An introduction to BPS algebras and invariants

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Quiver representations

Quivers

A *quiver* Q is the data of a set of vertices Q_0, a set of arrows Q_1, and a pair of morphisms $s, t\colon Q_1 \to Q_0$ taking an arrow to its source/target.

Representations

A representation of Q of dimension $\gamma \in \mathbb{N}^{Q_0}$ is the data of a set of vector spaces ρ_i of dimension γ_i along with morphisms $\rho(a)\colon \rho_{s(a)} \to \rho_{t(a)}$.

- The *path algebra* $\mathbb{C}Q$ is a $\mathbb{C}$-algebra with basis given by paths (including lazy paths e_i of length 0 at each vertex i), and multiplication given by concatenation of paths.
- A $\mathbb{C}Q$-module ρ corresponds to a Q-representation (set $\rho_i = e_i \cdot \rho$).
- For simplicity, to start with we assume that Q is symmetric (e.g. as many arrows from i to j as from j to i) and not say much about stability conditions on Q.

Moduli spaces

Fix a quiver Q and a dimension vector $\gamma \in \mathbb{N}^{Q_0}$. Set

$$\mathbb{A}_{Q,\gamma} := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma_{s(a)}}, \mathbb{C}^{\gamma_{t(a)}}).$$

This is an affine space, parameterising all γ-dimensional Q-reps. It is acted on by the gauge group

$$\mathrm{GL}_\gamma := \prod_{i \in Q_0} \mathrm{GL}(\mathbb{C}^{\gamma_i})$$

by change of basis, and two representations are isomorphic iff they lie in the same orbit. We define

$$\mathfrak{M}_\gamma(Q) := \mathbb{A}_{Q,\gamma} / \mathrm{GL}_\gamma$$

the stack theoretic quotient, and

$$\mathcal{M}_\gamma(Q) := \mathrm{Spec}(\Gamma(\mathbb{A}_{Q,\gamma})^{\mathrm{GL}_\gamma})$$

the coarse quotient.

The Jordan–Hölder map

- Points of $\mathfrak{M}_\gamma(Q)$ parameterise γ-dimensional $\mathbb{C}Q$-modules, while points of $\mathcal{M}_\gamma(Q)$ parameterise *semisimple* γ-dimensional $\mathbb{C}Q$-modules.
- There is a canonical *affinization* morphism $\mathtt{JH}\colon \mathfrak{M}_\gamma(Q) \to \mathcal{M}_\gamma(Q)$. At the level of points it takes a module ρ with Jordan–Hölder filtration

$$0 = \rho_0 \subset \rho_1 \subset \ldots \subset \rho_n = \rho$$

to $\bigoplus_{m=1}^{n}(\rho_m/\rho_{m-1})$, the semisimplification of ρ.

Cohomology

- Consider $\mathbb{Q}_{\mathrm{vir}} := \mathbb{Q}_{\mathfrak{M}_\gamma(Q)}[\dim(\mathfrak{M}_\gamma(Q))]$; the constant sheaf, shifted so that it is perverse (more on that later).
- Then we define

$$\begin{aligned}\mathcal{H}_{Q,\gamma} :=& \mathrm{H}(\mathfrak{M}_\gamma(Q), \mathbb{Q}_{\mathrm{vir}}) \\ \cong& \mathrm{H}_{\mathrm{GL}_\gamma}(\mathbb{A}_{Q,\gamma}, \mathbb{Q})[\dim(\mathfrak{M}_\gamma(Q))].\end{aligned}$$

- Its *Poincaré series* $\mathrm{p}(\mathcal{H}_{Q,\gamma}, q^{1/2}) := \sum_{m\in\mathbb{Z}}(-1)^m \dim(\mathcal{H}^m_{Q,\gamma})q^{m/2}$ is a formal Laurent series in $q^{1/2}$

Generating series

We define

$$\mathcal{Z}_Q(x) = \sum_{\gamma\in\mathbb{N}^{Q_0}} \mathrm{p}(\mathcal{H}_{Q,\gamma}, q^{1/2})x^\gamma \in \mathbb{Z}(\!(q^{1/2})\!)[\![x_1, \ldots, x_n]\!].$$

where $1, \ldots, n$ are the vertices of Q, and we write $x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}$.

A simple example

- Let Q be the quiver with one vertex and no arrows, so $\mathcal{H}_{Q,n} = \mathrm{H}_{\mathrm{GL}_n}(\mathrm{pt},\mathbb{Q})[-n^2]$.
- $\mathrm{p}(\mathcal{H}_{Q,n}, q^{1/2}) = (-q^{1/2})^{n^2}\prod_{m=1}^{n}(1-q^m)^{-1}$, since $\mathrm{H}_{\mathrm{GL}_n}(\mathrm{pt},\mathbb{Q})$ is a free commutative algebra in generators of cohomological degree $2,\ldots,2n$.
- $\mathcal{Z}_Q(x) = \sum_{n\geq 0}(-q^{1/2})^{n^2}\prod_{m=1}^{n}(1-q^m)^{-1}x^n = \prod_{n=1}^{\infty}(1-q^{n-1/2}x)$
- If we can write $\mathcal{Y}(x) = \prod_{0\neq\gamma\in\mathbb{N}^{Q_0}, i\in\mathbb{Z}}(1-q^{i/2}x^{\gamma})^{-\omega_{\gamma,i}}$ we define

$$\mathrm{Log}(\mathcal{Y}(x)) = \sum_{0\neq\gamma\in\mathbb{N}^{Q_0}, i\in\mathbb{Z}} \omega_{\gamma,i}q^{i/2}x^{\gamma}$$

$$\therefore \mathrm{Log}(\mathcal{Z}_Q(x)) = -\,q^{1/2}(1-q)^{-1}x$$

- We extract the Poincaré polynomials of the spaces of "basic" Q-representations by taking $(q^{1/2}-q^{-1/2})\,\mathrm{Log}(\mathcal{Z}_Q(x))$, and so for general Q we *define* the BPS invariants $\Omega_{Q,\gamma}(q^{1/2})$ via

$$\sum_{0\neq\gamma\in\mathbb{N}^{Q_0}} \Omega_{Q,\gamma}(q^{1/2})x^{\gamma} = (q^{1/2}-q^{-1/2})\,\mathrm{Log}(\mathcal{Z}_Q(x)).$$

Kontsevich–Soibelman Cohomological Hall algebra

- Let $\mathfrak{M}^{(2)}(Q)$ be the stack of short exact sequences of $\mathbb{C}Q$-modules

$$0 \to \rho' \to \rho \to \rho'' \to 0.$$

- There are morphisms $\pi_1, \pi_2, \pi_3 \colon \mathfrak{M}^{(2)}(Q) \to \mathfrak{M}(Q)$ taking such a short exact sequence to ρ', ρ, ρ'' respectively.
- We consider the correspondence diagram

$$\mathfrak{M}(Q)^{\times 2} \xleftarrow{\pi_1 \times \pi_3} \mathfrak{M}^{(2)}(Q) \xrightarrow{\pi_2} \mathfrak{M}(Q)$$

and define the associative product

$$\pi_{2,*} \circ (\pi_1 \times \pi_3)^* \colon \mathcal{H}_Q \otimes \mathcal{H}_Q \to \mathcal{H}_Q (:= \mathrm{H}(\mathfrak{M}(Q), \mathbb{Q}_{\mathrm{vir}})).$$

"Integrality"

- Since we assumed that Q is symmetric, (a slight twist of) $\mathcal{H}_Q$ is supercommutative.
- A theorem of Efimov states that there is an isomorphism

$$\mathcal{H}_Q \cong \mathrm{Sym}\left(\bigoplus_{0\neq\gamma\in\mathbb{N}^{Q_0}} V_{\mathrm{prim},\gamma}[-1]\otimes\mathbb{Q}[u]\right)$$

  where $V_{\mathrm{prim},\gamma}$ are cohomologically graded vector spaces, of finite total dimension, and u has cohomological degree 2.
- As a corollary

$$\sum_{0\neq\gamma\in\mathbb{N}^{Q_0}} \Omega_{Q,\gamma}(q^{1/2})x^{\gamma} := (q^{1/2}-q^{-1/2})\,\mathrm{Log}(\mathcal{Z}_Q(x))$$

$$= \sum_{0\neq\gamma\in\mathbb{N}^{Q_0}} \mathrm{p}(V_{\mathrm{prim},\gamma})x^{\gamma}$$

  and the BPS invariants are Laurent polynomials in $(-q^{1/2})$ with positive coefficients.

BPS cohomology

- For X a space, the category $\mathrm{Perv}(X) \subset \mathcal{D}_{\mathrm{c}}(X)$ is an abelian subcategory (and heart) of the bounded derived category of constructible $\mathbb{Q}$-sheaves on X, defined by inequalities regarding $\mathrm{codim}_X(\mathrm{supp}(\mathcal{H}^i(\mathcal{F})))$ for $\mathcal{F}$ a constructible complex. This category behaves in some ways better than the usual heart (e.g. invariant under Verdier duality).
- Recall the morphism $\mathtt{JH}\colon \mathfrak{M}_\gamma(Q) \to \mathcal{M}_\gamma(Q)$. The derived direct image $\mathtt{JH}_* \mathbb{Q}_{\mathrm{vir}}$ of the perverse sheaf $\mathbb{Q}_{\mathrm{vir}}$ is a complex of perverse sheaves, with cohomology in infinitely many degrees.

Theorem (-,Meinhardt)

Let Q be a symmetric quiver and let $V_{\mathrm{prim},\gamma}$ be as in Efimov's theorem. Then

$$V_{\mathrm{prim},\gamma} \cong \mathrm{H}(\mathcal{M}_\gamma(Q), {}^{\mathfrak{p}}\mathcal{H}^1(\mathtt{JH}_* \mathbb{Q}_{\mathrm{vir}})) =: \mathrm{BPS}_{Q,\gamma}$$

$$\therefore \mathrm{H}(\mathfrak{M}(Q), \mathbb{Q}_{\mathrm{vir}}) \cong \mathrm{Sym}(\mathrm{BPS}_Q[-1] \otimes \mathbb{Q}[u])$$

$$\therefore \Omega_{Q,\gamma}(q^{1/2}) = \mathrm{p}(\mathrm{BPS}_{Q,\gamma}, q^{1/2}).$$

Jacobi algebras

- Fix a quiver Q and choose a *potential* $W \in \mathbb{C}Q_{\mathrm{cyc}}$, a linear combination of cyclic paths in Q. If $W = a_1 \ldots a_l$ is a single cyclic path, and $a \in Q_1$, we define

$$\partial W/\partial a := \sum_{a_r = a} a_{r+1} \ldots a_l a_1 \ldots a_{r-1}$$

  and extend to arbitrary W linearly.
- Then we define

$$\mathrm{Jac}(Q, W) = \mathbb{C}Q/\langle \partial W/\partial a | a \in Q_1 \rangle$$

Example

If Q is the quiver with one vertex and three loops a, b, c, and $W = a[b, c]$, then $\mathbb{C}Q = \mathbb{C}\langle a, b, c\rangle$ and $\mathrm{Jac}(Q, W) \cong \mathbb{C}[a, b, c]$

More interesting examples

Let X be a toric Calabi–Yau threefold, then we can find a vector bundle $E = \bigoplus_{i \leq n} E_i$ with E_is distinct simple vector bundles, so that

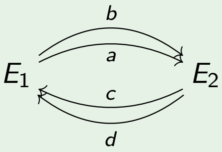
$$\mathrm{RHom}(E, -) \colon D^b(\mathrm{Coh}(X)) \to D^b_{\mathrm{fg}}(A\text{-mod})$$

is a derived equivalence, with $A := \mathrm{End}_X(E)$, and A a Jacobi algebra.

Example

Let $X = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ be the resolved conifold, $\pi \colon X \to \mathbb{P}^1$ the projection. Set $E_1 = \pi^* \mathcal{O}_{\mathbb{P}^1}$ and $E_2 = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$. Then $\mathrm{End}_X(E) = \mathrm{Jac}(Q, W_{\mathtt{KW}})$ where

$Q =$



and $W_{\mathtt{KW}} = acbd - adbc$.

More examples

- Let Q be extended Dynkin quiver, then under Mckay correspondence Q corresponds to a finite group $G_Q \subset \mathrm{Sl}_2(\mathbb{C})$. Let Y be the minimal resolution of the singularity $\mathbb{C}^2/G_Q$, and set $X = Y \times \mathbb{A}^1$. X is a noncompact CY3-fold.
- Let $\tilde{Q}$ be the tripled quiver, obtained from Q by adding one arrow a^* for each $a \in Q_1$, where a^* has the opposite orientation, and a loop ω_i at each $i \in Q_0$.
- Set $\tilde{W} = \sum_{a\in Q_1}[a, a^*]\sum_{i\in Q_0}\omega_i$.
- Then there is a derived equivalence

$$D^b(\mathrm{Coh}(X)) \to D^b_{\mathrm{fg}}(\mathrm{Jac}(\tilde{Q}, \tilde{W})\text{-mod}).$$

In fact the quiver with potential $(\tilde{Q}, \tilde{W})$ obtained by applying the above construction to *arbitrary* Q turns out to have interesting BPS invariants...

BPS invariants for Jacobi algebras

- Denote by $\mathfrak{M}_\gamma(\mathrm{Jac}(Q,W)) \subset \mathfrak{M}_\gamma(Q)$ the substack of $\mathbb{C}Q$-modules satisfying the Jacobi relations. Then $\mathfrak{M}_\gamma(\mathrm{Jac}(Q,W))$ is the critical locus of the function $\mathrm{Tr}(W)$.
- We consider $\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}} \in \mathrm{Perv}(\mathfrak{M}_\gamma(Q))$, the *perverse sheaf of vanishing cycles*. This is a sheaf supported on the critical locus of $\mathrm{Tr}(W)$, that measures how bad the singularities of $\mathrm{crit}(\mathrm{Tr}(W))$ are.
- Set $\mathcal{H}_{Q,W,\gamma} := \mathrm{H}(\mathfrak{M}_\gamma(Q), \phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}})$.
- We define $\mathcal{Z}_{Q,W}(x) = \sum_{\gamma\in\mathbb{N}^{Q_0}} \chi_{\mathrm{w}}(\mathcal{H}_{Q,W,\gamma}, q^{1/2})x^\gamma$. Here χ_{w} is the *weight series*, which depends on the mixed Hodge structure on the cohomology of $\mathcal{H}_{Q,W,\gamma}$. It is the same as the Poincaré series if the cohomology is *pure* which it will be in all examples in this talk.
- As before, we *define* the refined BPS invariants $\Omega_{Q,W,\gamma}(q^{1/2})$ via

$$\sum_{0\neq\gamma\in\mathbb{N}^{Q_0}} \Omega_{Q,W,\gamma}(q^{1/2})x^\gamma = (q^{1/2} - q^{-1/2})\,\mathrm{Log}(\mathcal{Z}_{Q,W}(x)).$$

Integrality for Jacobi algebras

Perverse filtration

Recall the morphism $\mathtt{JH}\colon \mathfrak{M}_\gamma(Q) \to \mathcal{M}_\gamma(Q)$. We denote by ${}^{\mathfrak{p}}\tau^{\leq n}\,\mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}$ the nth truncation of the derived direct image of the vanishing cycle sheaf, for the perverse t-structure. There is a natural morphism

$${}^{\mathfrak{p}}\tau^{\leq n}\,\mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}} \to \mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}$$

and so a natural morphism

$$\mathfrak{P}^n := \mathrm{H}(\mathcal{M}(Q), {}^{\mathfrak{p}}\tau^{\leq n}\,\mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}) \xrightarrow{\iota_n} \mathrm{H}(\mathcal{M}(Q), \mathtt{JH}_*\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}) = \mathcal{H}_{Q,W}$$

Theorem (-,Meinhardt)

The multiplication on $\mathcal{H}_{Q,W}$ respects the above perverse filtration. $\mathfrak{P}^n = 0$ for $n \leq 0$. There is an isomorphism of algebras

$$\begin{aligned}\mathrm{Gr}^{\mathfrak{P}}\,\mathcal{H}_{Q,W} &\cong \mathrm{Sym}\left(\mathfrak{P}^1 \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})\right)\\ &\cong \mathrm{Sym}\left(\mathrm{H}(\mathcal{M}(Q), {}^{\mathfrak{p}}\mathcal{H}^1(\mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}))[-1] \otimes \mathbb{Q}[u]\right)\end{aligned}$$

BPS Lie algebra

From the isomorphism

$$\mathrm{Gr}^{\mathfrak{P}}\,\mathcal{H}_{Q,W} \cong \mathrm{Sym}\left(\mathrm{H}(\mathcal{M}(Q), {}^{\mathfrak{p}}\mathcal{H}^1(\mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}))[-1] \otimes \mathbb{Q}[u]\right)$$

we deduce
$(q^{1/2} - q^{-1/2})\,\mathrm{Log}(\sum_{\gamma\in\mathbb{N}^{Q_0}} \chi_{\mathrm{w}}(\mathcal{H}_{Q,W,\gamma})x^\gamma) = \sum_{\gamma\in\mathbb{N}^{Q_0}} \chi_{\mathrm{w}}(\mathrm{BPS}_{Q,W,\gamma})x^\gamma$
where we define *BPS cohomology*

$$\mathrm{BPS}_{Q,W,\gamma} := \mathrm{H}(\mathcal{M}_\gamma(Q), {}^{\mathfrak{p}}\mathcal{H}^1(\mathtt{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}})).$$

So $\Omega_{Q,W,\gamma}(q^{1/2})$ is a Laurent polynomial in $(-q^{1/2})$.

Lie algebra structure

Since the perverse filtration starts in degree 1, and $\mathrm{Gr}^{\mathfrak{P}}\,\mathcal{H}_{Q,W}$ is supercommutative, the first piece of the filtration:

$$\mathfrak{g}_{Q,W} := \mathfrak{P}^1 = \mathrm{BPS}_{Q,W}[-1] \subset \mathcal{H}_{Q,W}$$

is closed under the commutator Lie bracket, i.e. after a shift, the BPS cohomology carries a Lie algebra structure.

Examples

1. Let Q be the three loop quiver, with potential $a[b,c]$. Then $\mathrm{BPS}_{Q,W,n} \cong \mathbb{Q}[3]$ for all n. As a result there is an isomorphism

$$\bigoplus_{n\geq 0} \mathrm{H}(\mathfrak{M}(\mathbb{C}[a,b,c]),\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}) \cong \mathrm{Sym}\left(\bigoplus_{n\geq 1} \mathrm{H}(\mathrm{pt}/\mathbb{C}^*,\mathbb{Q})[2]\right)$$

the Lie bracket on $\mathfrak{g}_{Q,W} = \bigoplus_{n\geq 1} \mathrm{BPS}_{Q,W,n}[-1]$ vanishes for degree reasons.

2. Let $Q, W_{\mathtt{KW}}$ be the quiver with potential giving the noncommutative conifold. Then

$$\mathfrak{g}_{Q,W_{\mathtt{KW}},(m,n)} \cong \begin{cases} \mathbb{Q}[-1] & \text{if } |m-n|=1 \\ \mathbb{Q}\oplus\mathbb{Q}[2] & \text{if } m=n \\ 0 & \text{otherwise.} \end{cases}$$

Somebody should work out the Lie algebra structure!

3. For $\tilde{Q}, \tilde{W}$ a tripled quiver with potential, $\mathfrak{g}^0_{\tilde{Q},\tilde{W}} \cong \mathfrak{n}^-_{Q_{\mathrm{re}}}$ where Q_{re} is obtained by removing vertices supporting loops.

BPS cohomology in general

- Let $\mathcal{C}$ be a 3-Calabi–Yau category, e.g. coherent sheaves on a compact CY 3-fold, or local systems on a real orientable 3-manifold, or the category of semistable $\mathrm{Jac}(Q,W)$-representations of fixed slope (wrt a stability condition ζ).
- We assume that there is a good moduli space $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$, and that $\mathfrak{M}_{\mathcal{C}}$ comes with a coherent choice of orientation data o. By work of BBDJS, B-BBBJ, Brav+Dyckerhoff, PTVV, $\mathfrak{M}_{\mathcal{C}}$ carries a perverse sheaf $\phi_{\mathcal{C},o}$, which in the case of semistable $\mathrm{Jac}(Q,W)$-modules is $\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}$ as before.
- We *define* the BPS cohomology

$$\mathrm{BPS}_{\mathcal{C}} := \mathrm{H}(\mathcal{M}_{\mathcal{C}}, {}^{\mathfrak{p}}\mathcal{H}^1(p_*\phi_{\mathcal{C},0}))$$

- For the category of ζ-semistable $\mathrm{Jac}(Q,W)$-modules, $\mathrm{BPS}_{\mathcal{C},\gamma}$ will depend on γ *and* ζ if Q isn't symmetric $\Rightarrow$ wall crossing...

BPS cohomology for the stack of Higgs bundles

- Let C be a smooth projective curve, let $X = \mathrm{Tot}(\mathcal{O}_C \oplus \omega_C)$.
- Let $\mathcal{C}$ be the category of semistable coherent sheaves on X, of pure dimension one, of fixed slope. These correspond to semistable Higgs bundles with an endomorphism.
- The stack of objects $\mathfrak{M}_{\mathcal{C}}$ has a good moduli space $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$, as well as a natural choice of orientation.

Theorem (Kinjo-Koseki)

If we define $\mathrm{BPS}_{\mathcal{C}} = \mathrm{H}(\mathcal{M}_{\mathcal{C}}, {}^{\mathfrak{p}}\mathcal{H}^1(p_*\phi_{\mathcal{C},o}))$ *then there is an isomorphism*

$$\mathrm{H}(\mathfrak{M}_{\mathcal{C}}, \phi_{\mathcal{C},o}) \cong \mathrm{Sym}\left(\mathrm{BPS}_{\mathcal{C}}[-1] \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})\right).$$

Moreover the summand $\mathrm{BPS}_{\mathcal{C},(r,d)}$ *given by fixing rank and degree is isomorphic to the cohomology of the fine moduli space of degree* 1 *rank* r *Higgs bundles (the Dolbeault moduli space in nonabelian Hodge theory).*

Theorem in progress (with Kinjo): $\mathcal{H}_{\mathcal{C}}$ is a (deformable) universal enveloping algebra $\mathrm{U}(\mathfrak{g}^{\mathrm{Dol}}_{\mathrm{BPS}}(C) \otimes \mathbb{Q}[u])$, or "curve Yangian".