Rank r DT Theory from Rank 1

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The talk is based on:

- Joint project with Richard Thomas

- [FT1] Curve counting and S-duality, arXiv:2007.03037
- [FT2] Rank r DT theory from rank 0, arXiv:2103.02915
- [FT3] Rank r DT theory from rank 1, arXiv:2108.02828

- [F] Explicit formulae for rank zero DT invariants and the OSV conjecture, arXiv:2203.10617

Overview

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$$\Delta_H(E) = \left(\mathrm{ch}_1(E).H^2\right)^2 - 2H^3 \, \mathrm{ch}_0(E) \, \mathrm{ch}_2(E).H \geq 0.$$

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- Assume X is a Calabi-Yau 3-fold: $K_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

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1. $J(\alpha) \in \mathbb{Q}$ is unchanged by deformation of the Calabi-Yau 3-fold X.
2. If $\tau, \tilde{\tau}$ are two (weak) stability conditions on X, there is an explicit change of stability condition formula giving $J^{\tau}(\alpha)$ in terms of the $J^{\tilde{\tau}}(\beta)$.

Rank r DT Theory from rank 1

Theorem 1.1 (Feyzbakhsh-Thomas)

Let $(X, \mathcal{O}_X(1))$ be a Calabi-Yau 3-fold satisfying the conjectural BMT inequality BG. Then for fixed $v \in K(X)$ of rank ≥ 0,

$$J(v) \; = F\big(J(\alpha_1), J(\alpha_2), \dots\big)$$

is a universal polynomial in invariants $J(\alpha_i)$, with all α_i of rank 1. If X also satisfies the MNOP conjecture then we can replace the $J(\alpha_i)$ by the Gromov-Witten invariants of X.

Weak Stability Conditions

- The μ_H-slope of a coherent sheaf E on X is

$$\mu_H(E) := \begin{cases} \frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3} & \text{if } \mathrm{ch}_0(E) \neq 0, \\ +\infty & \text{if } \mathrm{ch}_0(E) = 0. \end{cases}$$

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- Denote the maximum slope in the Harder-Narasimhan filtration by $\mu_H^+(E)$ and minimum by $\mu_H^-(E)$.
- Let $\mathcal{D}(X) := D^b\mathrm{Coh}(X)$. For any $b \in \mathbb{R}$, define

$$\mathcal{A}(b) := \left\{E^{-1} \xrightarrow{d} E^0 \ : \ \mu_H^+(\ker d) \leq b\,,\ \mu_H^-(\operatorname{cok} d) > b\right\} \subset \mathcal{D}(X)$$

- T. Bridgeland showed that $\mathcal{A}(b)$ is the heart of a bounded t-structure on $\mathcal{D}(X)$.

Weak Bridgeland stability conditions

- For $w > \frac{b^2}{2}$, define the slope

$$\nu_{b,w}(E) = \begin{cases} \frac{\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1^b(E).H^2} & \text{if } \mathrm{ch}_1^b(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^b(E).H^2 = 0 \end{cases}$$

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- If $E \in \mathcal{A}(b)$, then $\mathrm{ch}_1^b(E).H^2 \geq 0$.
- We say $E \in \mathcal{D}(X)$ is $\nu_{b,w}$-(semi)stable if and only if
  - $E[k] \in \mathcal{A}(b)$ for some $k \in \mathbb{Z}$, and
  - For all non-trivial subobjects $F \hookrightarrow E[k]$ in $\mathcal{A}(b)$, we have

$$\nu_{b,w}(F)\ (\leq)\ \nu_{b,w}(E[k]/F)$$

- $\nu_{b,w}$-stability satisfies Harder-Narasimhan property.

Weak Stability Conditions

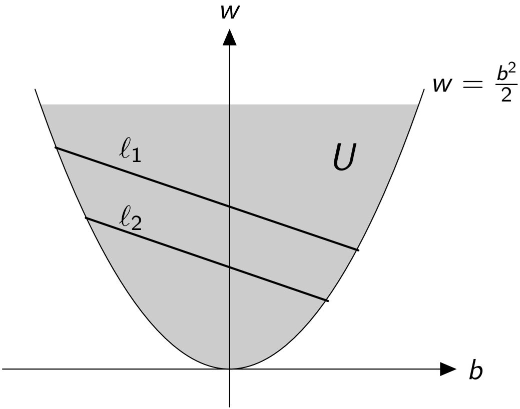

Figure: (b, w)-plane & walls for an object $E \in \mathcal{D}(X)$

Weak Stability Conditions

- **Wall and Chamber Decomposition:**
  For any fixed $v \in K(X)$, there exists a set of line $\{\ell_i\}_{i \in I}$ in $\mathbb{R}^2$ such that the segments $\ell_i \cap U$ (called "walls") are locally finite and satisfy

  1. The $\nu_{b,w}$-(semi)stability of any $E \in \mathcal{D}(X)$ of class v is unchanged as (b, w) varies within any connected component (called a "*chamber*") of $U \setminus \bigcup_{i \in I} \ell_i$.
  2. For any wall $\ell_i \cap U$ there is a strictly $\nu_{b,w}$-semistable object $E \in \mathcal{D}(X)$ of class v along the wall ℓ_i which is unstable in one of the adjacent chambers.

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Conjecture 2.1 (Bogomolov-Gieseker inequality (Bayer-Macrì-Toda))

Any $\nu_{b,w}$-semistable $E \in \mathcal{D}(X)$ satisfies

$$0 \leq Q_{b,w}(E) \; = \; (C_1^2 - 2C_0C_2\,)w + \; (3C_0C_3 - C_1C_2\,)b + (2C_2^2 - 3C_1C_3),$$

where $C_i := \mathrm{ch}_i(E).H^{3-i}$.

Weak Stability Conditions

BMT Conjecture has been proved in the following cases:

- X is projective space $\mathbb{P}^3$ [Macrì], the quadric threefold [Schmidt] or, more generally, any Fano threefold of Picard rank one [Li],
- X an abelian threefold [Maciocia-Piyaratne], a Calabi-Yau threefold of abelian type, a Kummer threefold [Bayer-Macrì-Stellari], or a product of an abelian variety and $\mathbb{P}^n$ [Koseki],
- X with nef tangent bundle [Koseki],
- X is a quintic threefold [Li],
- X is Calabi-Yau threefold of complete intersection of quadratic and quartic hypersurfaces [Liu]
- X is a general weighted hypersurface in the weighted projective spaces $\mathbb{P}(1,1,1,1,2)$ or $\mathbb{P}(1,1,1,1,4)$ [Koseki].

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- For $n \gg 0$, take any section $s\colon \mathcal{O}_X(-n) \to E$ and consider $\mathrm{cok}(s)$,

$$E \to \mathrm{cok}(s) \to \mathcal{O}_X(-n)[1]$$

$$v_n := \mathrm{ch}(\mathrm{cok}(s)) = \left(r - 1,\ D + nH,\ \beta - \frac{n^2}{2}H^2,\ m + \frac{n^3}{6}H^3\right)$$

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- There is a line ℓ_{JS} in $\mathbb{R}^2$ such that for $(b_0, w_0) \in \ell_{JS} \cap U$:
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it has the same ν_{b_0,w_0}-slope as $\mathcal{O}_X(-n)[1]$.
$\Rightarrow \ell_{JS}$ is a wall for objects of class v_n which is called Joyce-Song wall.

Joyce-Song wall

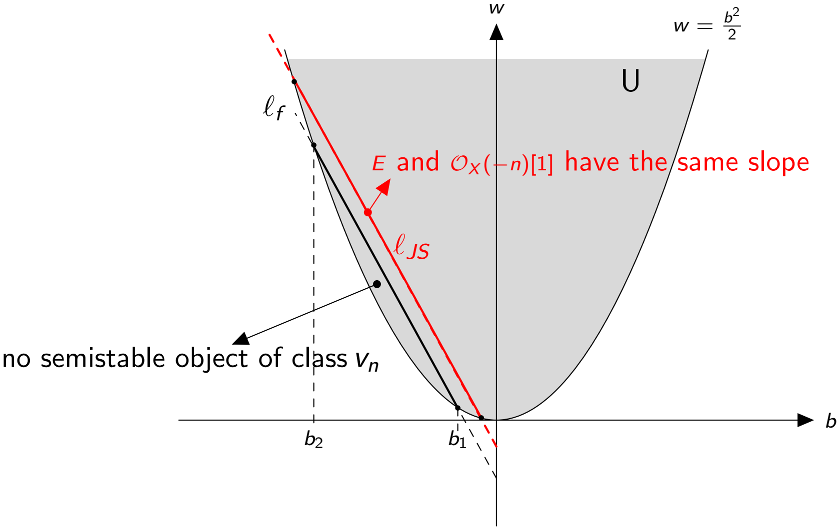

Figure: Walls for objects of class v_n

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- If $r = 1$, there is no wall for class v_n above ℓ_{JS}, so the cokernel of a Joyce-Song pair is slope-semistable, i.e. the map

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- If $r > 1$, there could be walls above and below the JS wall.

Higher rank

Theorem 3.1

Given $(b, w) \in U$ and a $\nu_{b,w}$-semistable object $F \in \mathcal{A}(b)$ of class v_n. If $b < \mu(F)$, then F is a sheaf.

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  - If $r > 1$ and $\mathcal{H}^{-1}(F) \neq 0$, then F gets destabilised by a sequence $F_1 \hookrightarrow F \twoheadrightarrow F_2$ where one of the F_i is a sheaf and the other one is an object of type v_n with rank smaller than $r - 1$:

$$\mathrm{ch}(F_j) = \left(r', D' + nH, \beta' - \frac{n^2}{2}H^2, m' + \frac{n^3}{6}H^3\right)$$

where $D'.H^2$, $\beta'.H$ and m' lie in bounded intervals determined by $D.H^2$, $\beta.H$ and m.

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- Type (1) The destabilising objects E_1, E_2 are sheaves and ℓ lies in the "safe areas" of E_1 and E_2.
- Type (2) One of the destabilising objects is a tilt-semistable sheaf and the other one is of type v_n for lower rank.

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- By applying Joyce's Ringel-Hall algebra technology to any $(b, w) \in U$ and a class $\alpha \in K(X)$ with $\nu_{b,w}(\alpha) < +\infty$, we can define

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  which 'counts' $\nu_{b,w}$-semistable objects of class α.
- Wall-crossing formula: suppose (b, w^+) and (b, w^-) are points on two sides of a wall for class α.

$$J_{b,w^+}(\alpha) = J_{b,w^-}(\alpha) + \sum_{\substack{m \geq 2,\ \alpha_1, \dots, \alpha_m \in C(\mathcal{A}(b)), \\ \sum_{i=1}^m \alpha_i = \alpha,\ \nu_{b,w_0}(\alpha_i) = \nu_{b,w_0}(\alpha)\ \forall i}} C_{+,-}(\alpha_1, \dots, \alpha_m) \prod_{i=1}^m J_{b,w^-}(\alpha_i).$$

Joyce-Song wall

- When $m = 2$,

$$C_{+,-}(\alpha_1, \alpha_2) + C_{+,-}(\alpha_2, \alpha_1) \; = (-1)^{\chi(\alpha_1,\alpha_2)-1}\chi(\alpha_1, \alpha_2).$$

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- Let $(b, w^{\pm})$ be points just above and below ℓ_{JS}.

$$J_{b,w^+}(v_n) = J_{b,w^-}(v_n) + (-1)^{\chi(\mathsf{v}(n))-1} \chi(\mathsf{v}(n)) \cdot J_{b,\infty}(\mathsf{v}) \cdot \# H^2(X, \mathbb{Z})_{tors} + \dots .$$

We have included one of the $m = 2$ terms with $\alpha_1, \ \alpha_2 = \mathsf{v} \ , \ \big[\mathcal{O}(-n)[1]\big]$.

- There is no wall for objects of class v either on or above ℓ_{JS}, so $J_{b,w^-}(\mathsf{v}) = J_{b,\infty}(\mathsf{v})$.
- $J_{b,w^-}\big[\mathcal{O}(-n)[1]\big] = \# \operatorname{Pic}_0(X) = \# H^2(X, \mathbb{Z})_{tors}$.
- All other terms involve only $J_{b,w^-}(\alpha_i)$ where $0 \le \operatorname{rank}(\alpha_i) \le r - 1$.

Wall-crossing formulae

- From $(b, w^{\pm})$, after passing finitely many walls we can reach below ℓ_f or large volume limit $w \gg 0$. Hence there is a universal formula

$$J_{b,w^{\pm}}(v_n) = F_{b,w^{\pm}}(J_{b,\infty}(\alpha_i)) \quad \text{in classes } \alpha_i \text{ with } 0 \leq \text{rank}\,(\alpha_i) \leq \text{rank}\,(v_n)$$

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- The JS wall-crossing formula yields a universal formula

$$J_{b,\infty}(\mathrm{v}) = F\big(J_{b,\infty}(\beta_j)\big) \text{ with } 0 \leq \text{rank}(\beta_j) \leq r-1 \ \forall\, j.$$

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- For any class $\alpha \in K(X)$ with non-negative rank,

  $$J_{b,\infty}(\alpha) = J(\alpha) + F_{\alpha}\big(J(\alpha_k)\big) \text{ with } 0 \leq \text{rank}(\alpha_k) < \text{rank}(\alpha) \ \forall\, j.$$

Rank 2 class $v = (2, H, \tilde{\beta}, \tilde{m})$ when $\text{Pic}(X) = \mathbb{Z}.H$

- Along any wall for class v_n except JS wall: one of the destabilising factors E_1 is of rank 1 and the other E_i's for $i > 1$ are all of rank zero.

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$$DT(x,y,z) := \sum_{e^{kH}(1,0,-\beta,-m) \in M_{v,n}} I_{m,\beta}\; x^k\, y^{-\beta+\frac{1}{2}k^2H^2}\, z^{-m-k\beta H+\frac{1}{6}k^3H^3}\,.$$

$$A_{v,n} := DT(x,y,z) \cdot \prod_{\substack{\alpha = (0,\, kH,\, \beta,\, m) \in K(X) \\ 0 < k \leq n \\ \frac{\beta.H}{kH^3} < \mu(\ell_{\mathrm{JS}})}} \exp\left((-1)^{\chi_\alpha}\, \chi_\alpha\, J(\alpha)\, x^k y^\beta z^m\right)$$

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Theorem 4.1 (F)

The coefficient of $x^{n+1} y^{\tilde{\beta} - \frac{n^2H^2}{2}} z^{\tilde{m} + \frac{n^3H^3}{6}}$ in the series $\frac{(-1)^{\chi(\mathcal{O}_X(-n), v)+1}}{\chi(\mathcal{O}_X(-n), v)} A_{v,n}$ is equal to $J(v)$.

Thank you!