS states } = { BPS crystals }

   act $\Uparrow \Downarrow$ bootstrap

3. BPS algebra = quiver Yangian $\text{Y}(Q,W)$

BPS quiver Yangians from colored crystals

IIA string in generic toric CY_3 X

1. $(Q, W) \quad \longleftarrow \quad$ Input

$$\Downarrow \quad \text{define}$$

2. { BPS states } = { BPS crystals }

$$\text{act} \quad \Uparrow \Downarrow \quad \text{bootstrap}$$

3. BPS algebra = quiver Yangian $\text{Y}(Q, W) \quad \longleftarrow \quad$ Output

BPS quiver Yangians from colored crystals

IIA string in generic toric CY_3 X

1. (Q, W) $\longleftarrow$ Input

   $\Downarrow$ define

2. { BPS states } = { BPS crystals }

   act $\Uparrow\Downarrow$ bootstrap

3. BPS algebra = quiver Yangian $\text{Y}(Q, W)$ $\longleftarrow$ Output

Advantages

1. Explicit algebraic relations
2. Applies to generic toric Calabi-Yau threefolds
3. Easily generalized to trigonometric and elliptic versions
4. Easy to describe representations (by subcrystals or framings of Q).

General representations

So far: BPS crystals are molten crystals from canonical crystal

1. BPS states from NC chamber
2. vacuum representation of quiver Yangians

General representation of (shifted) quiver Yangian

1. Molten crystals from subcrystals of full crystal
2. Can describe other chambers and open BPS counting.
3. Different framings of the same quiver

Outline

1. Introduction
2. BPS crystals
3. BPS algebras
4. Representations
5. Summary

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Worldvolume theory on D-brane bound state

IIA string on a toric CY$_3$ X

- $\frac{1}{2}$-BPS sector with D6/D2/D0 brane on holomorphic 6/2/0 cycles of X

$$\#(\mathrm{D6},\ \mathrm{D4},\ \mathrm{D2},\ \mathrm{D0}) = (1, 0, m_i, n)$$

world volume theory: $\mathcal{N} = 4$ quiver quantum mechanics (Q, W)

(Brane tiling: *Hanany-Vegh '05, Feng-He-Kennaway-Vafa '05*)

1. Quiver $Q = (Q_0, Q_1)$

$$Q_0 = \{\text{vertex } a\} \qquad a : \ U(N_a) \text{ gauge group}$$

$$Q_1 = \{\text{arrow } I : a \to b\} \qquad \Phi_I : \ \text{bi-fundamentals } (\overline{N_a}, N_b)$$

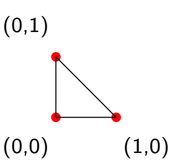
2. superpotential $W = \sum \pm \prod \Phi_I$ (with each Φ_I appearing twice with $\pm$)
3. $(Q, W) \Longleftrightarrow$ periodic $\tilde{Q} = (Q_0, Q_1, Q_2)$

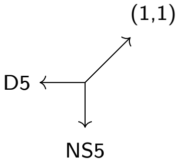
$$\text{with} \quad W = \mathrm{Tr}\Big(\sum_{F \in Q_2^+} \prod_{I \in F} \Phi_I - \sum_{F \in Q_2^-} \prod_{I \in F} \Phi_I \Big)$$

$$\Longrightarrow \quad \text{F-term constraints } \partial W \equiv \{ \frac{\partial}{\partial \Phi_I} W = 0 \mid I \in Q_1 \}$$

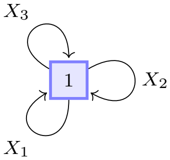
toric CY$_3$ $X \to (Q,W) \to$ periodic quiver: $\mathbb{C}^3$

- toric diagram and (p,q)-web diagram

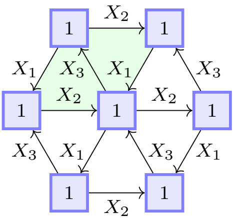




- From (Q,W) to periodic quiver



$\Longleftrightarrow$



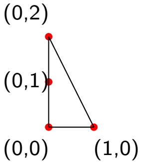
$$W = \mathrm{Tr}[-X_1\, X_2\, X_3 + X_1\, X_3\, X_2]$$

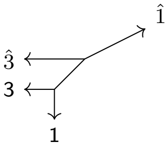
$(\Longrightarrow$ F-term constraints:

$[X_1, X_2] = [X_2, X_3] = [X_3, X_1] = 0)$

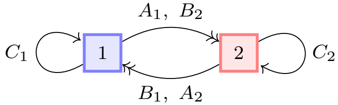
toric CY$_3$ $X \to (Q,W) \to$ periodic quiver: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$

- toric diagram and (p,q)-web diagram





- From (Q,W) to periodic quiver



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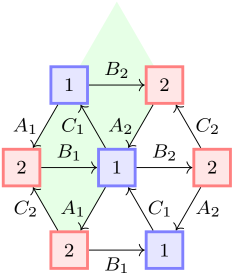


$$W = \mathrm{Tr}[-C_m\,A_m\,B_m + C_m\,B_{m+1}\,A_{m+1}]$$

$(\Longrightarrow$ F-term constraints:

$$A_m B_m = B_{m+1} A_{m+1}$$
$$B_m C_m = C_{m+1} B_m$$
$$C_m A_m = A_m C_{m+1})$$

From BPS states to BPS crystals

- Framing (from D6 on CY_3)

$$\hat{Q} = \{\hat{Q}_0, \hat{Q}_1\} : \qquad \begin{cases} \hat{Q}_0 = Q_0 \cup \{\infty\} \\ \hat{Q}_1 = Q_1 \cup \{\infty \to a_{\mathfrak{f}}\} \end{cases} \qquad (\text{set } a_{\mathfrak{f}} = 1)$$

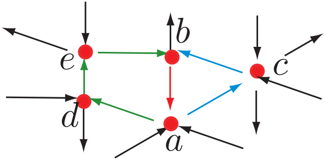
- config. of $\{\Phi_I\}$ solving F-term condition:

representation of $\mathcal{A}(\hat{Q}, W) = \frac{\text{path algebra of } \hat{Q}}{F\text{-term constraint}}$

  - Path algebra

$$\Phi_I \cdot \Phi_J \neq 0 \quad \longrightarrow \quad \text{target}(I) = \text{source}(J)$$

  - Path equivalent from F-term constraint



    - $[a \to d \to e \to b] \sim [a \to c \to b]$ (two F-term equivalent fields)
    - Superpotential $W \ni \text{Tr}(\Phi_{ba}\Phi_{ad}\Phi_{de}\Phi_{eb} - \Phi_{ba}\Phi_{ac}\Phi_{cb})$.
    - F-term relation $\partial W / \partial \Phi_{ba} = \Phi_{ad}\Phi_{de}\Phi_{eb} - \Phi_{ac}\Phi_{cb} = 0$

BPS states via path algebra of quiver

Szendrői '07, Mozgovoy-Reineke '07

- In terms of the factor algebra $\mathcal{A}(Q,W) = \frac{\text{path algebra of } Q}{F\text{-term constraint}}$

a D-brane bound state with charge $(1, 0, m_j, n)$ in toric CY_3 X

$\Updownarrow$

a $U(1)^2$-inv. solution (of F/D-term) in quiver QM (Q, W) with rank $\{N_a\}$

$\Updownarrow$

a $U(1)^2$-inv. $\hat{\theta}$-stable module of $\mathcal{A}(\hat{Q}, W)$ of dim $\{N_a\}$

$\Updownarrow$

an ideal $(\subset \mathcal{A}(Q,W)|_{a_{\mathfrak{f}}})$ of $\mathcal{A}(Q,W)$ of dim $\{N_a\}$

BPS states via path algebra of quiver

Szendröi '07, Mozgovoy-Reineke '07, Ooguri-Yamazaki '08

- In terms of the factor algebra $\mathcal{A}(Q,W) = \frac{\text{path algebra of } Q}{F\text{-term constraint}}$

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  $$\Updownarrow$$

  an ideal $(\subset \mathcal{A}(Q,W)|_{a_{\mathfrak{f}}})$ of $\mathcal{A}(Q,W)$ of dim $\{N_a\}$

- $\mathcal{A}(Q,W)|_{a_{\mathfrak{f}}} \Rightarrow$ full 3D crystal $\mathcal{C}_{a_{\mathfrak{f}}}(Q,W)$ from uplift of periodic quiver $\tilde{Q}$ (starting from $a = a_{\mathfrak{f}}$)
- An ideal $(\subset \mathcal{A}(Q,W)|_{a_{\mathfrak{f}}})$ of $\mathcal{A}(Q,W) \Rightarrow$ a "molten crystal" of $\mathcal{C}_{a_{\mathfrak{f}}}(Q,W)$

BPS states via path algebra of quiver

Szendröi '07, Mozgovoy-Reineke '07, Ooguri-Yamazaki '08

- In terms of the factor algebra $\mathcal{A}(Q,W) = \frac{\text{path algebra of } Q}{F\text{-term constraint}}$

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  $\Updownarrow$

  a $U(1)^2$-inv. solution (of F/D-term) in quiver QM (Q,W) with rank $\{N_a\}$

  $\Updownarrow$

  a $U(1)^2$-inv. $\hat{\theta}$-stable module of $\mathcal{A}(\hat{Q},W)$ of dim $\{N_a\}$

  $\Updownarrow$

  an ideal $(\subset \mathcal{A}(Q,W)|_{a_\mathfrak{f}})$ of $\mathcal{A}(Q,W)$ of dim $\{N_a\}$

  $\Updownarrow$

  a 3D molten crystal K from $\mathcal{C}_{a_\mathfrak{f}}(Q,W)$ with $\{N_a$ number of $\boxed{a}\}$

- $\mathcal{A}(Q,W)|_{a_\mathfrak{f}} \Rightarrow$ full 3D crystal $\mathcal{C}_{a_\mathfrak{f}}(Q,W)$ from uplift of periodic quiver $\tilde{Q}$ (starting from $a = a_\mathfrak{f}$)
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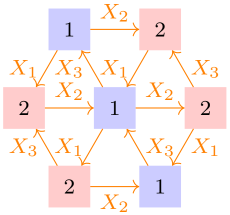
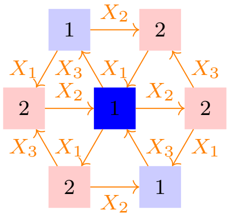
BPS crystal from uplifting $\tilde{Q}$: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$



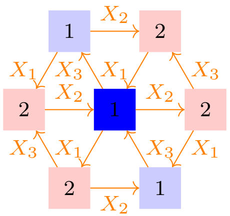
Origin of crystal

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$



path ⇒ atom

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o}$ ⇒ atom $\boxed{a}$



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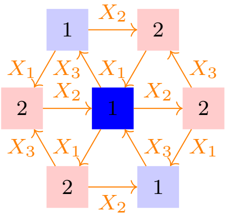

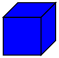

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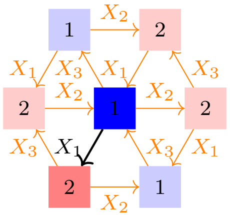

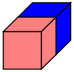

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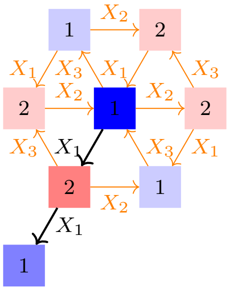
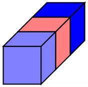
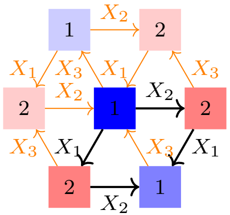
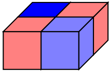




Path equivalence

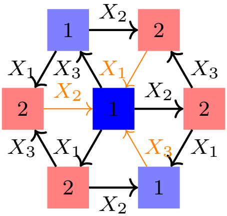
1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths

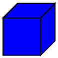




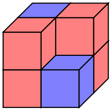
Depth of an atom

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths
4. depth $=$ number of closed loop in the path





depth $= 0$



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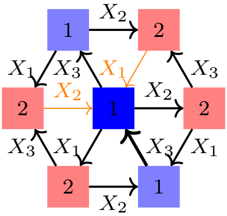
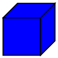
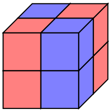




depth $= 0$

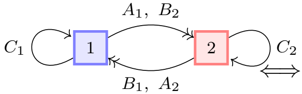


depth $= 1$

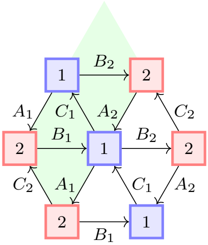
From periodic quiver to full crystal $\mathcal{C}_{a_\mathfrak{f}}(Q,W)$: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$

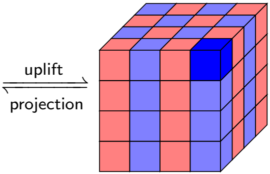
$$\mathcal{A}(Q,W)|_{a_\mathfrak{f}} \Rightarrow \text{full 3D crystal } \mathcal{C}_{a_\mathfrak{f}}(Q,W)$$

(Q,W) periodic quiver full crystal



$$W = \mathrm{Tr}[-C_m\,A_m\,B_m + C_m\,B_{m+1}\,A_{m+1}]$$

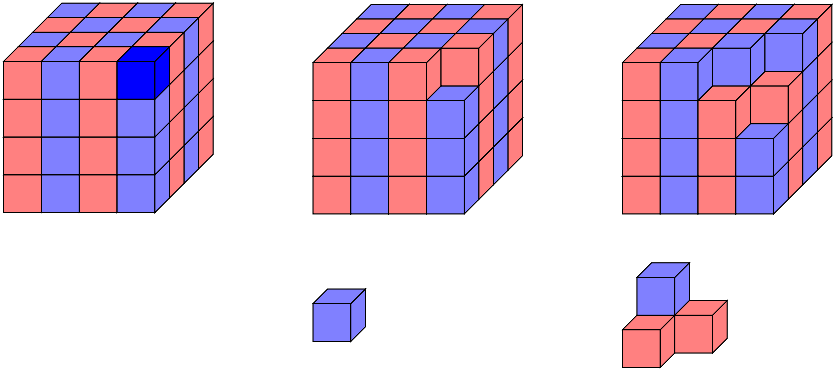




Question: what are the ideals $(\subset \mathcal{A}(Q,W)|_{a_\mathfrak{f}})$ of $\mathcal{A}(Q,W)$?

Molten crystals $\Longrightarrow$ ideals: $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$

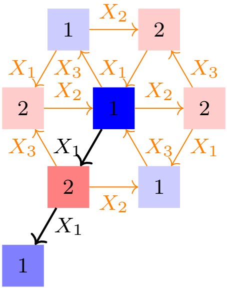
Trivial   ideal of $(1,0)$-dim   ideal of $(2,2)$-dim

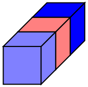


- Molten crystal K: finite subset removed from tip of full crystal $\mathcal{C}_{\mathfrak{a}_\mathfrak{f}(Q,W)}$
- Grown from empty space subject to "melting rule"

Melting rule

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths
4. depth = number of closed loop in the path
5. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$

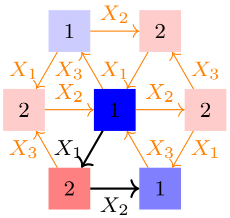


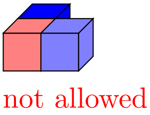


allowed

Melting rule

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o}$ $\Rightarrow$ atom $\boxed{a}$
3. equivalence of paths
4. depth $=$ number of closed loop in the path
5. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$





Melting rule

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
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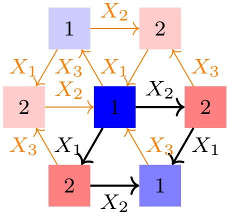
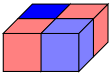




allowed

Equivariant weight of arrows and atoms

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$
3. equivalence of paths
4. depth $=$ number of closed loop in the path
5. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$

To derive BPS algebra from crystal, assign equivariant weights to atoms

L-Yamazaki '20

1. h_I: equivariant weight of arrow I
2. $h(\boxed{a})$: equivariant weight of atom $\boxed{a}$

Equivariant weight of arrows and atoms

1. set framed vertex $a_{\mathfrak{f}} = 1$ and choose origin $\mathfrak{o}$ (with color 1) in $\tilde{Q}$
2. path from $\mathfrak{o} \Rightarrow$ atom $\boxed{a}$ $\Longrightarrow$ $h(\boxed{a}) = \sum_{I \in \text{path}[\mathfrak{o} \to \boxed{a}]} h_I$
3. equivalence of paths $\Longrightarrow$ Loop constraint $\sum_{I \in L} h_I = 0$
4. depth $=$ number of closed loop in the path
   projection: same $h(\boxed{a})$ with different depth
5. Melting rule: if $\boxed{a} \notin \mathrm{K}$, then $I \cdot \boxed{a} \notin \mathrm{K}$

To derive BPS algebra from crystal, assign equivariant weights to atoms

L-Yamazaki '20

1. h_I: equivariant weight of arrow I
2. $h(\boxed{a})$: equivariant weight of atom $\boxed{a}$

Number of equivariant parameters

- number of $h_I = |Q_1| (= |Q_0| + |Q_2|)$

$(Q_0, Q_1, Q_2) = (\text{vertices}, \text{edges}, \text{faces})$

- Loop constraints (global symmetry)

$$\sum_{I \in L} h_I = 0$$

- Vertex constraints (gauge symmetry)

$$\sum_{I \in a} \text{sign}_a(I)\, h_I = 0$$

After loop and vertex constraints, the number of parameters $= 2$

L-Yamazaki '20

From BPS states to molten crystals

Szendrői '07, Mozgovoy-Reineke '07, Ooguri-Yamazaki '08

- Molten crystals

a D-brane bound state with charge $(1, 0, m_j, n)$ in toric CY_3 X

$$\Updownarrow$$

a $U(1)^2$-inv. solution (of F/D-term) in quiver QM (Q, W) with rank $\{N_a\}$

$$\Updownarrow$$

a 3D molten crystal K from $\mathcal{C}_{a_\mathfrak{f}}(Q, W)$ with $\{N_a$ number of $\boxed{a}\}$

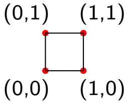
- periodic quiver $\underset{\text{projection}}{\overset{\text{uplift}}{\rightleftharpoons}}$ full 3D crystal $\longrightarrow$ {molten crystal K}
- Crystal generating function reproduces BPS partition function

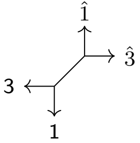
$$Z_{\text{crystal}}(p_a) \equiv \sum_{\mathrm{K}} \text{sign}(\mathrm{K}) \prod_{a \in Q_0} (p_a)^{|\mathrm{K}|_a}$$

$$= Z_{\text{NCDT}}(q_a)$$

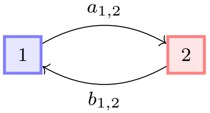
toric CY$_3$ $X \to (Q,W) \to$ periodic quiver: Resolved conifold

- toric diagram and (p,q)-web diagram



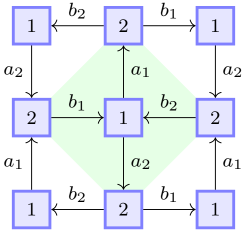


- From (Q,W) to periodic quiver

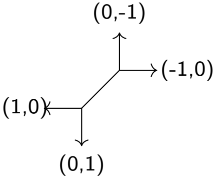


$$W = \mathrm{Tr}[a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1]$$
$$(\Longrightarrow \text{F-term constraints:}$$
$$b_1 a_m b_2 = b_2 a_m b_1$$
$$a_1 b_m a_2 = a_2 b_m a_1)$$

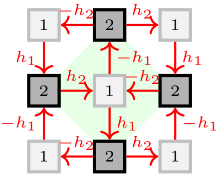
$\Longleftrightarrow$

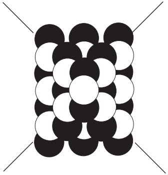


Toric CY_3 $\Longrightarrow$ periodic quiver $\Longrightarrow$ 3D crystal

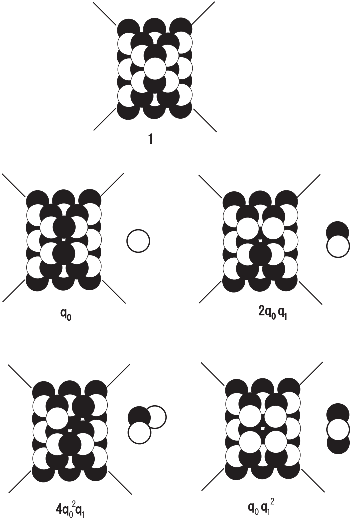


conifold





From full crystal to molten crystal (resolved conifold)



$$
\begin{aligned}
&Z_{\text{crystal}}(q_0, q_1) \\
=&\frac{\prod_{k=1}^{\infty}(1+q_1(q_0q_1)^k)^k(1+\frac{1}{q_1}(q_0q_1)^k)^k}{\prod_{k=1}^{\infty}(1-(q_0q_1)^k)^2k} \\
=&1+q_0+{\color{blue}2}q_0q_1+{\color{blue}4}q_0^2q_1+q_0q_1^2+\dots
\end{aligned}
$$

Young '07

Outline

1 Introduction

2 BPS crystals

3 BPS algebras

4 Representations

5 Summary

Deriving BPS algebra

$$\text{BPS algebra} \cdot \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}$$
$$\alpha \;\; |\text{K}_1\rangle \;\; = |\text{K}_2\rangle$$

generators: consider "primative BPS states" Δ

$$\begin{cases} \text{raising} & e & : & |\text{K}\rangle \to |\text{K} + \Delta\rangle \\ \text{Cartan} & \psi & : & |\text{K}\rangle \to |\text{K}\rangle \\ \text{lowering} & f & : & |\text{K}\rangle \to |\text{K} + \Delta\rangle \end{cases}$$

1. Determine Δ
2. Fix action of (e, ψ, f) on $|\text{K}\rangle \in \mathcal{H}_{\text{BPS}}$
3. Find $\{F(e,\psi,f)|F(e,\psi,f)|\text{K}\rangle = 0\}$

$$\text{BPS algebra} = \frac{< e, \psi, f >}{\{F(e,\psi,f) = 0\}}$$

Deriving BPS algebra

$$\text{BPS algebra} \cdot \mathcal{H}_{\text{BPS}} \to \mathcal{H}_{\text{BPS}}$$
$$\alpha \quad |\text{K}_1\rangle \quad = |\text{K}_2\rangle$$

generators: consider "primative BPS states" Δ

$$\begin{cases} \text{raising} & e^{(a)}: & |\text{K}\rangle \to |\text{K} + \boxed{a}\rangle \\ \text{Cartan} & \psi^{(a)}: & |\text{K}\rangle \to |\text{K}\rangle \\ \text{lowering} & f^{(a)}: & |\text{K}\rangle \to |\text{K} + \boxed{a}\rangle \end{cases}$$

1. Determine Δ

$|\text{K}\rangle$: BPS crystal
Δ: atoms $\boxed{a}$ in BPS crystal

2. Fix action of (e, ψ, f) on $|\text{K}\rangle \in \mathcal{H}_{\text{BPS}}$

3. Find $\{F(e,\psi,f)|F(e,\psi,f)|\text{K}\rangle = 0\}$

$$\text{BPS algebra} = \frac{< e, \psi, f >}{\{F(e,\psi,f) = 0\}}$$

Ansatz for action of BPS algebra on BPS crystal

L-Yamazaki '20

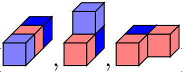
$\mathcal{H}_{\text{BPS}}$: $\infty-$dim $\quad\Longrightarrow\quad (e_i^{(a)}, \psi_i^{(a)}, f_i^{(a)})$ $\infty-$dim

$$e^{(a)}(z) \equiv \sum_j \frac{e_j^{(a)}}{z^{j+1}} \qquad \psi^{(a)}(z) \equiv \sum_j \frac{\psi_j^{(a)}}{z^{j+1}} \qquad f^{(a)}(z) \equiv \sum_j \frac{f_j^{(a)}}{z^{j+1}} \qquad a \in Q_0$$

Ansatz for action

$$\begin{cases} \text{Cartan:} & \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi_{\mathrm{K}}(z)|\mathrm{K}\rangle \\ \text{raising:} & e^{(a)}(z)|\mathrm{K}\rangle = \sum \frac{E(\mathrm{K}\to\mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \\ \text{lowering:} & f^{(a)}(z)|\mathrm{K}\rangle = \sum \frac{F(\mathrm{K}\to\mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \end{cases}$$

Example: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ with $|\mathrm{K}\rangle =$ 

- $|\mathrm{K}+\boxed{a}\rangle = \{$  $\}$
- $|\mathrm{K}-\boxed{a}\rangle =$ 

Demand: $|\mathrm{K}\pm\boxed{a}\rangle$ are valid crystal states

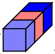
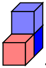
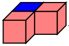
(otherwise $E(\mathrm{K}\to\mathrm{K}+\boxed{a})$ and $F(\mathrm{K}\to\mathrm{K}-\boxed{a})$ should vanish)

Ansatz for action of BPS algebra on BPS crystal

L-Yamazaki '20
($\mathbb{C}^3$, affine Yangian of $\mathfrak{gl}_1$: *Tsymbaliuk '14*)

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv (\frac{1}{z})^{\delta_{a,1}} \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u - h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add}(\mathrm{K})} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z - h(\boxed{a})} |\mathrm{K} + \boxed{a}\rangle \,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem}(\mathrm{K})} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z - h(\boxed{a})} |\mathrm{K} - \boxed{a}\rangle \,, \end{cases}$$

Example: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ with $|\mathrm{K}\rangle =$ 

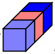
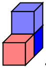
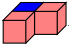
- $|\mathrm{K} + \boxed{a}\rangle = \{$  ,  ,  $\}$
- $|\mathrm{K} - \boxed{a}\rangle =$ 

Ansatz for action of BPS algebra on BPS crystal

L-Yamazaki '20
($\mathbb{C}^3$, affine Yangian of $\mathfrak{gl}_1$: *Tsymbaliuk '14*)

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv (\frac{1}{z})^{\delta_{a,1}} \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u-h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add}(\mathrm{K})} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem}(\mathrm{K})} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \,, \end{cases}$$

Example: $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ with $|\mathrm{K}\rangle =$ 

- $|\mathrm{K}+\boxed{a}\rangle = \{$ , ,  $\}$
- $|\mathrm{K}-\boxed{a}\rangle =$ 

How to fix $\varphi^{a\Leftarrow b}(u)$?

Ansatz for action of BPS algebra on BPS crystal *L-Yamazaki '20*

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle\,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv (\frac{1}{z})^{\delta_{a,1}} \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u-h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add}(\mathrm{K})} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle\,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem}(\mathrm{K})} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle\,, \end{cases}$$

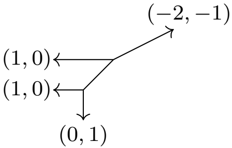
Demand:

1. Irreducibility (cyclic from any $|\mathrm{K}\rangle$)
2. "Melting Rule" satisfied

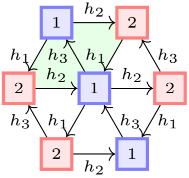
$\Longrightarrow$ Bond factor: $\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}$

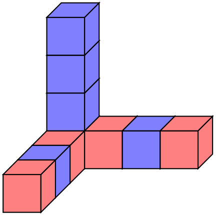
$\Longrightarrow$ $\{\text{poles of } \Psi^{(a)}_{\mathrm{K}}(z)\} \quad \simeq \quad \mathrm{Add}^{(a)}(\mathrm{K}) \cup \mathrm{Rem}^{(a)}(\mathrm{K})$

$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$



$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$



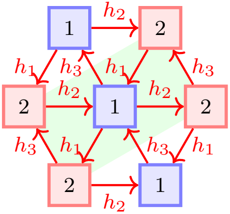


- After loop and vertex constraints $h_1+h_2+h_3=0$
- building blocks of $\Psi^{(a)}_{\mathrm{K}}(u)$

$$\varphi^{1\Leftarrow 1}(u)=\varphi^{2\Leftarrow 2}(u)=\frac{u+h_3}{u-h_3}$$
$$\varphi^{1\Leftarrow 2}(u)=\varphi^{2\Leftarrow 1}(u)=\frac{(u+h_1)(u+h_2)}{(u-h_1)(u-h_2)}$$

- Now: check irreducibility (cyclic for any $|\mathrm{K}\rangle$) and melting rule

$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal

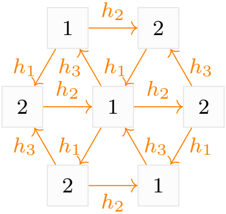


$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}$ crystal: vacuum

vacuum $|\emptyset\rangle$

1. Charge functions

$$\begin{cases} \Psi^{(1)}_{\mathrm{K}}(z) = \dfrac{1}{z} \\ \Psi^{(2)}_{\mathrm{K}}(z) = 1 \end{cases}$$



$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: vacuum $\Longrightarrow$ level-1

vacuum $|\emptyset\rangle$

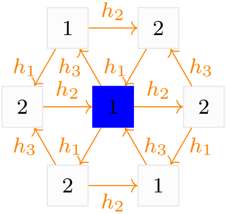
1. Charge functions

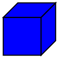
$$\begin{cases} \Psi^{(1)}_{\text{K}}(z) = \frac{1}{z} \\ \Psi^{(2)}_{\text{K}}(z) = 1 \end{cases}$$

2. Pole for $\boxed{1}$: $z=0$ $\Longrightarrow$ $e^{(1)}(z)|\emptyset\rangle = \frac{\#}{z}|\boxed{1}\rangle$

   Pole for $\boxed{2}$: none $\Longrightarrow$ $e^{(2)}(z)|\emptyset\rangle = 0$

3. $f^{(1)}(z)|\emptyset\rangle = f^{(2)}(z)|\emptyset\rangle = 0$



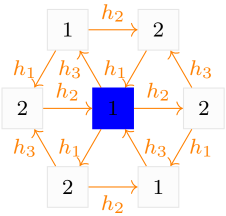


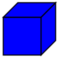
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-1

1-atom state $|\boxed{1}\rangle \quad \Longrightarrow \quad h(\boxed{1})=0$

1. Charge functions

$$\begin{cases} \psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1})) = \dfrac{1}{z}\cdot\dfrac{z+h_3}{z-h_3} \\ \psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1})) = \dfrac{(z+h_1)(z+h_2)}{(z-h_1)(z-h_2)} \end{cases}$$





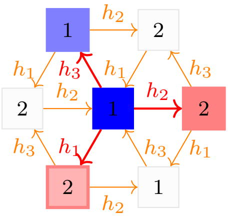
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-1 $\Longrightarrow$ level-2

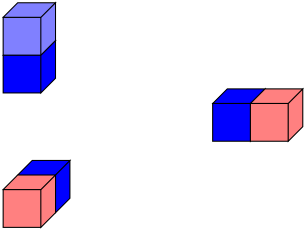
1-atom state $|\boxed{1}\rangle \quad \Longrightarrow \quad h(\boxed{1})=0$

1. Charge functions

$$\begin{cases} \psi^{(1)}_{\mathrm{K}}(z)=\psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))=\dfrac{1}{z}\cdot\dfrac{z+h_3}{z-h_3} \\ \psi^{(2)}_{\mathrm{K}}(z)= \qquad \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))=\dfrac{(z+h_1)(z+h_2)}{(z-h_1)(z-h_2)} \end{cases}$$

2. Pole for $\boxed{1}$: $z=0$ and $z=h_3$
   Pole for $\boxed{2}$: $z=h_1$ and $z=h_2$
3. $f^{(1)}(z)|\boxed{1}\rangle=|\emptyset\rangle$ and $f^{(2)}(z)|\boxed{1}\rangle=0$



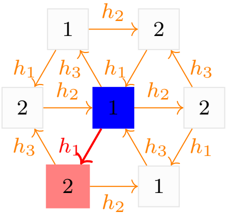


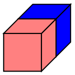
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-2

2-atoms state $|\boxed{1}_0\boxed{2}_1\rangle \quad \Longrightarrow \quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1$

1. Charge function

$$\begin{cases} \Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2})) \\ \qquad = \dfrac{(1)}{\cancel{z}}\cdot\dfrac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\dfrac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}} \\ \Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2})) \\ \qquad = \dfrac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\dfrac{(z+h_3-h_1)}{\cancel{(z+h_2)}} \end{cases}$$



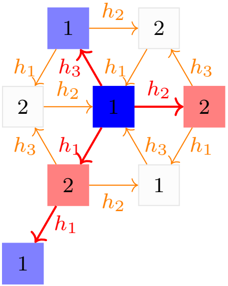


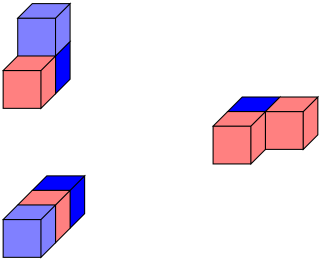
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: level-2 $\Longrightarrow$ level-3

2-atoms state $|\boxed{1}_0\boxed{2}_1\rangle \quad\Longrightarrow\quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1$

1. Charge function

$$\begin{cases} \Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2})) \\ \qquad = \dfrac{(1)}{\cancel{z}}\cdot\dfrac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\dfrac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}} \\ \Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2})) \\ \qquad = \dfrac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\dfrac{(z+h_3-h_1)}{\cancel{(z+h_2)}} \end{cases}$$



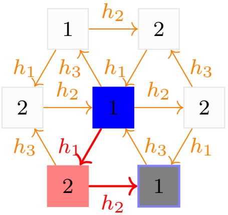


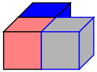
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: Melting Rule

2-atoms state $|\boxed{1}_0\boxed{2}_1\rangle \quad \Longrightarrow \quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1$

1. Charge function

$$\begin{cases}\Psi^{(1)}_{\mathrm{K}}(z)=\psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2}))\\ \qquad =\frac{(1)}{\cancel{z}}\cdot\frac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\frac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}}\\ \Psi^{(2)}_{\mathrm{K}}(z)=\varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2}))\\ \qquad =\frac{(z+h_1)\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\frac{(z+h_3-h_1)}{\cancel{(z+h_2)}}\end{cases}$$



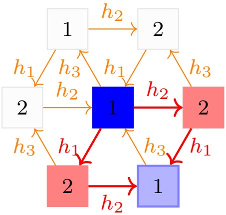


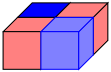
$(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C}$ crystal: Melting Rule

3-atoms state $|\boxed{1}_0\boxed{2}_1\boxed{2}_2\rangle \quad\Longrightarrow\quad h(\boxed{1}_0)=0,\ h(\boxed{2}_1)=h_1\ \ h(\boxed{2}_2)=h_2$

1. Charge function

$$
\begin{cases}
\Psi^{(1)}_{\mathrm{K}}(z) = \psi_0(z)\cdot\varphi^{1\Rightarrow 1}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2}_1))\cdot\varphi^{2\Rightarrow 1}(z-h(\boxed{2}_2)) \\
\qquad = \frac{(1)}{\cancel{z}}\cdot\frac{\cancel{(z+h_3)}}{(z-h_3)}\cdot\frac{\cancel{(z)}(z+h_2-h_1)}{(z-2h_1)\cancel{(z+h_3)}}\cdot\frac{(z)(z+h_1-h_2)}{(z-2h_2)(z+h_3)} \\
\Psi^{(2)}_{\mathrm{K}}(z) = \varphi^{1\Rightarrow 2}(z-h(\boxed{1}))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2}_1))\cdot\varphi^{2\Rightarrow 2}(z-h(\boxed{2}_2)) \\
\qquad = \frac{\cancel{(z+h_1)}\cancel{(z+h_2)}}{(z-h_1)(z-h_2)}\cdot\frac{(z+h_3-h_1)}{\cancel{(z+h_2)}}\cdot\frac{(z+h_3-h_2)}{\cancel{(z+h_1)}}
\end{cases}
$$





Poles of $\Psi^{(a)}_{\mathrm{K}}(z)$ encode the positions of $\boxed{a} \in \mathrm{Add}(\mathrm{K})$ and $\mathrm{Rem}(\mathrm{K})$

1. Each $\boxed{b}$ in K contributes a factor of $\varphi^{a\Leftarrow b}(z - h(\boxed{b}))$ to $\Psi^{(a)}_{\mathrm{K}}(z)$
2. $h(\boxed{b}) \equiv \sum_{I \in \mathrm{path}[\mathfrak{o}\to\boxed{b}]} h_I$
3. $$\boxed{\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}}$$
4. loop constraint $\sum_{I\in\mathrm{loop}\ L} h_I = 0$

Poles are always pushed to the surface of crystal !

"Melting rule" is automatically implemented

Quadratic relations in BPS algebra

L-Yamazaki '20

$$\begin{cases} \psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \,, \qquad \Psi^{(a)}_{\mathrm{K}}(u) \equiv \left(\frac{1}{z}\right)^{\delta_{a,1}} \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u - h(\boxed{b})) \\ e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Add}(\mathrm{K})} \frac{\pm\sqrt{\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z - h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \,, \\ f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\,\in\,\mathrm{Rem}(\mathrm{K})} \frac{\pm\sqrt{(-1)^{|a|}\mathrm{Res}_{u=h(\boxed{a})}\Psi^{(a)}_{\mathrm{K}}(u)}}{z - h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \,, \end{cases}$$

$$\begin{aligned} \psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z) \,, \\ \psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z) \,, \\ e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z) \,, \\ \psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z) \,, \\ f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z) \,, \\ \left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w} \,, \end{aligned}$$

- Algebraic relations confirmed directly from $\mathcal{N}=4$ quiver QM (using mathematica, up to some level) *Galakhov-Yamazaki '20*

Relations in terms of modes

To compare to other algebras, convert to relations in terms of modes

1. Read off mode expansion of $(e^{(a)}(z), \psi^{(a)}(z), f^{(a)}(z))$ from algebra's action on crystals:

   Using

$$\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}}(u+h_I)}{\prod_{I\in\{b\to a\}}(u-h_I)}$$

$$\text{w/o compact 4-cycle} \quad \Rightarrow \quad \text{non-chiral quiver} \quad \Rightarrow \quad \text{homogenous } \varphi^{a\Leftarrow b}(u)$$

$$\text{w compact 4-cycle} \quad \Rightarrow \quad \text{chiral quiver} \quad \Rightarrow \quad \text{inhomogenous } \varphi^{a\Leftarrow b}(u)$$

$\Longrightarrow$

$$e^{(a)}(z) = \sum_{j=0}^{\infty} \frac{e_j^{(a)}}{z^{j+1}} \quad \text{and} \quad f^{(a)}(z) = \sum_{j=0}^{\infty} \frac{f_j^{(a)}}{z^{j+1}}$$

$$\psi^{(a)}(z) = \begin{cases} \sum_{j=0}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/o compact 4-cycle)} \\ \sum_{j=-\infty}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/ compact 4-cycle)} \end{cases}$$

Relations in terms of modes

To compare to other algebras, convert to relations in terms of modes

2. Plug in mode expansions to algebraic relations and take singular terms:

$$\left[\psi_n^{(a)}\,,\,\psi_m^{(b)}\right]=0\,,\qquad\qquad \left[e_n^{(a)}\,,\,f_m^{(b)}\right\}=\delta^{a,b}\,\psi_{n+m}^{(a)}\,,$$

$$\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma_{|b\to a|-k}^{b\to a}\,[\psi_n^{(a)}\,e_m^{(b)}]_k=\sum_{k=0}^{|a\to b|}\sigma_{|a\to b|-k}^{a\to b}\,[e_m^{(b)}\,\psi_n^{(a)}]^k\,,$$

$$\ldots$$

$$[A_n\,B_m]_k\equiv\sum_{j=0}^{k}(-1)^j\binom{k}{j}A_{n+k-j}\,B_{m+j}\,,\quad [B_m\,A_n]^k\equiv\sum_{j=0}^{k}(-1)^j\binom{k}{j}B_{m+j}\,A_{n+k-j}\,.$$

$\sigma_k^{a\to b}\equiv k^{\text{th}}$ elementary symmetric sum of the set $\{h_I\}$ with $I\in\{a\to b\}$

Serre relations

- Demanding that

  vacuum character of algebra = **generating function** of crystal

  gives additional cubic or higher relations *L-Yamazaki '21*

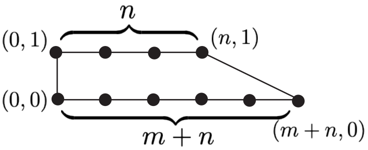
- Reproduce Serre relations for affine Yangian of $\mathfrak{gl}_{n|m}$

$$\text{e.g. } \mathbb{C}^3: \quad \mathrm{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\, e(z_1)\, e(z_2)\, e(z_3)$$
$$\sim \mathrm{Sym}_{(z_1,z_2,z_3)}(z_2-z_3)\, f(z_1)\, f(z_2)\, f(z_3) \sim 0$$

- Open problem: classify Serre relations for general quiver Yangians

Compare with known algebras

- Toric CY_3 $xy = z^m w^n$



$$\text{quiver Yangian} \quad = \quad \text{affine Yangian of } \mathfrak{gl}_{m|n}$$

Ueda '19

- For general toric CY_3, quiver Yangian is new algebra

Generalize to trigonometric and elliptic version

Galakhov-L-Yamazaki '21

- bond factor

$$\varphi^{a\Leftarrow b}(u) \equiv (-1)^{|b\to a|\chi_{ab}} \frac{\prod_{I\in\{a\to b\}} \zeta(u+h_I)}{\prod_{J\in\{b\to a\}} \zeta(u-h_J)}$$

- rational $\longrightarrow$ trigonometric $\longrightarrow$ elliptic

$$\zeta(u) \equiv \begin{cases} u & \text{(rational)} \quad \Longrightarrow \quad \text{quiver Yangians} \\ \sim \sinh \beta u & \text{(trig.)} \quad \Longrightarrow \quad \text{toroidal quiver algebras} \\ \sim \theta_q(u) & \text{(elliptic)} \quad \Longrightarrow \quad \text{elliptic quiver algebras} \end{cases}$$

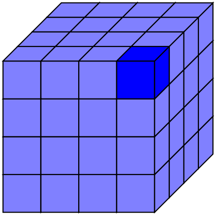
- Bootstrap from crystal representation before central extension
- Turn on central extension and fix by consistency
- Confirm from gauge theory (2D $(2,2)$ and 3D $\mathcal{N}=2$ theory)

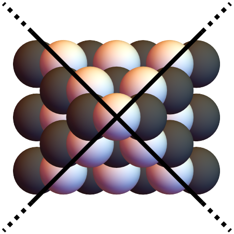
Outline

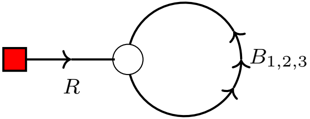
1. Introduction
2. BPS crystals
3. BPS algebras
4. Representations
5. Summary

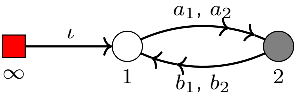
So far: canonical crystal

$\mathbb{C}^3$ | resolved conifold





$Q_0 =$


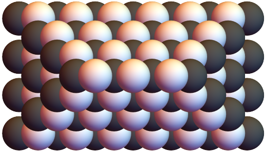
$Q_0 =$


vacuum charge function $\psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,\mathfrak{o}}}$

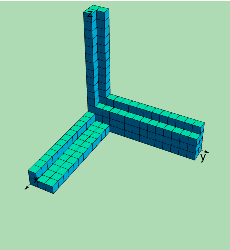
From canonical crystal to other crystals

- The canonical crystal corresponds to counting of closed BPS invariants in the non-commutative DT chamber.
- Can have crystal with other shapes

wall crossing to other chambers



Open BPS states



- Can consider arbitrary subcrystals of canonical crystal

Subscrystal ${}^\sharp\mathcal{C}$

1. How to describe an arbitrary subcrystal ${}^\sharp\mathcal{C}$?

2. What is their relations to quiver Yangian?

3. What is their relation to the quiver?

Subscrystal $^\sharp\mathcal{C}$

1. How to describe an arbitrary subcrystal $^\sharp\mathcal{C}$?

   $\Longrightarrow$ superposition of positive/negative canonical crystals

2. What is their relations to quiver Yangian?

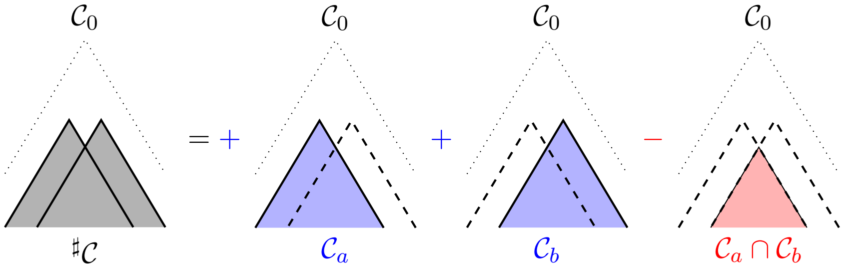
   $\Longrightarrow$ non-vacuum representations of (shifted) quiver Yangians

3. What is their relation to the quiver?

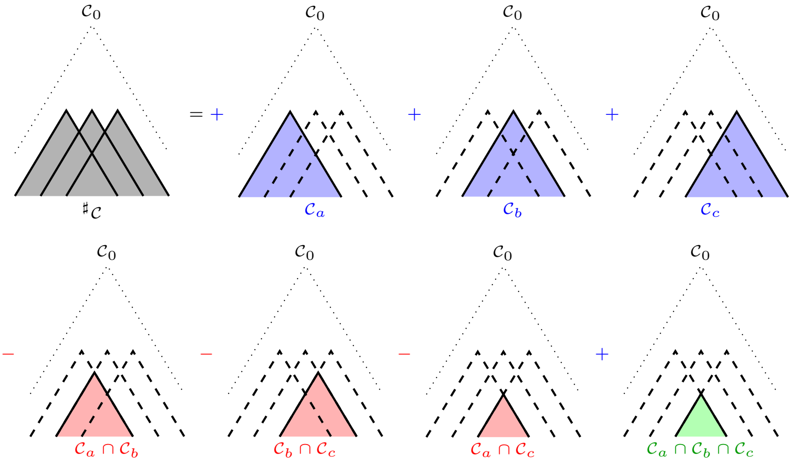
   $\Longrightarrow$ different framing of the original quiver

Decomposing subcrystal $^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

- step-1: determine the positions of positive crystals
- step-2: determine the overlaps of positive crystals
  $\Longrightarrow$ add negative crystals to cancel the overlaps

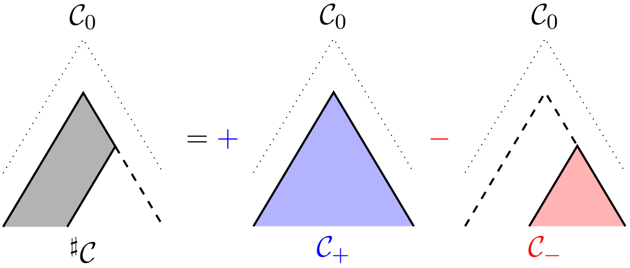


- step-3: determine the overlaps of negative crystals
  $\Longrightarrow$ add positive crystals to cancel overlaps of negative crystals
- step-4: continue until $^\sharp\mathcal{C}$ is reproduced (inclusion-exclusion principle)



Decomposing subcrystal ${}^\sharp\mathcal{C}$ into positive/negative $\mathcal{C}_0$

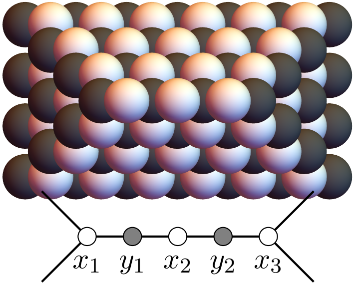
- (optional) final step: truncate by adding negative crystals



Simply-connected subcrystal can be decomposed into superpositions of positive/negative crystals.

Galakhov-L-Yamazaki '21

Crystal decomposition — infinite chamber for conifold



- positive crystal: starts at 1 at x_1, x_2, x_3
- negative crystal: starts at 2 at y_1, y_2

From subcrystal $^\sharp\mathcal{C}$ to ground state charge function $^\sharp\psi$

Galakhov-L-Yamazaki '21

- charge function of arbitrary crystal

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \,, \quad \Psi^{(a)}_{\mathrm{K}}(u) \equiv {}^\sharp\psi^{(a)}_0(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{a\Leftarrow b}(u - h(\boxed{b}))$$

- General representations

- contribution from ground state

sub-crystal $^\sharp\mathcal{C}$: $\quad {}^\sharp\psi^{(a)}_0(z) = \dfrac{\prod_{\beta=1}^{\mathtt{s}_-^{(a)}}(z - z_{-\beta}^{(a)})}{\prod_{\alpha=1}^{\mathtt{s}_+^{(a)}}(z - \mathfrak{p}_\alpha^{(a)})}$

positive crystal staring at $\boxed{a}$: pole $\mathfrak{p}^{(a)} = h(\boxed{a})$

negative crystal staring at $\boxed{a}$: zero $z_-^{(a)} = h(\boxed{a})$

c.f. canonical crystal $\mathcal{C}_0$: $\quad \psi^{(a)}_0(z) = \left(\frac{1}{z}\right)^{\delta_{a,\mathfrak{o}}}$

Shifted quiver Yangian

Galakhov-L-Yamazaki '21

- mode expansion of original quiver Yangian

$$\psi^{(a)}(z) = \begin{cases} \sum_{j=-1}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/o compact 4-cycle)} \\ \sum_{j=-\infty}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1}} & \text{(w/ compact 4-cycle)} \end{cases}$$

- change of ground state charge function

$$\psi_0^{(a)}(z) = \left(\frac{1}{z}\right)^{\delta_{a,1}} \quad \Longrightarrow \quad {}^\sharp\psi_0^{(a)}(z) = \frac{\prod_{\beta=1}^{\mathtt{s}_-^{(a)}} (z - z_{-\beta}^{(a)})}{\prod_{\alpha=1}^{\mathtt{s}_+^{(a)}} (z - \mathfrak{p}_\alpha^{(a)})}$$

- mode expansion of shifted quiver Yangian

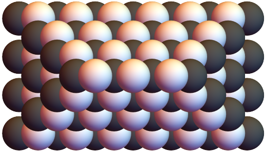
$$\psi^{(a)}(z) = \begin{cases} \sum_{j=-1}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1+\mathtt{s}^{(a)}}} & \text{(w/o compact 4-cycle)} \\ \sum_{j=-\infty}^{\infty} \frac{\psi_j^{(a)}}{z^{j+1+\mathtt{s}^{(a)}}} & \text{(w/ compact 4-cycle)} \end{cases}$$

$$\mathtt{s}^{(a)} \equiv \mathtt{s}_+^{(a)} - \mathtt{s}_-^{(a)}$$

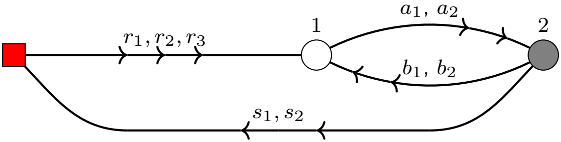
From subcrystal to framed quiver

Galakhov-L-Yamazaki '21

1. For starting atom $\boxed{a}$ of each positive crystal, add an arrow $\infty \to a$
2. For starting atom $\boxed{a}$ of each negative crystal, add an arrow from $a \to \infty$
3. Add terms to superpotential



$Q_3^{(i)} =$



,

$$W_3^{(i)} = \mathrm{Tr}\left[b_2 a_2 b_1 a_1 - b_2 a_1 b_1 a_2 + \sum_{i=1}^{2} s_i \left(a_2 r_i - a_1 r_{i+1} \right) \right].$$

Outline

1 Introduction

2 BPS crystals

3 BPS algebras

4 Representations

5 Summary

Summary of construction

Given a toric Calabi-Yau threefold X, consider the BPS sector of D-brane system of type IIA string on X

1. Quiver quantum mechanics (Q, W)

$$\Downarrow \quad \text{define}$$

2. { BPS states } = { colored crystals }

$$\text{act} \quad \Uparrow \Downarrow \quad \text{bootstrap}$$

3. BPS quiver Yangian $Y(Q, W)$

Summary of construction

Given a toric Calabi-Yau threefold X, consider the BPS sector of D-brane system of type IIA string on X

1. Quiver quantum mechanics (Q, W) $\longleftarrow$ Input

   $\Downarrow$ define

2. { BPS states } = { colored crystals }

   act $\Uparrow\Downarrow$ bootstrap

3. BPS quiver Yangian $Y(Q, W)$ $\longleftarrow$ Output

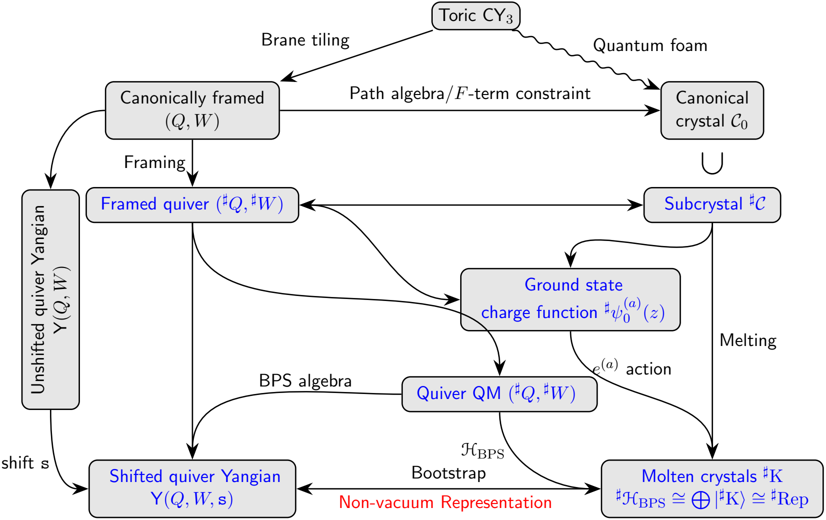
Summary: BPS algebra for general toric Calabi-Yau X_3

1. periodic quiver $(Q,W) \Longrightarrow \varphi^{a\Leftarrow b}(z-w)$ and $|a|$
2. quiver Yangian $Y(Q,W)$

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\;, \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,\psi^{(a)}(z)\;, \\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)\,e^{(b)}(w)\,e^{(a)}(z)\;, \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\;, \\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{a\Leftarrow b}(z-w)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\;, \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\;,
\end{aligned}
$$

3. Advantages
   - Explicit algebraic relations
   - Applies to ALL toric Calabi-Yau threefolds
   - Easily generalized to trigonometric and elliptic versions
   - Easy to describe representations (by subcrystals and framed quivers)

Subcrystal representation, shifted quiver Yangians, and framed quiver



Cover all other cyclic chambers and open BPS states questions....

Q1: relation to Kontsevich-Soibelman's CoHA

- Want to show CoHA $\sim$ Borel subalgebra (with e's) of quiver Yangians
- Maybe along the line of *Rapcak-Soibelman-Yang-Zhao '18* (for $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$)
  Rapcak-Soibelman-Yang-Zhao '20 (for $\hat{\mathcal{Y}}(\mathfrak{gl}_{n|m})$)
- How about general toric CY_3 (w/ compact 4-cycle)?
  More direct approach?

Q2: relations to vertex operator algebras

- $\mathbb{C}^3$: $\mathcal{Y}[\widehat{\mathfrak{gl}_1}] \cong \mathrm{UEA}[\mathcal{W}_{1+\infty}]$ *Procházka '15; Gaberdiel-Gopakumar-L-Peng '17*
  $\mathcal{W}_{1+\infty}$ algebra: $\mathcal{W}$ algebra with one field per spin $s = 1, 2, 3, \cdots, \infty$, only two parameters

$$W^{(s)}(z) = \sum_{n\in\mathbb{Z}} \frac{W_n^{(s)}}{z^{n+s}} \qquad s = 1, 2, 3, \ldots$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
spin-5	$\ldots$	X_{-3}	X_{-2}	$X_{-1} \sim e_4$	$X_0 \sim \psi_5$	$X_1 \sim f_4$	X_2	X_3
spin-4	$\ldots$	U_{-3}	U_{-2}	$U_{-1} \sim e_3$	$U_0 \sim \psi_4$	$U_1 \sim f_3$	U_2	U_3
spin-3	$\ldots$	W_{-3}	W_{-2}	$W_{-1} \sim e_2$	$W_0 \sim \psi_3$	$W_1 \sim f_2$	W_2	W_3
spin-2	$\ldots$	L_{-3}	L_{-2}	$L_{-1} \sim e_1$	$L_0 \sim \psi_2$	$L_1 \sim f_1$	L_2	L_3
spin-1	$\ldots$	J_{-3}	J_{-2}	$\mathbf{J}_{-1} \sim e_0$	$J_0 \sim \psi_1$	$\mathbf{J}_1 \sim f_0$	J_2	J_3

- Calabi-Yau without compact 4-cycles (non-chiral quivers):

  affine Yangian of $\mathfrak{g} \cong$ UEA[$\mathfrak{g}$-extended $\mathcal{W}_{1+\infty}$ algebra]

  (w/ $\mathfrak{g}$=$\mathfrak{gl}_{n|m}$ or $D(2,1|\alpha)$)

- Q: Calabi-Yau with compact 4-cycles (chiral quivers) $\sim$ some VOA?

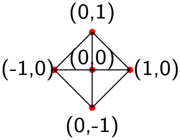
Q3: algebra isomorphism from quiver mutation

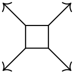
In general, the map from toric CY$_3$ X to (Q, W) is one-to-many

related as Seiberg duality (quiver mutation) *Beasley-Plesser '01*

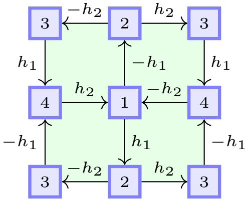
- E.g. $X = K_{\mathbb{P}^1 \times \mathbb{P}^1}$

X:

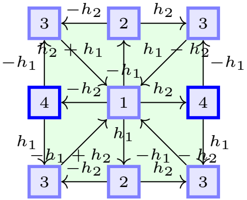




periodic quivers:



mutate by vertex 4



Conjecture: quiver Yangins related by quiver mutation are isomorphic

L-Yamazaki '20

(Evidence: for $\hat{\mathcal{Y}}(\mathfrak{gl}_{n|m})$, mutation $\to$ same $\mathfrak{gl}_{n|m}$ with different Dynkin diagrams)

More questions

- What are those new sub-crystal representations?
- Derive R-matrices for general quiver Yangians?

  (for Bethe/Gauge correspondence)
- Generalize to other geometries?

Thank you for your attention !