Topological String on Non-Commutative Resolution

Work in with Katz, Schimannek, Sharpe work in progress
T.S. 2108-[illegible]

Motivation: One parameter CY W $b_3(W)=4$ 17[illegible]

Picard Fuchs $\to$ AESZ List on 10^3 CY operators $\mathcal{L}^{(4)}$ 1704.00[illegible]

How to intepret their degerations?...

Simplest #K+ are hypergeometric

$\Theta = z\frac{d}{dz}$ $\quad \mathcal{L}^{(4)} = \Theta^4 - \mu^{-1} z \prod_{k=1}^{4}(\Theta + a_k)$ $\quad \{\mu^{(i)}, \{a_k^{(i)}\}\}$ $i=1$

Eg $\mu^{-1} = 5^5$ $\quad a_k = \frac{k}{5}$ $\quad k=1,2,3,4$ Quintic

Riemann Symbol $\qquad \mathcal{M}_{cs}(W) = \mathbb{P}^1 \setminus \{0, \mu, \infty\}$

$$\mathcal{P}\begin{pmatrix} z=0 & \mu & \infty \\ 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 2 & a_5 \end{pmatrix}$$

completly degenerate — Mum Point m

1 log deg — Conifold c

depend on $\{a_k\}$ — Orbifold-, Conifold-, K-Mum points

Main players today: $\mathcal{M}_1$ $\quad \mu^{-1} = 2^{16}$ $\quad \{a\} = \{1/8, 3/8, 5/8, 7/8\}$
$\Downarrow$
$z=\infty$
$\mathbb{Z}_8$-orbifold

$\mathcal{M}_2$ $\quad \mu^{-1} = 2^8$ $\quad a_k = \frac{1}{2}$ $\forall k$
$z=\infty$

Object of Key Interest in most of the talk $\hookrightarrow$ 2 Mum point

Geometry: M_1 double cover of $\mathbb{P}^3(\underline{x})$
branched at deg 8 pol

$$w^2 = p_8(\underline{x})$$

$$h_{21} = \binom{8+4-1}{8} - \underset{\substack{\uparrow \\ \# GL(4)}}{16} = 149, \quad h_{11} = 1 \quad \Rightarrow \chi = -296$$

Mirror W_1 e.g. by Batyrev's construction

M_2: Vanishing locus of 4 qudric in $\mathbb{P}^7$

$$\chi = 2(h_{11} - h_{21}) = -128 \quad h_{11} = 1 \quad \Rightarrow h_{21} = 65$$

Mirror W_2: $\widehat{M_2/(\mathbb{Z}/4\mathbb{Z})^3}$: Invariant constraints

$$P_j = x_i^2 + y_j^2 - 2\psi\, x_{i+1} x_{j+1} \qquad j \in \mathbb{Z}/(4\mathbb{Z})$$

$$\left(z = \frac{1}{(2\psi)^8}\right)$$

Recall Standard Mum point in $W \Leftrightarrow$ large volume geometry in some Mirror M $\quad \mathcal{L}\Pi = 0$

$$\Pi = X^0(z(t)) \begin{pmatrix} 2F_0 - t\partial_t F_0 \\ \partial_t F_0 \\ 1 \\ t \end{pmatrix} \leftarrow \text{triple log}$$

genus 0

$$F_0 = \text{Cubic}(t) + \sum_{\beta \in H_2(M,\mathbb{Z})}{}' n_0^\beta \, \mathrm{Li}_3(Q^\beta), \qquad Q = e^t$$

E.g. for W_2: $X^0 = \sum_{n=0}^{\infty} \binom{2n}{n}^4 z^n$

$t = \frac{X_1}{X_0} \leftarrow$ log sol

higher genus $N_{\beta,\chi} = \#[1, 0, \beta, \chi]$ DT invariant

$\Uparrow$ MNOP

$$F(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = C(t,\lambda) + \sum_{g=0}^{\infty} \sum_{\substack{\beta \in H_2 \\ \neq 0}} \left(\sum^{K-1}\right) \sum_{m=1}^{\infty} \frac{n_g^\beta}{m} \Big(2\sinh \underset{\substack{\uparrow \\ \text{asymptotic}}}{\lambda \frac{m}{2}}\Big)^{2g-2} \underset{\substack{\uparrow \\ \text{convergent}}}{Q^{\beta \cdot m}}$$

$e^{2\pi i m b/K}$

Arguments (apart from wall crossing) why n, N should not be understand locally in $\mathcal{M}_{cs}(W)$

1) Way to compute them: OG 12125 HKQ

- Holomorphic morphic anomaly
- BCOV almost holomorphic "modular ring"
- discrete boundary condition cusp behaviour,

· gap at conifold $\quad F_g(t_D) = \frac{(-1)^{g-1} B_{2g}}{2g(2g-2)\, t_D^{2g-2}} + \mathcal{O}(t_D^0)$

· regularity at orbifold, etc.

2) $\mathcal{M}_2^{I}$ $\quad n_0^{\beta} \underset{\beta \to \infty}{\sim} \left(\frac{2\pi\, X(\mu)}{t(\mu)} \right) \frac{e^{2\pi \beta\, t(\mu)}}{k^3 \log^2 k}$ $\quad$ non-archimedian analysis in $\mathcal{M}_{cs}(W)$

$\omega_0 = 1.07072$ $\quad$ [2203.09426] BKSZ

$t = 1.20812$

$$X_0(\mu) = -64\, L(f,2)/(2\pi)^2 = \frac{\omega_p^-}{8(2\pi i)^3} = 1.118636$$

$$\left[X_1(\mu) = \frac{16\, L(f,1)}{2\pi} = i \frac{\omega_p^+}{(2\pi i)^3} = 0.902728678 \right]$$

$$t(\mu) = \frac{16\, L(f,2)}{2\pi} / X_0(\mu)$$

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} a_n \chi(n)\, n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + \chi(p)\, p^{4-1-2s}}$$

$f = q - 4q^3 - 2q^5 + \dots \in S_4(\Gamma_0(8))$ $\qquad W_2 f = -1$

Hecke eigen form

f follows from local Hasse Weyl zeta function

$$Z(\mathcal{M}_2^{p}, p) = \exp\left(\sum_{n=1}^{\infty} \# \mathcal{M}_2^{p}(\mathbb{F}_{p^n}) \frac{T^n}{n} \right)$$

$$= \prod_{r=0}^{2d} P_r(X_p, T)^{(-1)^{r-1}}$$

Dwork: $[\nabla_{GW}, F_{rp}^{*}] = p$

$\deg P_r = b_r$ $\quad$ can be calculated from non-archimedean p-adic analys of the periods

$$P_3(W_2/\mathbb{F}_p, T) = \det(1 - T\, F_{rp}^{*}\, H^3(\overline{W_2}, \mathbb{Q}_\ell)) = 1 + \alpha_p T + \beta_p p T^2 + \alpha_p p^3 T^3 + p^6 T^4$$

$$= \big|_{z=\mu} (1 - \chi(p))(1 - a_p T + p^3 T^2)$$

Nature of the 2nd MUM point of $\mathcal{M}_2^{II}$

Claim: $\mathcal{M}_2^{II}$ corresponds to a non-perturbative resolution of the following degeneration of $\mathcal{M}_1$

$$w^2 = \det A_{8\times 8}(\underline{x})$$
↖ symmetric

M_1 has ordinary double points iff corank of $A_{n\times n}$ $r \geq 1$

$$[\det A]_r = \left(\prod_{k=0}^{r-1} \binom{n+k}{r-k} \Big/ \binom{2k+1}{k} \right) c_1(L)^{\binom{r+1}{2}} = 84 \cdot 3^3$$

$2 \cdot 42 = 2 \cdot 2 \cdot 3 \cdot 7$

$k=0$
28

$\Rightarrow n_s = 84$ terminal singularities

$\Rightarrow$ only small analytic res

How many complex structure deformations preserve the nodes?

$$A \to S^T A S$$
↖ $SL(8,\mathbb{C})$

$$n^s_{def} = \frac{8\cdot 9}{2}\cdot 4 - (8\cdot 8 - 1) - 16 = 65 = h_{21}(\mathcal{M}_2)$$

$$n^s_{def} + n_s = 149$$
↖ deformation localized at each node

$\mathcal{M}_1^{sing}$ $\mathbb{Q}$-factorial

Clemens & J. Werner's thesis theorems imply

"Double solids" 83

that in $\hat{\mathcal{M}}_1 = X$ each $[C_p] \cong \mathbb{P}^1$, $p = 1 \ldots 84$ are $\mathbb{Z}/2\mathbb{Z}$ torsion classes

$\Rightarrow$ Non-trivial flat B-field $\cong H^2(X, U(1))$ in the presence of Torsion

Bockstein Homomorphism

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}) \to H^2(X, U(1)) \xrightarrow{c} H^3(X,\mathbb{Z}) \to H^3(X,\mathbb{R})$$

$c(B) = \varsigma \neq 0 \Rightarrow$ non trivial B-field

$$\{ B + iJ \mid B \in H^2(X,\mathbb{R}) / H^2(X,\mathbb{Z})_\varsigma \,,\ J \gg 0 \}$$

Assume $H^3(X,\mathbb{Z})_{\text{tors}} = \mathbb{Z}_K$ $\quad \tilde{B} = B - B'' \quad k\tilde{B} = 0$

$\forall$ curve classes $\beta \in H_2(X,\mathbb{Z})$ we get a possible weight

$$\langle \tilde{B}, \beta \rangle = e^{2\pi i b/K} \qquad b = 0, \ldots, K-1$$

$\Rightarrow n_g^{\beta, b}$ $\quad \beta \in H_2(X,\mathbb{Z})$ decomposes in $b = 0, 1, \ldots, K-1$ sectors

and the BPS Formula changes as indicated.

Let us look at the constant map $\beta = 0$ of $(*)$

E.g. $g = 0$ we have $n_0^0 = \frac{\chi}{2}$ $\Rightarrow$

$$(*) \Rightarrow F_0(X)\Big|_{\text{const}} = -\frac{\overset{\overset{\displaystyle Li_3(1)}{\shortparallel}}{\zeta(3)}}{(2\pi i)^3} \frac{\chi}{2}$$

For $M_1^{K=2}$ $\quad b = 0, 1$ we get two sectors

$$Li_3(-1) = -\frac{3}{4}\zeta(3)$$

$$F_0(M_1^{K=2})\Big|_{\text{const}} = -\frac{\zeta(3)}{(2\pi i)^3}\left(\frac{\chi}{2} + \frac{3}{4} n_s\right) \quad \text{result of analytic cont}$$

$g \geq 2$ $\quad (*)$ $\quad F_g\Big|_\beta = (-1)^{g+1} \frac{B_{2g}}{2g\,(2g-2)!} Li_{3-2g}(q^\beta)\, n_g^\beta$

$$Li_{3-2g}(-1) = \frac{1-2^{2g-2}}{2g-2} B_{2g-2}$$

$$F_g(M_1^{K=2})\Big|_{\beta=0} = (-1)^g \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}\left(\frac{\chi}{2} + (1-2^{2g-2}) n_s\right)$$

- The latter is a nontrivial boundary condition for $F_g(t)$.

- In the model M_2 one can use the Wave function transformation to evaluate $F(\lambda, t)$ at $z=0$ (Mum I), $z = 2^8$, $z = \infty$ (Mum II)

- Interesting subtlety: The generators of the a.h. modular Ring can be taken to be BCOV propagators [0708.2886 AL]

E.g. $S^{\varphi\varphi} \subset \Gamma\left(\mathrm{Sym}^2(T^{1,0}M) \otimes \mathcal{L}^{-2}\right)$ Kähler Line bundle

at Mum II with local exponents $\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$

We need $\varphi = z^{-1/2}$

$$(\mathcal{L} \otimes \bar{\mathcal{L}}) \Rightarrow e^{K} \to e^{K + \sqrt{z} + \sqrt{\bar{z}}} \Rightarrow$$ induces transformation of the global ambiguitie

to enjoy all boundary con

$\Rightarrow$ enumerative result p. 10 zoom

$$n_g^{\beta 0}(M_2^{II}) + n_g^{\beta 1}(M_2^{II}) = n_g^{\beta}(M_1) \qquad n^{\beta}_{g_{max}-1}$$

$g=2$, $\beta=2$

$$504 + 360 = 864 \;\text{"="}\; (-1)^{\dim M + 1}\left(e(C) + (2g-2)\, e(M)\right) = 864$$

Checks based on minimal assumptions and geometric interpretation of GV invariants by $SL(2)_L \otimes SL(2)_R$ Lefshetz action on Jacobian Fibration out Deformation space of $g=$ curve made by Sheldon Katz

Remarks : 1)

Rødland example: two canonical Mum

$$D^b(\mathcal{M}^{I}_{Rød}) = D^b(\mathcal{M}_{Rød}^{II}) \qquad \text{equivalence}$$

Pfaffian: rank 4 locus of 7×7 matrix in $\mathbb{P}^{20}$

HS in $G(2,7)$

Our example:

$$D^b(\mathcal{M}_2) \cong D^b(\hat{\mathcal{M}}^s_1, b)$$

homological projective dual 2008

Addington

Kuznetsov Ad Math

Căldăraru

$$D^b(\mathbb{P}^3, S) \Longleftrightarrow D^b(\mathcal{M}_2)$$

Clifford Algebra

2) One can make Kähler, but non CY blow ups $c_1(\hat{\hat{\mathcal{M}}}{}^s_1) \neq 0$. Conjecture R.P. GW theory makes sense for curves β

$$\int_\beta c_1(\hat{\mathcal{M}}_1) = 0 \quad \Rightarrow \text{vir dim } \mathcal{M}_{\beta g}(\hat{\mathcal{M}}_1) = 0$$

3) Torsion and non-commutative Resolutions are prevalent in genus one fibrations

$[\mathbb{P}^1]_{\mathbb{Z}_N}$ appear in the analytic resolution of singular Jacobian fibrations $K = 2, 3, 4, 5$

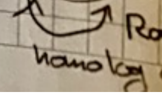
[*] [2108.09311] Thorsten Instantons p 87 [*] $K = 5$

Elliptic curve Don Zagier AMS 2009 and $\Gamma_1(5)$ with discr. $\Delta = (1 - 11 z_1 - z_1^2)$

monodromy

PF ops and expansions p 64 [*] $X^{540}(11169)$

$x_0 = \pm 1$ $1/5$ $4/5$ $2/5$ $3/5$



Mum point

$\cong$ Crop

Rødland homolog dual

$n_g^{\beta,0}$	$\beta=1$	2	3	4	5	6
$g=0$	14752	64415616	711860273440	11596528004344320	233938237312624658400	5403936140181888393638272
1	0	20160	10732175296	902646044328864	50712027457008177856	2461377693242784849884352
2	0	504	−8275872	6249833130944	2700746768622436448	376922599978113825644184
3	0	0	−88512	−87429839184	10292236849965248	19650836158735384901936
4	0	0	0	198065872	−337281112359424	127720125422251398968
5	0	0	0	157306	6031964134528	−1760771999464321184
6	0	0	0	1632	−43153905216	72538234118612304
7	0	0	0	24	18764544	−2014447575952656
8	0	0	0	0	177024	33618983785016
9	0	0	0	0	0	−268869372720
10	0	0	0	0	0	459490472
11	0	0	0	0	0	238896
12	0	0	0	0	0	4536
13	0	0	0	0	0	0

Table 1. Non-vanishing Gopakumar-Vafa numbers for the non-commutative resolution of degenerate double cover of $\mathbb{P}^3$ (denoted $X_8(1^4,4)$) with $\mathbb{Z}_2$ charge 0 up to $\beta=6$, calculated at the $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ MUM point of four quadrics in $\mathbb{P}^7$ (denoted $X_{2,2,2,2}(1^9)$).

$n_g^{\beta,1}$	$\beta=1$	2	3	4	5	6
$g=0$	14752	64419296	711860273440	11596528020448992	233938237312624658400	5403936140182066358700128
1	0	21152	10732175296	902646048376992	50712027457008177856	2461377693242840073388128
2	0	360	−8275872	6249834146800	2700746768622436448	376922599978124915164776
3	0	6	−88512	−87429664640	10292236849965248	19650836158738175255854
4	0	0	0	198149928	−337281112359424	127720125422926728320
5	0	0	0	144144	6031964134528	−1760771999299008316
6	0	0	0	2520	−43153905216	72538234051130296
7	0	0	0	0	18764544	−2014447551562602
8	0	0	0	0	177024	33618973175488
9	0	0	0	0	0	−268866517172
10	0	0	0	0	0	458933912
11	0	0	0	0	0	284538
12	0	0	0	0	0	2496
13	0	0	0	0	0	40

Table 2. Analog GV numbers as in Table 1 but for $\mathbb{Z}_2$ charge 1. Note that $\sum_{q=0}^{1} n_g^{\beta,q}(X_{nc}) = n_g^{\beta}(X_8(1^4,4))$ $\forall g,\beta$, the latter are tabulated in [12], and 14752 is the number of lines in $\mathbb{P}^3$ that are 4 times tangent to a generic degree 8 surface [13, 14]. Further note that the Castelnovo bound for $n_g^{\beta,i}(X_{nc})$ together with the one $n_g^{\beta}(X_{2,2,2,2}(1^9))$ at north pole of the same moduli space $\mathcal{M}_{cs}(X_{2,2,2,2}(1^9)) = \mathbb{P}^1 \setminus \{0, z_{con}, \infty\}$ listed also in [12] provide additional boundary data to fix the holomorphic ambiguities and evaluate $n_g^{\beta,i}(X_{nc})$, $n_g^{\beta}(X_{2,2,2,2}(1^9))$ and $n_g^{\beta}(X_8(1^4,4))$ to higher genus then before.

Following 3 pages from https://arxiv.org/abs/2108.09311 by T. Schimmanek:

$n^{(g)}_{(d_1,1),0}$	$g=0$	1	2	3	4
$d_1=0$	3	0	0	0	0
1	−180	−6	0	0	0
2	29270	342	9	0	0
3	40818484	−57484	−492	−12	0
4	4354611955	−81811190	85065	630	15
5	215303747352	−8954366490	122915960	−112040	−756

Table 37: Base degree 1 Gopakumar-Vafa invariants for $X_0^{(5)}$ with $\mathbb{Z}_5$ charge 0.

$n^{(g)}_{(d_1,1),\pm1}$	$g=0$	1	2	3	4
$d_1=0$	0	0	0	0	0
1	−200	0	0	0	0
2	28850	400	0	0	0
3	40815550	−56500	−600	0	0
4	4354596400	−81802600	83350	800	0
5	215303680450	−8954314100	122899250	−109400	−1000

Table 38: Base degree 1 Gopakumar-Vafa invariants for $X_0^{(5)}$ with $\mathbb{Z}_5$ charge ±1.

$n^{(g)}_{(d_1,1),\pm2}$	$g=0$	1	2	3	4
$d_1=0$	0	0	0	0	0
1	−250	0	0	0	0
2	28200	500	0	0	0
3	40810800	−54900	−750	0	0
4	4354571400	−81788800	80600	1000	0
5	215303571950	−8954229700	122872350	−105300	−1250

Table 39: Base degree 1 Gopakumar-Vafa invariants for $X_0^{(5)}$ with $\mathbb{Z}_5$ charge ±2.

References

[1] M. Kreuzer and H. Skarke, "Complete classification of reflexive polyhedra in four-dimensions," *Adv. Theor. Math. Phys.* **4** (2002) 1209–1230, `arXiv:hep-th/0002240`.

[2] P. Candelas, A. Dale, C. Lütken, and R. Schimmrigk, "Complete intersection calabi-yau manifolds," *Nuclear Physics B* **298** no. 3, (1988) 493–525, `https://www.sciencedirect.com/science/article/pii/0550321388903525`.

C.3 $X_0^{(5)}$

$n^{(0)}_{(d_1,d_2),0}$	$d_2=0$	1	2	3	4
$d_1=0$	0	3	-6	27	-192
1	90	-180	450	-2880	25740
2	90	29270	-117260	1030120	-11796650
3	90	40818484	14923920	-182269500	2713157820
4	90	4354611955	-9987106460	44828505450	-590900946660
5	90	215303747352	1554493980250	-8542333784568	120753741368778

Table 34: Genus 0 Gopakumar-Vafa invariants for $X_0^{(5)}$ with $\mathbb{Z}_5$ charge 0.

$n^{(0)}_{(d_1,d_2),\pm1}$	$d_2=0$	1	2	3	4
$d_1=0$	0	0	0	0	0
1	100	-200	500	-3200	28600
2	100	28850	-115600	1016225	-11641250
3	100	40815550	14950200	-182549900	2717025075
4	100	4354596400	-9986764325	44823849500	-590823324900
5	100	215303680450	1554497429950	-8542398268200	120755025918450

Table 35: Genus 0 Gopakumar-Vafa invariants for $X_0^{(5)}$ with $\mathbb{Z}_5$ charge ±1.

$n^{(0)}_{(d_1,d_2),\pm2}$	$d_2=0$	1	2	3	4
$d_1=0$	0	0	0	0	0
1	125	-250	625	-4000	35750
2	125	28200	-113050	994700	-11400375
3	125	40810800	14993100	-183006850	2723321475
4	125	4354571400	-9986212275	44816327125	-590697868950
5	125	215303571950	1554503015325	-8542502642700	120757104858075

Table 36: Genus 0 Gopakumar-Vafa invariants for $X_0^{(5)}$ with $\mathbb{Z}_5$ charge ±2.

The Picard-Fuchs system at the large volume point associated to $X_1^{(5)}$ is, in the mirror dual algebraic coordinates z_1, z_2, generated by the operators

$$\begin{aligned}\mathcal{D}_1 =&\Theta_1^2 - 3\Theta_1\Theta_2 + 7\Theta_2^2 + z_1\left(-3 - 11\Theta_1 - 11\Theta_1^2\right)\\ &- z_1^2\left(1+\Theta_1+\Theta_2\right)\left(1+\Theta_1+2\Theta_2\right) - z_2\left(1+\Theta_1+2\Theta_2\right)\left(14+15\Theta_1+14\Theta_2\right)\,,\\ \mathcal{D}_2 =&\Theta_2^3 - z_2\left(1+\Theta_1+\Theta_2\right)\left(1+\Theta_1+2\Theta_2\right)\left(2+\Theta_1+2\Theta_2\right)\,,\end{aligned} \tag{8.3}$$

and the discriminant takes the form

$$\Delta = (1 - 11z_1 - z_1^2)^3 + \mathcal{O}(z_2)\,, \tag{8.4}$$

with the roots in the large base limit given by (5.52),

$$z_+ = -\frac{1}{2}\left(11 + 5\sqrt{5}\right)\,,\quad z_- = -\frac{1}{2}\left(11 - 5\sqrt{5}\right)\,. \tag{8.5}$$

To study the large volume limit corresponding to $X_{0,\text{n.c.}1}^{(5)}$, we resolve the triple tangency at $z_- = z_2 = 0$ by choosing coordinates v_1, v_2 such that

$$z_1 = \frac{5}{2}\left(25 - 11\sqrt{5}\right)\left(\frac{1}{5\sqrt{5}} - v_1\right)\,,\quad z_2 = -\frac{v_1^3 v_2}{\left(\frac{1}{5\sqrt{5}} - v_1\right)^3}\,. \tag{8.6}$$

The particular normalization can again be found by constructing an appropriate elliptic fibration, using the results from [26], and studying the Higgs transitions with B-fields. On the other hand, the correct choice of coordinates to find the large volume limit $X_{0,\text{n.c.}2}^{(5)}$ inside the triple tangency at $z_+ = z_2 = 0$ is given by w_1, w_2 with

$$z_1 = -\frac{5}{2}\left(25 + 11\sqrt{5}\right)\left(\frac{1}{5\sqrt{5}} + w_1\right)\,,\quad z_2 = \frac{w_1^3 w_2}{\left(\frac{1}{5\sqrt{5}} + w_1\right)^3}\,. \tag{8.7}$$

Due to their size, we do not provide the generators of the respective Picard-Fuchs systems. However, the coefficients of the operators are not rational but contained in $\mathbb{Q}[\sqrt{5}]$ and the leading terms of the genus zero free energies are

$$\begin{aligned}F_{0,X_{0,\text{n.c.}1}^{(5)}} =&\frac{1}{3!}c_{ijk}t^it^jt^k + p_2(t^1,t^2) - 5q_1\left(10+z_-\right) + 3q_2 - \frac{5}{8}q_1^2\left(79+7z_-\right)\\ &+ 10q_1q_2\left(10+z_-\right) - \frac{45}{8}q_2^2 + \mathcal{O}(q^3)\,,\\ F_{0,X_{0,\text{n.c.}2}^{(5)}} =&\frac{1}{3!}c_{ijk}t^it^jt^k + p_2(t^1,t^2) - 5q_1\left(10+z_+\right) + 3q_2 - \frac{5}{8}q_1^2\left(79+7z_+\right)\\ &+ 10q_1q_2\left(10+z_+\right) - \frac{45}{8}q_2^2 + \mathcal{O}(q^3)\,,\end{aligned} \tag{8.8}$$

with the triple intersection numbers in both cases being

$$c_{111} = 9\,,\quad c_{112} = 3\,,\quad c_{122} = 1\,,\quad c_{222} = 0\,. \tag{8.9}$$

As we by now expect, these are the same as those of the smooth deformation $X_0^{(1)}$ of $X_0^{(5)}$. Comparing the expansions (8.8), it turns out that the free energies of $X_{0,\text{n.c.}1}^{(5)}$ and $X_{0,\text{n.c.}2}^{(5)}$ are