Stability structures in holomorphic Morse-Novikov theory

DT invariants appear in two different contexts:
1) CY3 categories, 2) holomorphic Floer theory
in arbitrary dimension

Goal of the lecture: a totally elementary and controllable example, simpler than 4dim wall-crossing (Kontsevich-Soibelman & Joyce-Song), but a bit more complicated than 2dim wall-crossing (Cecotti-Vafa).

Morse-Novikov theory

X: smooth compact manifold

$\alpha_{\mathbb{R}}$: closed real 1-form on X

generically $\alpha_{\mathbb{R}}$ has isolated simple zeroes

$$\alpha_{\mathbb{R}} \approx \sum_{i=1}^{\dim X} (\pm\, dx_i)^2$$

pick a generic Riemannian metric g on X

$\rightsquigarrow$ $\xi = g^{-1}\alpha_{\mathbb{R}}$ gradient flow

infinitely many (isolated) gradient lines

Novikov's idea: count each line with the weight $T^{\underbrace{\int_x^{x'} \alpha_{\mathbb{R}}}_{>0}}$ $\Longrightarrow$ $d^2=0$

Ring $\quad \mathbb{Q}[[T^{\mathbb{R}}]] \quad = \left\{ \sum a_{\lambda_i} T^{\lambda_i} \;\middle|\; \begin{array}{l} a_\lambda \in \mathbb{Q} \quad 0 \le \lambda_1 < \lambda_2 < \dots \\ \lim \lambda_i = +\infty \end{array} \right\}$

Field $\quad \mathbb{Q}((T^{\mathbb{R}})) \;=:\; \mathcal{N}ov$

$\rightsquigarrow$ Finite-dimensional complex over Novikov field

Theorem $\quad H^\bullet(\dots)$ do not depend on the choice of generic metric g

$\approx$ $H^\bullet\big(X,$ universal local system of rank 1, with coefficients in $\mathcal{N}ov$, holonomy around loop $\circlearrowleft \gamma$ $\big)$

canonically

$= T^{\int_\gamma \alpha_{\mathbb{R}}}$

More canonical isomorphisms

$\alpha_{\mathbb{R}} \overset{\text{using } g}{\rightsquigarrow}$ vector field $\xi = g^{-1}\alpha_{\mathbb{R}} \rightsquigarrow$ open convex cone in $H^1(X, \mathbb{R})$

$$= \left\{ [\tilde{\alpha}_{\mathbb{R}}] \;\middle|\; d\tilde{\alpha}_{\mathbb{R}} = 0,\ \langle \tilde{\alpha}_{\mathbb{R}}, \xi \rangle|_{\forall x \notin \text{zeroes } \alpha_{\mathbb{R}}} > 0 \right\}$$

$\forall$ non-archimedean field K, $\forall\ \rho : \pi_1(X, x) \to GL(1, K) = K^\times \xrightarrow[\log|\cdot|]{} \mathbb{R}$

$$\searrow H_1(X, \mathbb{Z}) \nearrow$$

$\rightsquigarrow$ class in $H^1(X, \mathbb{R})$

$\Rightarrow$ Conclusion: we can calculate $H^\bullet(X, \mathcal{L})$ $\forall \mathcal{L} \in$ non-archimedean tube domain $\subset H^1(X, K^\times)$

Holomorphic Morse-Novikov theory

$X = X_{\mathbb{C}}$ compact complex variety $\dim_{\mathbb{R}} X = 2n$

complex Morse type:

$$\alpha_{\mathbb{C}} \approx d\left(\sum_{i=1}^{n} z_i^2\right)$$

$\alpha_{\mathbb{C}}$: holomorphic closed 1-form

$\forall\ \theta \in \mathbb{R}/2\pi\mathbb{Z} \rightsquigarrow$ real closed 1-form $\operatorname{Re}\left(e^{-i\theta}\alpha_{\mathbb{C}}\right)$

real Morse type of index n

$$\theta = 0 : d\left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2\right)$$

$z_i = x_i + \sqrt{-1}\, y_i$

$\rightsquigarrow$ No differential for generic metric

Canonical basis in $H^\bullet(X, \mathcal{L})$

local system of rank 1 in a non-archimedean tube domain

if we use only hermitean metrics and if $\theta \neq \operatorname{Arg} \int_\gamma \alpha_{\mathbb{C}}\ \ \forall \gamma \in H_1(X, \text{Zeroes}(\alpha_{\mathbb{C}}))$

assuming that the local system is trivialized at

$\boxed{\text{No walls in the contractible space of hermitean metrics}}$

Reason: $e^{-i\theta}\,\alpha_{\mathbb{C}}\big|_{\text{gradient line for any hermitean metric}}$

is pointwise strictly positive, $\in \mathbb{R}_{>0} \subset \mathbb{C}$

Generalization: $\alpha_{\mathbb{C}}$ has degenerate, or even non-isolated zeroes

$$\mathrm{Zeroes}(\alpha_{\mathbb{C}}) = \coprod Z_j$$

connected components

near Z_j $\alpha_{\mathbb{C}} = d f_j$ $\quad f_j|_{Z_j} = 0$

normalization

$$H^\bullet_\theta := \bigoplus_j H^\bullet\left(\varepsilon\text{- neighborhood of } Z_j,\ f^{-1}(\mathbb{R}_{>0}\cdot e^{i\theta});\ \mathbb{Z}\right)$$

local system of ~~abelian groups~~ $D^b(\text{f.g. } \mathbb{Z}\text{-mod})$ $/\ S^1_\theta = \mathbb{R}/2\pi\mathbb{Z}$

Claim: if $\theta \notin$ Stokes directions $\iff$ $\operatorname{Arg}\int_\gamma \alpha_\hbar \neq \theta \quad \forall\, \gamma \in H_1(X, \mathrm{Zeroes}(\alpha); \mathbb{Z})$

$\rightsquigarrow$ canonical identification

$$H^\bullet_\theta \overset{L}{\underset{\mathbb{Z}}{\otimes}} K \simeq H^\bullet(X, \mathcal{L})$$

Even more: assume that X is not even a complex manifold, just smooth C^∞

$\alpha_{\mathbb{C}}$: complex-valued closed 1-form

$$\alpha_{\mathbb{C}} = \underbrace{\alpha_1}_{\mathrm{Re}(\alpha_{\mathbb{C}})} + \sqrt{-1}\,\underbrace{\alpha_2}_{\mathrm{Im}(\alpha_{\mathbb{C}})}$$

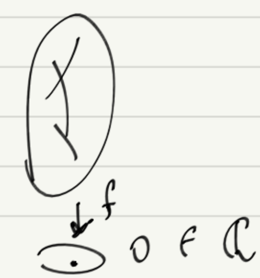
Axiom: outside of $\mathrm{Zeroes}(\alpha_{\mathbb{C}})$, (α_1, α_2) are $\mathbb{R}$-linearly independent pointwise.

$\Rightarrow$ $X - \mathrm{Zeroes}(\alpha_{\mathbb{C}})$ carries foliation of codimension 2, with transversal translational structure

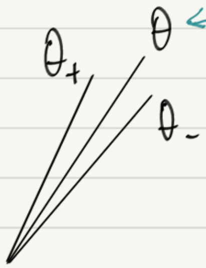
$\Rightarrow$ $GL(2,\mathbb{R})$-action on $\{(\alpha_1, \alpha_2)\}$

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

locally $f = \int^{1}(\alpha_{\mathbb{C}})$ has only one critical value, $=0$.



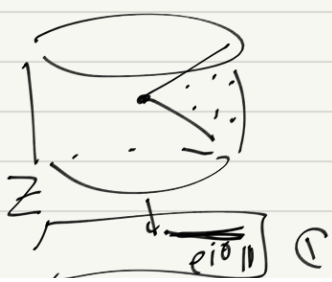
What happens when we cross Stokes direction? We get two different canonical identifications.



$$\bigoplus_{j\in\pi_0(\mathrm{Zeroes}(\alpha_{\mathbb{C}}))} H^\bullet(Z_j, \varphi_f \mathbb{Z}_X)\otimes K \;\underset{\theta_+}{\overset{\theta_-}{\rightrightarrows}}\; H^\bullet(X,\mathcal{L})$$

with both arrows isomorphisms ($\sim$).

$\Longrightarrow$ the ratio is an element of $\displaystyle\bigoplus_{j\in\pi_0(\mathrm{Zeroes}(\alpha_{\mathbb{C}}))} \bigoplus_{i=0}^{2\dim_{\mathbb{C}} X} \mathrm{Aut}\left(H^i(Z_j, \varphi_f \mathbb{Z}_X)\otimes \mathbb{Z}[[C_\theta]]\right)$



$C_\theta \subset \Gamma := H_1(X, \mathrm{Zeroes}(\alpha_{\mathbb{C}}); \mathbb{Z})$ a strict convex submonoid

$$Z(C_\theta) \subset e^{i\theta}\mathbb{R}_{\geq 0}$$

Claim: we get a stability structure in Γ- graded Lie algebra (more precisely, a local system of graded algebras over S^1_θ)

$$\mathrm{Mat}(N \times N, \mathbb{Z}[T_1^{\pm 1}, \dots, T_k^{\pm 1}] \qquad k = \mathrm{rk}\, \Gamma$$

$N = \#$ of critical points (in the holomorphic Morse case)

or, in the general case for given $i \in \{0, \dots, 2\dim_{\mathbb{C}} X\}$: $\quad N = \sum_j \mathrm{rk}\, H^i(Z_j, \phi_f \mathbb{Z}_X)$

Wall-crossing formula: if we vary $(X, \alpha_{\mathbb{C}})$ preserving $\alpha_{\mathbb{C}}$ near $\mathrm{Zeroes}(\alpha_{\mathbb{C}})$.

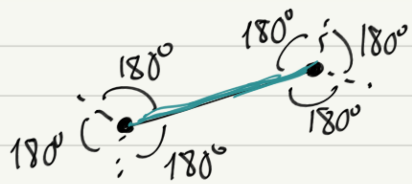
Remark: one can generalize the whole story to complex-valued closed 1-forms on non-compact manifolds, need some some constraints on the behaviour at infinity.

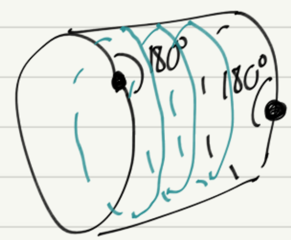
Apply to quadratic differentials on complex curves
(get abelian differentials on ramified double covers)

$$\beta \in \Gamma(C, K_C^{\otimes 2}) \rightsquigarrow \alpha_\sigma = \beta^{1/2} \in \Gamma(\tilde{C}, K_{\tilde{C}})$$

$$X := \tilde{C} \overset{2:1}{\longrightarrow} C$$

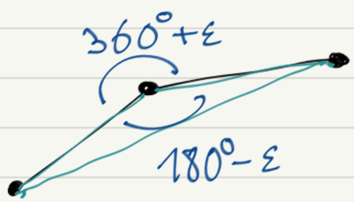
Traditional story (works only for quadratic differentials with simple zeroes and arbitrary poles), - DT invariants for A-model on the associated noncompact 3CY variety (Bridgeland-Smith, Haiden): BPS states correspond to saddle connections and cylinders filled by closed geodesics.



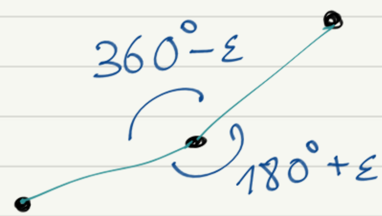


$\Rightarrow$ Stability structure in the $\mathbb{Z}^{2g}$-graded Lie algebra of Hamiltonian vector fields on the symplectic algebraic torus $(\mathbb{C}^\times)^{2g}$

Two events (varying curve with quadratic differential):

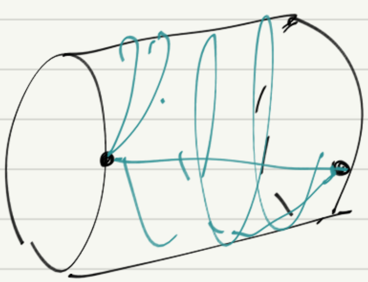


3 saddle connections

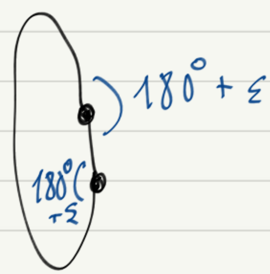


2 saddle connections

Two different wall-crossing identities: in hamiltonian vector fields, and in matrix-valued Laurent polynomials



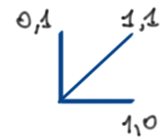
infinitely many saddle connections, and a cylinder filled by closed geodesics

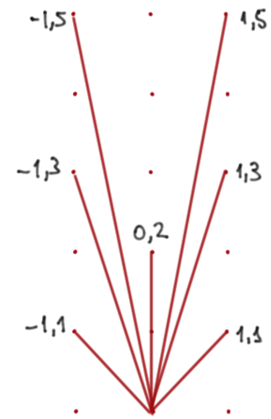


2 saddle connections

$$T_{a,b} : (x,y) \mapsto \left(x \cdot (1-(-1)^{ab}x^a y^b)^{-b}, y \cdot (1-(-1)^{ab}x^a y^b)^{a}\right)$$

$$\Longrightarrow \; T_{1,0} \circ T_{0,1} = T_{0,1} \circ T_{1,1} \circ T_{1,0} \,,$$





$$T_{1,1} \circ T_{-1,1} = T_{-1,1} \circ T_{-1,3} \circ T_{-1,5} \circ \cdots \circ T_{0,2}^2 \circ \cdots \circ T_{1,5} \circ T_{1,3} \circ \cdots \circ T_{1,1}$$

A_2

$$\begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & xy\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$\hat{A}_1$

$$\begin{pmatrix}1 & xy\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 0\\ -x^{-1}y & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ -x^{-1}y & 1\end{pmatrix} \cdot \begin{pmatrix}1 & 0\\ -x^{-1}y^3 & 1\end{pmatrix} \cdot \cdots \cdot \begin{pmatrix}1-y^2 & 0\\ 0 & \frac{1}{1-y^2}\end{pmatrix} \cdot \cdots \cdot \begin{pmatrix}1 & xy^3\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}1 & xy\\ 0 & 1\end{pmatrix}$$

How can we calculate all these numbers of saddle conbections "at once"?

Key notion: generalized square-tiled surfaces

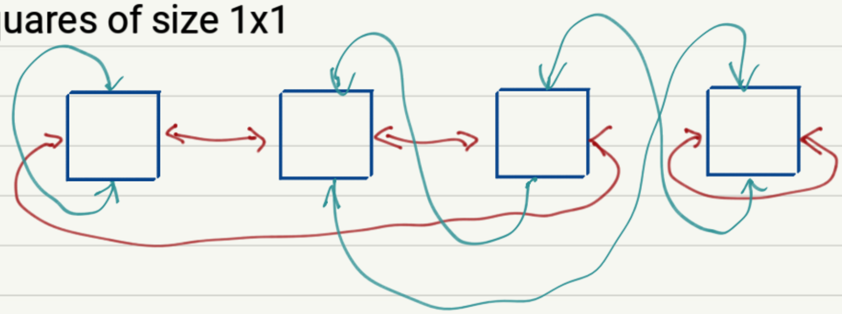
Want $[\alpha_{\mathbb{C}}] \in H^1(X, \mathrm{Zeroes}(\alpha_{\mathbb{C}}); \mathbb{C})$ to belong to the image of

$$H^1(X, \mathrm{Zeroes}(\alpha_{\mathbb{C}}); \mathbb{Z}+\sqrt{-1}\mathbb{Z}).$$

$$\Longleftrightarrow \quad \begin{tikzcd} Z: H_1(X, \mathrm{Zeroes}(\alpha_{\mathbb{C}})) \arrow[rr] \arrow[dr] & & \mathbb{C} \\ & \mathbb{Z}+\sqrt{-1}\mathbb{Z} \arrow[ur, hook] & \end{tikzcd}$$

Example: X is curve of genus 2, $\alpha_{\mathbb{C}}$ is an abelian differential (holomorphic 1-form) with two simple zeroes.

minimal square tiled: 4 squares of size 1x1



almost vertical

almost horizontal

$\rightsquigarrow$ Two rational identifications $H^\bullet(\text{Zeroes}, \varphi_f \mathbb{Z}_X) \otimes K \simeq H^\bullet(X, \mathcal{L})$

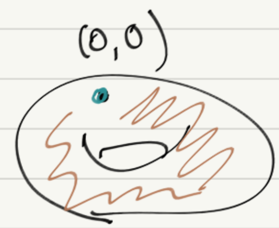
$\rightsquigarrow \Pi$ Automorphisms associated with Stokes directions $\in GL(2,$ rational functions$)$

$$\begin{bmatrix} 1+\frac{(a^3f-a^2f)(b^2g-b^3g)+(a-a^3f)(b^3g-b)}{(1-a^3f)(1-b^3g)} & \frac{(a-a^3f)b^2+(a^3f-a^2f)b}{(1-a^3f)(1-b^3g)}-\frac{(a^2f-af)b}{(1-a^3f)(1-b)} \\ \frac{a^2(b^3g-b)+a(b^2g-b^3g)}{(1-a^3f)(1-b^3g)}-\frac{a(bg-b^2g)}{(1-a)(1-b^3g)} & 1+\frac{a^2b^2+ab}{(1-a^3f)(1-b^3g)}-\frac{a^3fb}{(1-a^3f)(1-b)}-\frac{ab^3g}{(1-a)(1-b^3g)} \end{bmatrix}$$

In general, if $Z_{\alpha_{\mathbb{C}}} : \Gamma = H_1(X, \mathrm{Zeroes}(\alpha_{\mathbb{C}}); \mathbb{Z}) \xrightarrow[\int \alpha_{\mathbb{C}}]{} \mathbb{Z} + \sqrt{-1}\,\mathbb{Z} \subset \mathbb{C}$

$$\rightsquigarrow \int_{x_0 \in \mathrm{Zeroes}(\alpha_{\mathbb{C}})}^{?} \alpha_{\mathbb{C}} : X \xrightarrow{pr} S^1 \times S^1$$

$$\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$



the only one critical value

fibration outside (0,0).

$$H^\bullet(X, \mathbb{Z}) = \mathbb{H}^\bullet(S^1 \times S^1; pr_* \mathbb{Z})$$

constructible sheaf on $(S^1 \times S^1, (0,0))$

$$\pi_1(S^1 \times S^1 - (0,0)) = \mathrm{Free}\langle A, B \rangle$$

V_g : generic stalk

V_s : special stalk

$A, B \in \mathrm{Aut}(V_g)$ $\quad j : V_s \longrightarrow V_g^{ABA^{-1}B^{-1}}$

$R\Gamma(S^1 \times S^1, pr_* \mathcal{L})$:

contractible subcomplexes for generic A,B

$$\begin{tikzcd}
 & V_g \arrow[dr, "1-A"] & \\
V_s \arrow[ur, "(1-B)\circ j"] \arrow[dr, "-(1-A)\circ j"'] & & V_g \\
 & V_g \arrow[ur, "1-B"'] &
\end{tikzcd}$$

$$0 \qquad\qquad 1 \qquad\qquad 2$$

$\Rightarrow \quad R\Gamma(S^1 \times S^1, pr_* \mathcal{L})$

qis $\wr\wr$ $\qquad\qquad$ $\wr\wr$ qis

$$V_s \xrightarrow{(1-B)\circ j} V_g \qquad\qquad V_s \xrightarrow{-(1-A)\circ j} V_g$$

$$0 \qquad\qquad 1$$

assume for simplicity $V_s = 0$

$\Rightarrow$ comparison of identifications = $(1-B)^{-1}(1-A)$

If $(1-B)$ and $(1-AB)$ are both invertible then

$$(1-B)^{-1}(1-A) = (1-AB)^{-1}(1-A) \cdot (1-B)^{-1}(1-BA)$$

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version: $(1-AB)$ and $(1-BA)$ invertible (like in vanishing/nearby cycles) then

$$(1-A)(1-BA)^{-1}(1-B) = (1-B)(1-AB)^{-1}(1-A)$$

New meaning: a nooncommutative expression which is *symmetric* in A, B!

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rewriting:

$$h(B,A) = h(AB,A) \cdot h(B,BA), \quad h(B,A) := (1-B)^{-1}(1-A)$$

Iterate:

$$\begin{gathered} h(B,A) = \\ h(AB,A)h(B,BA) = \\ h(AAB,A)h(AB,ABA)h(BAB,BA)h(B,BBA) = \\ \dots \end{gathered}$$

$$h(\textcolor{red}{\frac{0}{1}}, \textcolor{blue}{\frac{1}{1}}) =$$

$$h(\textcolor{red}{\frac{1}{2}}, \textcolor{blue}{\frac{1}{1}})h(\textcolor{red}{\frac{0}{1}}, \textcolor{blue}{\frac{1}{2}}) =$$

$$h(\textcolor{red}{\frac{2}{3}}, \textcolor{blue}{\frac{1}{1}})h(\textcolor{red}{\frac{1}{2}}, \textcolor{blue}{\frac{2}{3}})h(\textcolor{red}{\frac{1}{3}}, \textcolor{blue}{\frac{1}{2}})h(\textcolor{red}{\frac{0}{1}}, \textcolor{blue}{\frac{1}{3}}) =$$

$$h(\frac{3}{4},\frac{1}{1})h(\frac{2}{3},\frac{3}{4})h(\frac{3}{5},\frac{2}{3})h(\frac{1}{2},\frac{3}{5})h(\frac{2}{5},\frac{1}{2})h(\frac{1}{3},\frac{2}{5})h(\frac{1}{4},\frac{1}{3})h(\frac{0}{1},\frac{1}{4}) = \\ \cdots$$

where

$$\frac{0}{1} := B, \quad \frac{1}{1} := A$$

$$\frac{p+p'}{q+q'} := \frac{p'}{q'}\cdot\frac{p}{q}, \qquad \frac{p+p'}{q+q'} := \frac{p}{q}\cdot\frac{p'}{q'}$$

Replace approximately
$h(B,A) \rightsquigarrow \tilde{h}(B,A) := (1-A)(1-B)^{-1}$

$$h(\frac{3}{4},\frac{1}{1})h(\frac{2}{3},\frac{3}{4})h(\frac{3}{5},\frac{2}{3})h(\frac{1}{2},\frac{3}{5})h(\frac{2}{5},\frac{1}{2})h(\frac{1}{3},\frac{2}{5})h(\frac{1}{4},\frac{1}{3})h(\frac{0}{1},\frac{1}{4}) \rightsquigarrow$$

$$\tilde{h}(\frac{3}{4},\frac{1}{1})\tilde{h}(\frac{2}{3},\frac{3}{4})\tilde{h}(\frac{3}{5},\frac{2}{3})\tilde{h}(\frac{1}{2},\frac{3}{5})\tilde{h}(\frac{2}{5},\frac{1}{2})\tilde{h}(\frac{1}{3},\frac{2}{5})\tilde{h}(\frac{1}{4},\frac{1}{3})\tilde{h}(\frac{0}{1},\frac{1}{4}) =$$

$$(1-\frac{1}{1})(1-\frac{3}{4})^{-1}(1-\frac{3}{4})(1-\frac{2}{3})^{-1}(1-\frac{2}{3})\ldots(1-\frac{0}{1})^{-1} =$$

$$(1-\frac{1}{1})\frac{3}{4}\frac{2}{3}\frac{3}{5}\frac{1}{2}\frac{2}{5}\frac{1}{3}\frac{1}{4}(1-\frac{0}{1})^{-1}$$

where

$$\frac{p}{q} := (1-\frac{p}{q})^{-1}(1-\frac{p}{q})$$

In the limit

$$h(\frac{0}{1}, \frac{1}{1}) = (1 - \frac{1}{1}) \cdot \prod_{\frac{p}{q} \in (0,1) \searrow} \frac{p}{q} \cdot (1 - \frac{0}{1})^{-1}$$

Proof of $(1-A)(1-BA)^{-1}(1-B) = (1-B)(1-AB)^{-1}(1-A)$:

invert:

$$(1-B)^{-1}(1-BA)(1-A)^{-1} \stackrel{?}{=} (1-A)^{-1}(1-AB)(1-B)^{-1}$$

denote $X := 1 - A, \quad Y := 1 - B$:

$$Y^{-1}(1-(1-Y)(1-X))X^{-1} \stackrel{?}{=} X^{-1}(1-(1-X)(1-Y))Y^{-1}$$

$$Y^{-1}(Y + X - YX)X^{-1} \stackrel{?}{=} X^{-1}(X + Y - XY)Y^{-1}$$

$$X^{-1} + Y^{-1} - 1 = Y^{-1} + X^{-1} - 1 \quad \blacksquare$$

General framework: holomorphic Floer theory, quantization, resurgence
(ongoing project: M.K. & Y. Soibelman & ...)

$(M_{\mathbb{C}}, \omega_{\mathbb{C}})$ complex algebraic symplectic manifold
(? symplectic leaf of a compact Poisson variety ?)

$L_{\mathbb{C}} \subset M_{\mathbb{C}}$ an algebraic Lagrangian subvariety (possibly singular)

$\rightsquigarrow Z : \Gamma := H_2(M_{\mathbb{C}}, L_{\mathbb{C}}; \mathbb{Z}) \xrightarrow{\int \omega_{\mathbb{C}}} \mathbb{C}$

$\rightsquigarrow \forall \hbar \in \mathbb{C}^{\times}$

$0 < |\hbar| \ll 1$

$\mathrm{Fukaya}_{local}(U_{\varepsilon}(L_{\mathbb{C}}); \frac{1}{\hbar}\omega_{\mathbb{C}}) \hookrightarrow \mathrm{Fukaya}(M_{\mathbb{C}}, \frac{1}{\hbar}\omega_{\mathbb{C}})$

fully faithful embedding

complex Planck constant such that $\hbar \notin$ Stokes ray

$\Leftrightarrow \mathrm{Arg}\,\hbar \neq \mathrm{Arg}\,\underbrace{Z(\gamma)}_{\neq 0} \quad \forall \gamma \in \Gamma$

Reason: no pseudo-holomorphic discs with boundary on $L_{\mathbb{C}}$

Wall-crossing structure: countable dimensional Γ- graded Lie algebra $\mathfrak{g}$:

$$\tilde{\mathfrak{g}} := HH^1\left(\underbrace{\mathrm{Fukaya}_{\mathrm{local}}\left(U_\varepsilon(L_{\mathbb{C}}), \tfrac{1}{\hbar}\omega_{\mathbb{C}}\right)}\right) / \mathbb{Q} \simeq \bigoplus_{\tilde{\gamma} \in H_1(L_{\mathbb{C}}, \mathbb{Z})} \tilde{\mathfrak{g}}_{\tilde{\gamma}}$$

category over $\mathbb{Q}$ of finite type

acts $\circlearrowright$ $\otimes$ rk=1 local systems on $L_{\mathbb{C}}$

$(\mathbb{C}^\times)^{\mathrm{rk}\, H_1(L_{\mathbb{C}})}$

$$\mathfrak{g} := \bigoplus_{\gamma \in H_2(M_{\mathbb{C}}, L_{\mathbb{C}}; \mathbb{Z})} \mathfrak{g}_\gamma \qquad \mathfrak{g}_\gamma := \tilde{\mathfrak{g}}_{\partial(\gamma)} \qquad \partial : H_2(M_{\mathbb{C}}, L_{\mathbb{C}}; \mathbb{Z}) \to H_1(L_{\mathbb{C}}, \mathbb{Z})$$

boundary map

Specializes to hamiltonian vector fields on $H^1(L_{\mathbb{C}}, \mathbb{C}^\times)$ $\rightsquigarrow$ "usual" DT invariants

if $L_{\mathbb{C}}$ is smooth

Special case: $M_{\mathbb{C}} = T^* X_{\mathbb{C}} \qquad L_{\mathbb{C}} = L_1 \cup L_2$

$$\left.\begin{array}{l} L_1 = \text{zero section} = X_{\mathbb{C}} \subset T^* X_{\mathbb{C}} \\ L_2 = \operatorname{graph}(\alpha_{\mathbb{C}}) \subset T^* X_{\mathbb{C}} \end{array}\right\}$$

both are unobstructed for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$

endow L_1 with the trivial local system of rank 1, and L_2 with a generic local system of rank 1

$$\begin{array}{ccccc} \mathrm{Fukaya}_{\mathrm{local}}(U_\varepsilon L_1) & \hookrightarrow & \mathrm{Fukaya}(U_\varepsilon(L_1 \cup L_2)) & \hookleftarrow & \mathrm{Fukaya}_{\mathrm{local}}(U_\varepsilon L_2) \\ & & \downarrow & & \\ & & \mathrm{Fukaya}(M_{\mathbb{C}}) & & \end{array}$$

$\theta \neq$ Stokes

$$\mathrm{Hom}_{\mathcal{F}(L_1 \cup L_2)}(L_1, (L_2, \rho)) \simeq H^\cdot(\mathrm{Zeroes}(\alpha_{\mathbb{C}}), \text{vanishing cycles}) \simeq \mathrm{Hom}_{\mathcal{F}(T^* X_{\mathbb{C}})}(L_1, (L_2, \rho)) \simeq H^\cdot(X_{\mathbb{C}}, \mathbb{Z})$$