# 3d Coulomb Branch & Magnetic Quivers


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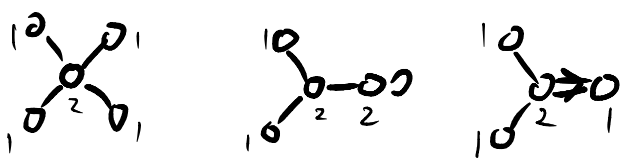

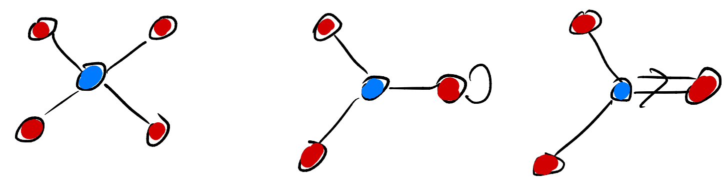


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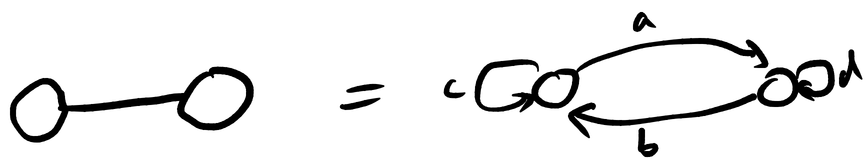
# Disclaimer 1:

All my "quivers" are what was called "tripled" in the first talk of the workshop!




$+ W = adb - bca$

# Disclamers 2, 3, ... :

* More of a review talk, for non-experts
* No citations
* Most things are not my own work (I'll say if they are)
* I'm a physicist! I will try to be understandable for mathematicians, but don't know the words...

If you do know them, please tell them to me.

# Plan:

1) Introduction / Set Up
2) 3d $\mathcal{N}=4$ Coulomb Branches
3) Applications to other HK problems

# Set Up:

$$\boxed{3d\ \mathcal{N}=4\ \text{QFT}} \xleftarrow{\text{compactify}} \begin{cases} 6d\ \mathcal{N}=(1,0) \\ 5d\ \mathcal{N}=1 \\ 4d\ \mathcal{N}=2 \end{cases}$$

- 3d Minkowski
- 8 Supercharges

R-Symmetry $= SU(2)_H \times SU(2)_C$

Theory $T \longrightarrow$ Moduli Space of Vacua $\mathcal{M}(T)$

## Moduli Space:

$\mathcal{M} \circlearrowleft$ acted on by $SU(2)_H \times SU(2)_C$

contains two distinct hyper-Kähler subspaces

**Higgs Branch $\mathcal{H}$:**

$\mathcal{M} \supset \mathcal{H} \circlearrowleft SU(2)_H$

(inv. under $SU(2)_C$)

**Coulomb Branch $\mathcal{C}$:**

$\mathcal{M} \supset \mathcal{C} \circlearrowleft SU(2)_C$

(inv. under $\mathcal{C}$)

What we will look at today!

$\mathcal{M}$ is a difficult object!

→ Can we at least compute $\mathcal{H}$ & $e$?

---

Let's take $T$ to be gauge theory w/

gauge group $G$ ← Vector multiplet $\in$ adj$(G)$

matter representation $R$ ← Hypermultiplet $\in R$ of $G$

Vector multiplet → Vector field $A$ + 3 real scalars $\varphi_i$

Hypermultiplet → 4 real scalars (or 1 quaternionic)

## Higgs Branch: same as in 4d $\mathcal{N}=2$

1987: Hitchin, Karlhede, Lindström, Roček

If $T$ is gauge theory w/

gauge group $G$ ← Vector multiplet $\in$ adj($G$)

matter representation $R$ ← Hypermultiplet $\in R$ of $G$

scalars in Hypermultiplet

$$\mathcal{H} = \mathbb{H}^{\dim R} /\!/\!/ G$$

Higgs Branch is HK quotient

(if $T$ quiver, then $\mathcal{H}$ is Nakajima Quiver Variety)

## Higgs Branch:

$$\mathcal{H} = \mathbb{H}^{\dim R} /\!/\!/ G$$

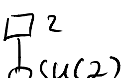
- 'classical' computation
- no quantum corrections ☆
- (relatively) easy

☆ The Higgs branch may consist of several components which intersect classically, but are 'torn apart' in quantum moduli space

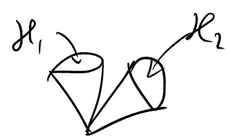
Example:

$G = SU(2)$

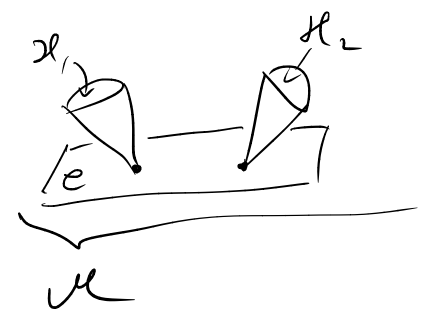
$R = 2[1]$




$$\mathcal{H} = \mathbb{H}^4 /\!/\!/ SU(2) = \mathbb{C}^2/\mathbb{Z}_2 \cup \mathbb{C}^2/\mathbb{Z}_2 =$$




$\longrightarrow$



# Coulomb Branch: different to 4d $\mathcal{N}=2$ (i.e. Seiberg-Witten)

## 1) Classical

$\varphi^i$ in Cartan subalg. of $G$

dual photon $\sigma$ ( $*dA = d\sigma$ )

$$\mathcal{C}_{\text{classical}} = \left(\mathbb{R}^3 \times S^1\right)^{\text{rank}(G)} \Big/ W_G$$

Weyl group of $G$

$\dim_{\mathbb{H}} \mathcal{C} = \text{rank}(G)$

$\rightarrow$ recieves quantum corrections

# Coulomb Branch:

## 2) Quantum

Holomorphic functions on Coulomb Branch (chiral ring)

$= \langle$ dressed monopole operators $\rangle$

Need to look at those guys!

## Monopole Operators: (dimensional reduction of t'Hooft line)

roughly: the dual photon $\sigma$ & one scalar, say $\varphi_3$ in Vplet conspire to make a ($\frac{1}{2}$-BPS) monopole operator $V$

* Monopole operators are labelled by a magnetic charge $m$ (element of the coweight lattice of $G$)

* If $\langle V_m \rangle \neq 0$ then the gauge group $G$ is broken to a subgroup $H_m < G$

→ Can 'dress' the monopole by combinations of $\underbrace{\varphi_1 + i\varphi_2}_{\text{complex scalar call } \varphi} \in Lie(H_m)$ (elements of $Z(UEA)$)

When will we
finally say something
mathematically tangible?

Remember the $SU(2)_C \subset R$-symmetry?

Pick $U(1) \subset SU(2)_C$. call $r(x)$ the $U(1)$-charge of $x$.
(spin of top component of $SU(2)_C$ rep)

<u>Have:</u>

* $r(\varphi) = \frac{1}{2}$

* $r(V_m) = -\sum_{\alpha} |\alpha(m)| + \frac{1}{2}\sum_{\rho} |\rho(m)|$
  $\alpha$ = pos. roots of $G$ $\qquad$ $\rho$ = weight of $R$

r gives us a grading of the chiral ring!

$\Rightarrow$ Can write down a Hilbert Series

# The Monopole Formula:

The Hilbert Series of the 3d $\mathcal{N}=4$ Coulomb Branch is given by:

"dressing function" normalisation

$$\boxed{H(t) = \sum_{m \text{ dom. weight of } G} P_m(t)\, t^{2r(m)}}$$

2013: Cremonesi, Hanany, Zaffaroni

could be refined by fugacities for top. symmetries

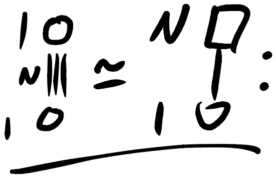
Where $P_m(t)$ is Hilbert Series of ring freely generated by Casimir invariants of $H_m \subset G$

# Other approaches to understanding CB:

* Direct construction of chiral ring through "abelianisation"
  (Bullimore, Dimofte, Gaiotto 2015)

* Mathematical definition of CB
  (Nakajima 2015 + Braverman, Finkelberg, Nakajima 2016)

# Canonical Example:




$$H(t) = \frac{1-t^{2N}}{(1-t^2)(1-t^N)^2} = PE\left[t^2 + 2t^N - t^{2N}\right]$$

$$\Rightarrow \mathcal{C} = \mathbb{C}^2/\mathbb{Z}_N$$

dim of global symmetry

$N=0$ : HS diverges

$N=1$ : $HS = \frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \dots$

$N=2$ : $HS = 1 + 3t^2 + \dots$

$N>2$ : $HS = 1 + 1t^2 + \dots$

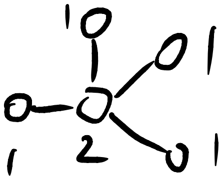
$t$-coefficient = # free

$t^2$-coefficient = dimension of global symmetry

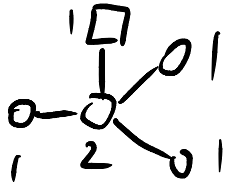
Problem for $N=0$. Free for $N=1$. Global Symm $= \underset{=A_1}{su(2)}$ for $N=2$.

$= u(1)$ for $N>2$.

# Second Example:



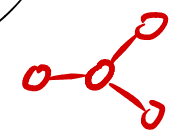

$\simeq$




$$: \quad HS(t) = 1 + 28t^2 + 300t^4 + \dots$$

global symmetry $= so(8) = D_4$

Dynkin Diagram



Looks just like our quiver!

# Balance:

Take a unitary quiver (as we have been so far) in its unframed version.

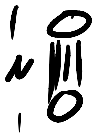
$$\left\{ \begin{array}{rl} \text{nodes} : & i \\ \text{gauge ranks} : & n_i \\ \#\text{edges } i\text{-}j : & e_{ij} \end{array} \right\}$$

define balance $b_i$ of the node $i$ as:

$$\boxed{b_i = \sum_j e_{ij} n_j - 2n_i}$$

# Examples:

1)



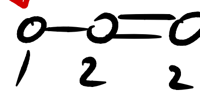

$N\cdot 1 - 2\cdot 1 = N-2$

same

3)

$1\cdot 2 - 2\cdot 1 = 0$

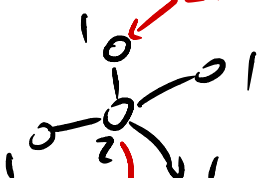
$1\cdot 1 + 2\cdot 2 - 2\cdot 2 = 1$




$2\cdot 2 - 2\cdot 2 = 0$

2)

$2\cdot 1 - 2\cdot 1 = 0$

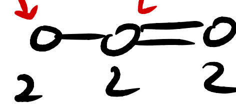



$4\cdot 1\cdot 1 - 2\cdot 2 = 0$

4)

$1\cdot 2 - 2\cdot 2 = -2$

$1\cdot 2 + 2\cdot 2 - 2\cdot 2 = 2$




$2\cdot 2 - 1\cdot 2 = 0$

Simply by inspecting the balances of nodes in the quiver, we can say things about the Coulomb Branch:

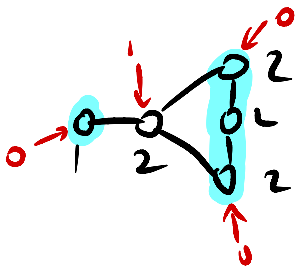
If one $\underline{b_i < -1}$ $\Rightarrow$ HS will diverge, CB is not a cone
(for physicists: UV & IR R-symmetry are different)

If one $\underline{b_i = -1}$ $\Rightarrow$ CB has a free part

Subset of $\underline{b_i = 0}$ $\Rightarrow$ Form Dynkin diagram which tells global symmetry (there can be enhancements)
Get CB generator in adj. of degree 2.

For one $\underline{b_i = b > 0}$ $\Rightarrow$ Get $U(1)$ in global symm + degree $2+b$ CB generators transforming in specific rep of global symmetry. (will try to explain)

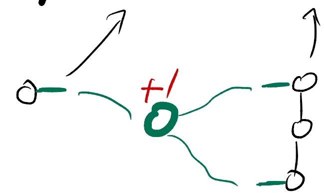
Example:




→ global symm $su(2) \times su(4)$

deg 2 gens in adj. $= [2]\otimes[0,0,0] + [0]\otimes[1,0,1]$

→ deg 3 gen in rep $[1]\otimes[1,0,1]$




<u>but</u> we miss 2 gens at degree 1.

don't see it with an easy rule

# Non-simply laced global symmetry, or a life beyond gauge theory

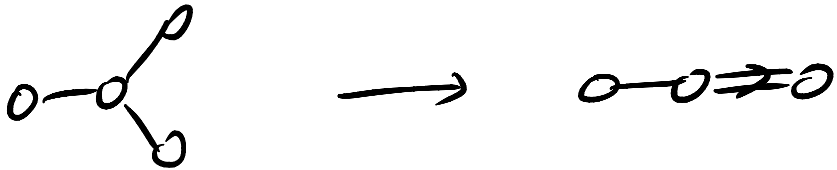
Can clearly have Coulomb branches with ADE symmetry!

What about non-simply laced, i.e. BCFG?

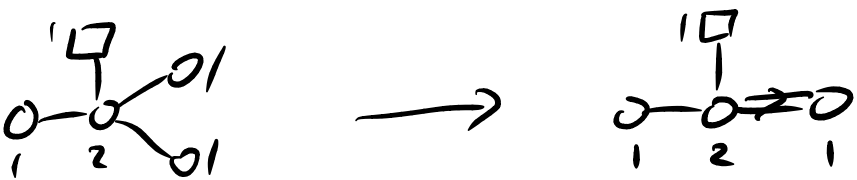
Exactly!

$\rightarrow$ non-simply laced quivers!

Take $D_4$ Dynkin diagram, can fold to $B_3$

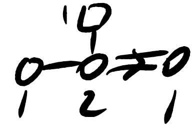


Same with quivers?




?

Can define the r-charge of $V_m$ appropriately!

* However quivers like  have no gauge theory interpretation.
* They can have a gauge theoretic 3d Mirror
* They arise from brane systems with orientifolds

Let's assume we're all on board with this...

we can now use quivers to construct HK cones with certain properties!

This goes under the name of

Magnetic Quiver !

Say we have the following wish:

Construct a HK one with the following:

- $\dim_{\mathbb{H}} = 16$
- global symmetry $C_5$
- a generator of degree 3 in $[0,0,0,0,1]_{C_5}$

→ find only one possible magnetic quiver:

$$1 - 2 - 3 - 4 \Leftarrow 5 - 2$$

$$
\begin{aligned}
PL(H)(t) = &\left.\begin{aligned} &+[20000]\,t^2 \\ &+[00001]\,t^3 \end{aligned}\right\} \text{our input} \\
&\left.\begin{aligned} &-[01000]\,t^4 \\ &-[10010]\,t^5 \\ &+([00200]-[20000]+[01000])\,t^6 \end{aligned}\right\} \text{information about relations} \\
&+O(t^7)
\end{aligned}
$$

# So what?

e.g. Higgs Branch of non-gauge theory (SCFT 6d, 5d, 4d)

e.g. hyper-Kähler implosion

- We encounter problems in Physics & Math, where little is known about a moduli space.
- Sometimes we can derive / construct (even guess) a magnetic quiver. e.g. by studying brane systems
- Using the Coulomb branch construction we can now compute several properties of our moduli space.

* By now there is a whole industry of computing magnetic quivers (not just at Imperial!)

e.g. 4d $\mathcal{N}=2$ SCFTs:

rank 1: Have magnetic quiver for Higgs branches of all known theories

rank 2: for 65/69 known theories (61/69 are unitary)

* There are many more types of quivers for which the Coulomb branch is understood.

e.g. - (unitary-) orthosymplectic quivers ($\exists$ non-simply laced)

- wreathed quivers

- ...

## More Stuff:

* For some HK cone (etc.) → get magnetic quiver

  → can use to study Hasse diagram of singular loci very easily (usually a hard problem) through quiver subtraction

* If you have a HK problem

  → why not think magnetic quiver?

Thank You!