D-branes on local $\mathbb{P}^2$ revisited

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Workshop on Quivers, Calabi-Yau threefolds and DT Invariants
Sorbonne U - Campus Pierre et Marie Curie
12 April 2022

Introduction: D-branes/BPS states on local $\mathbb{P}^2$

- **"Old story":** Douglas, M. R., Fiol, B., Römelsberger, C. The spectrum of BPS branes on a noncompact Calabi-Yau, 2000.
  - Not known at the time: the description of the Kähler moduli space as a modular curve, the precise mathematical notion of Bridgeland stability condition and the wall-crossing formula.
- **Present work:** revisit the story form a particular viewpoint given by scattering diagrams and flow trees.
  - Related: P. Bousseau, Scattering diagrams, stability conditions, and coherent sheaves on P2, 2019, arXiv:1909.02985.

Introduction: the general picture

- X Calabi-Yau 3-fold, IIA string on $\mathbb{R}^{1,3} \times X$
  - BPS spectrum of the resulting $\mathcal{N} = 2$ 4d theory: D0-D2-D4-D6 branes wrapping holomorphic cycles in X.
  - Dependence on the Kähler moduli $\sigma \in \mathcal{M}_K(X)$, expect

$$\widetilde{\mathcal{M}_K(X)} \to Stab(D^b Coh(X))$$

  - $\Omega(\gamma, \sigma)$: counts of BPS states of charge γ = DT invariants counting σ-stable objects of class γ in $D^b Coh(X)$. Jump as a function of σ described by the wall-crossing formula.
- Mirror: Y Calabi-Yau 3-fold, IIB string on $\mathbb{R}^{1,3} \times Y$
  - BPS spectrum of the resulting $\mathcal{N} = 2$ 4d theory: D3 branes wrapping special Lagrangians in Y.
  - Dependence on the complex moduli $\sigma \in \mathcal{M}_C(Y)$, expect

$$\widetilde{\mathcal{M}_C(Y)} \to Stab(Fuk(Y))$$

  - $\Omega(\gamma, \sigma)$: counts of BPS states of charge γ = DT invariants counting σ-stable objects of class γ in $Fuk(Y)$. Jump as a function of σ described by the wall-crossing formula.

Plan of the talk

1) For $X = K_{\mathbb{P}^2}$, the description of

$$\widetilde{\mathcal{M}_K(X)} \to Stab(D^b Coh(X))$$

2) Determination of the BPS spectrum $\Omega(\gamma, \sigma)$ by a one-parameter family of “scattering diagrams" made of flow trees in $\widetilde{\mathcal{M}_K(X)}$

Local $\mathbb{P}^2$

- $X = K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ non-compact Calabi-Yau 3-fold
  - Projection $\pi \colon X \to \mathbb{P}^2$
  - Zero section $\iota \colon \mathbb{P}^2 \hookrightarrow X$
- $D_{\mathbb{P}^2}(X)$: bounded derived category of sheaves on X set-theoretically supported on $\mathbb{P}^2$ (not necessarily scheme-theoretically!)
  - $\iota_* : D^b Coh(\mathbb{P}^2) \to D_{\mathbb{P}^2}(X)$ (not fully faithful nor essentially surjective!)
  - $\mathcal{O}(n) := \iota_* \mathcal{O}_{\mathbb{P}^2}(n)$ (D4-branes with n units of D2-charges)

The Kähler moduli space $\mathcal{M}_K(X)$ of local $\mathbb{P}^2$

- Mirror symmetry prediction/definition: the universal cover of the Kähler moduli space is the upper half-plane

$$\widetilde{\mathcal{M}_K(X)} = \mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$$

- Kähler moduli space: modular curve

$$\mathcal{M}_K(X) = \mathbb{H}/\Gamma_1(3)$$

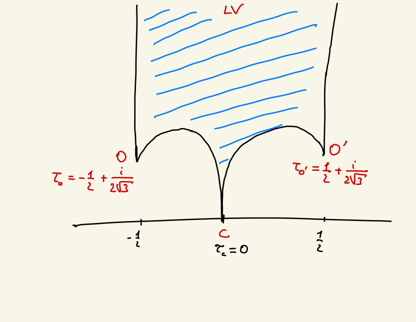
- Modular group:

$$\Gamma_1(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \right\}$$

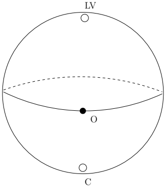
- Genus 0 modular curve with two cusps:
  - "Large volume": class of $\tau = i\infty$ and every $p/q \in \mathbb{Q}$ with $q = 0 \mod 3$
  - "Conifold": class of $\tau = p/q \in \mathbb{Q}$ with $q \neq 0 \mod 3$
- $\mathbb{Z}/3\mathbb{Z}$-orbifold point: class of $\tau_O = -\frac{1}{2} + \frac{i}{2\sqrt{3}}$

The Kähler moduli space $\mathcal{M}_K(X)$ of local $\mathbb{P}^2$

A fundamental domain F_C of $\Gamma_1(3)$ acting on $\mathbb{H}$:



The modular curve $\mathcal{M}_K(X) = \mathbb{H}/\Gamma_1(3)$:



Map to the space of stability conditions

- $Stab(D_{\mathbb{P}^2}(X))$: space of Bridgeland stability conditions on $D_{\mathbb{P}^2}(X)$, complex manifold of dimension 3
- Bayer-Macri (2009):

$$\widetilde{M_K(X)} = \mathbb{H} \to Stab(D_{\mathbb{P}^2}(X))$$

$$\tau \mapsto (\mathcal{A}(\tau), Z(\tau))$$

- Data for a stability condition:
  - $\mathcal{A}(\tau) \subset D_{\mathbb{P}^2}(X)$: abelian category, heart of a bounded t-structure
  - Central charge, additive map:

$$Z(\tau) : \Gamma = K_0(D_{\mathbb{P}^2}(X)) = \mathbb{Z}^3 \to \mathbb{C}$$

$$\gamma \mapsto Z_\gamma(\tau)$$

The mirror to X

- **Mirror:** family of mirror curves, induced by the universal elliptic curve

$$\{E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\}_{\tau \in \mathbb{H}} \to \mathbb{H} = \{\tau\}$$

- **Mirror symmetry prediction for the central charge:**

$$Z_\gamma(\tau) = \int_\gamma \lambda_\tau$$

$\gamma \in H_1(E_\tau, \mathbb{Z})$, λ_τ: particular meromorphic 1-form on E_τ.

- **New formula for the central charge:**

$$Z_E(\tau) = -r(E)T_D(\tau) + d(E)T(\tau) - ch_2(E)$$

where

$$T(\tau) = -\frac{1}{2} + \int_{\tau_O}^{\tau} C(u)du\,,\ T_D(\tau) = \frac{1}{3} + \int_{\tau_O}^{\tau} uC(u)du$$

where C: weight 3 modular form for $\Gamma_1(3)$, Eisenstein series:

$$C(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3} = \prod_{n\geq 1} \frac{(1-q^n)^9}{(1-q^{3n})^3} = 1 - 9\sum_{n\geq 1} \chi(n) \frac{q^n}{1-q^n}$$

Dirichlet character $\chi(n) = \begin{cases} \pm 1 & \text{if} \quad n = \pm 1 \mod 3 \\ 0 & \text{otherwise} \end{cases}$

What is the abelian category $\mathcal{A}(\tau)$?

- Basic requirement for a Bridgeland stability condition $(\mathcal{A}(\tau), Z(\tau))$:

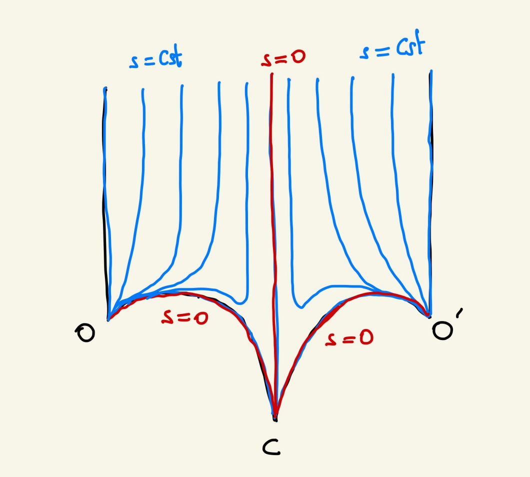
$$Z_E(\tau) \in \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0 \text{ or } z \in \mathbb{R}_{<0}\}$$

  for every $E \in \mathcal{A}(\tau)$ non-zero.
  - By definition of $Z(\tau)$: $\mathrm{Im}(Z_E(\tau)) = -r\mathrm{Im}(T_D) + d\mathrm{Im}(T)$
- For E a sheaf on X (in particular $r \geq 0$), $\mathrm{Im}(Z_E(\tau)) > 0$ is equivalent to

$$\mu = \frac{d}{r} > s := \frac{\mathrm{Im}(T_D)}{\mathrm{Im}(T)}$$

  - If $\mu(E) > s$, want $E \in \mathcal{A}(\tau)$.
  - If $\mu(E) \leq s$, want $E[1] \in \mathcal{A}(\tau)$ (clear if $<$, why if $=$?)
  - For τ in the fundamental domain F_C, away from the lower boundary, $\mathcal{A}(\tau) := \mathcal{A}_s$ tilt of the heart of coherent sheaves.

Curves $s = constant$



Proof of $Z_E(\tau) < 0$ if $\mu(E) = s$

- For polynomial stability conditions on surfaces

$$Z_E(s,t) := -\int_{\mathbb{P}^2} e^{-(s+it)H} ch(E)$$

(much of the focus in the math litterature, works for any surface), follows from the Bogomolov-Gieseker inequality

$$\Delta(E) := ch_2(E) - 2ch_0(E)ch_1(E) \geq 0$$

for every E slope-semistable torsion free sheaf.

- Here, "physics" stability condition on $X = K_{\mathbb{P}^2}$, need to know something stronger:

$$\Delta(E) \geq \frac{3}{8}$$

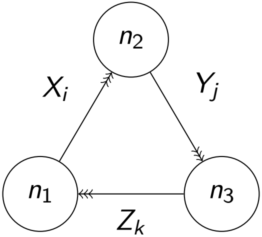
for every E slope-semistable torsion free sheaf on $\mathbb{P}^2$ which is not a line bundle (follows from Drézet-Le Potier study of sheaves on $\mathbb{P}^2$).

What if τ is on the lower boundary of F_C?

- On the OC arc, $s = 0$, $\mathcal{A}(\tau) = \mathcal{A}_0^+$.
- At the orbifold point O, $\mathrm{Im}\, T = 0$, s is not well-defined.

$$\mathcal{A}(\tau_O) = Coh_0(\mathbb{C}^3/(\mathbb{Z}/3\mathbb{Z})) = Rep^{nilp}(Q, W)$$

induced by the exceptional collection $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ on $\mathbb{P}^2$.



Potential $W = \sum_{i,j,k} \epsilon_{ijk} Z_k Y_j X_i$ with ϵ_{ijk} the totally antisymmetric tensor with $\epsilon_{123} = 1$.

Extension to general $\tau \in \mathbb{H}$ by $\Gamma_1(3)$ symmetry

- There exists a group homomorphism

$$\Gamma_1(3) \longrightarrow \mathrm{Aut}(D_{\mathbb{P}^2}(X))$$

  - Monodromy around the large volume point:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longmapsto (E \mapsto E \otimes \mathcal{O}(1))$$

  - Monodromy around the conifold point:

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \longmapsto (ST_{\mathcal{O}} : E \mapsto Cone(RHom(\mathcal{O}, E) \otimes \mathcal{O} \to E))$$

DT/BPS invariants

- To summarize:

$$\widetilde{M_K(X)} = \mathbb{H} \to Stab(D_{\mathbb{P}^2}(X))$$
$$\tau \mapsto (\mathcal{A}(\tau), Z(\tau))$$

- We can then do DT theory.
  - Moduli spaces

$$M(\gamma, \tau) = \{\tau\text{-semistable objects in } \mathcal{A}(\tau) \text{ of class } \gamma\}$$

  - DT/BPS invariants:

$$\Omega(\gamma, \tau) \in \mathbb{Z} \text{ or } \mathbb{Z}[y^{\pm}]$$

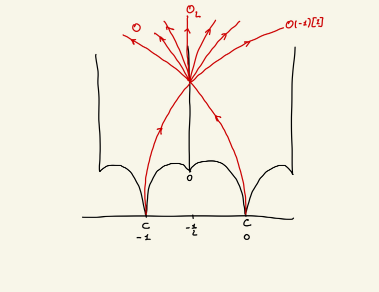
  - Wall-crossing as a function of $\tau \in \mathbb{H}$.
- Goal: study of the DT/BPS invariants using flow trees organized in "scattering diagrams" in $\widetilde{M_K(X)} = \mathbb{H}$
  - supergravity attractor picture
  - Kontsevich-Soibelman wall-structure on base of complex integrable systems.

Scattering diagrams

- Pick a phase $\psi \in \mathbb{R}/2\pi\mathbb{Z}$
  - For every $\gamma \in \Gamma$, consider the 1-dimensional locus, “rays":

$$\mathcal{R}_\gamma^+(\psi) := \{\tau \in \mathbb{H} \,|\, \mathrm{Arg}(Z_\gamma(\tau)) = \psi + \frac{\pi}{2}\,, \Omega(\gamma,\tau) \neq 0\} \subset \mathbb{H}$$

  - Orient rays such that $\mathrm{Im}(e^{-i\psi} Z_\gamma(\tau)) = |Z_\gamma(\tau)|$ increases.
  - “K-wall" in Gaiotto-Moore-Neitzke terminology.



- “Scattering diagram of phase ψ":

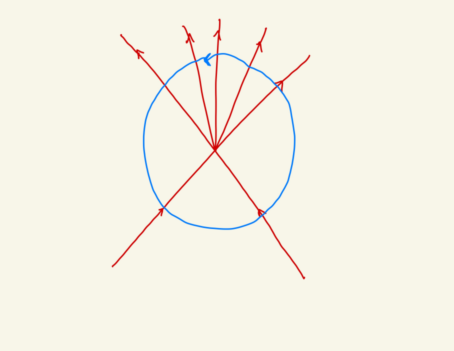
$$\mathcal{D}(\psi) = \bigcup_{\gamma \in \Gamma} (\mathcal{R}_\gamma^+(\psi), \exp(\overline{\Omega}(\gamma, \tau) z^\gamma)$$

  - Rational DT invariants

$$\overline{\Omega}(\gamma, \tau)(y) = \sum_{\gamma = k\gamma'} \frac{1}{k} \frac{1}{y^k - y^{-k}} \Omega(\gamma', \tau)(y^k)$$

  - Quantum torus variables: $z^\gamma z^{\gamma'} = y^{\langle \gamma, \gamma' \rangle} z^{\gamma + \gamma'}$, where $\langle -, - \rangle$: Euler form on Γ

- Reformulation of the wall-crossing formula: $\mathcal{D}(\psi)$ is consistent
  - for every point $\tau \in \mathbb{H}$, the ordered product of quantum torus elements of attached to the rays passing through τ is the identity.



Main result

Theorem

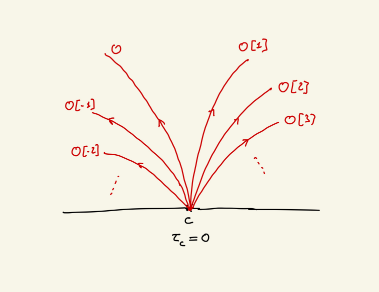
For every $\psi \in \mathbb{R}/2\pi\mathbb{Z}$, the scattering diagram $\mathcal{D}(\psi)$ can be uniquely reconstructed from:

- *Explicit initial rays coming from the conifold points.*
- *Scatterings imposed by the consistency condition.*

- Application: algorithmic reconstruction of the full BPS spectrum (except pure D0) at any point of the physical space of stability conditions.

Initial rays

At the conifold point $\tau_O = 0$, $Z_{\mathcal{O}}(\tau_O) = 0$. Infinitly many initial rays corresponding to the objects $\mathcal{O}[k]$, $k \in \mathbb{Z}$.



General conifold point: apply $\Gamma_1(3)$, spherical object E becoming massless, infinitly many initial rays corresponding to the objects $E[k]$, $k \in \mathbb{Z}$.

Reconstruction from initial rays

- Rays of $\mathcal{D}(\psi)$ are gradient flow lines of $\mathrm{Im}(e^{-i\psi} Z_\gamma(\tau))$. Then, $\mathrm{Re}(e^{-i\psi} Z_\gamma(\tau)) = Cst$, and we recover the field theory limit of attractor flow trees (Denef).
- Key point: for every $\gamma \in \Gamma$, the holomorphic function

$$\mathbb{H} \to \mathbb{C}$$

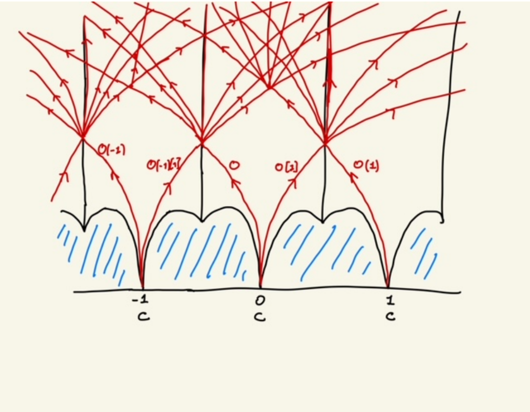
$$\tau \mapsto Z_\gamma(\tau)$$

  has no critical point on $\mathbb{H}$:

$$\frac{d}{d\tau} Z_\gamma(\tau) = (-r\tau + d) C(\tau) \neq 0$$

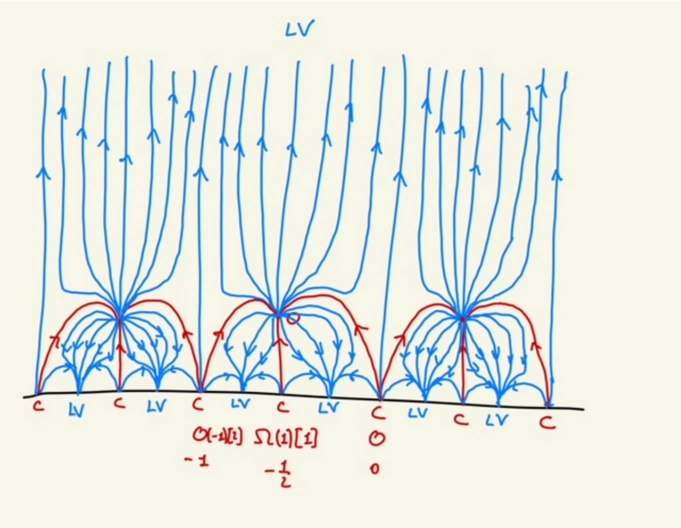
- Study of the boundary behavior: $C(\tau) \to 0$ when τ goes to a conifold point, not otherwise.
- For compact Calabi-Yau 3-folds, normalized non-holomorphic central charge $Z_\gamma^{norm} = e^{K/Z} Z_\gamma = \int_\gamma \Omega / \sqrt{|\int_Y \Omega \wedge \overline{\Omega}|}$, non-trivial "attractor points".

The scattering diagram $\mathcal{D}(\psi)$ for $\psi = 0$



Recovers the scattering diagram of arXiv:1909.02985, computing the BPS spectrum at large volume, and so the scattering diagram of Carl-Pumperla-Siebert counting holomorphic curves in $(\mathbb{P}^2, E)$ for mirror symmetry (hyperkähler rotated version of the special Lagrangians).

The scattering diagram $\mathcal{D}(\psi)$ for $\psi = \frac{\pi}{2}$



The scattering diagram $\mathcal{D}(\psi)$ for $\psi = \frac{\pi}{2}$

Theorem

The local picture of $\mathcal{D}(\pi/2)$ near the orbifold point τ_O can be identified with the slice $\theta_1 + \theta_2 + \theta_3 = 0$ of the stability scattering diagram in $\mathbb{R}^3$ for (Q, W).

Corollary

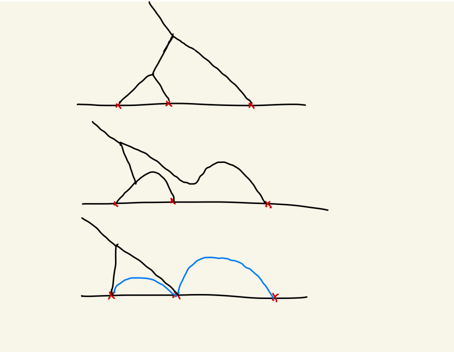
The stability scattering diagram of (Q, W) equals the corresponding cluster scattering diagram (up to the D0 wall), and so the attractor invariants of (Q, W) are trivial (up to D0 branes).

Alternative algebraic proof of the Corollary: see Descombes's talk.

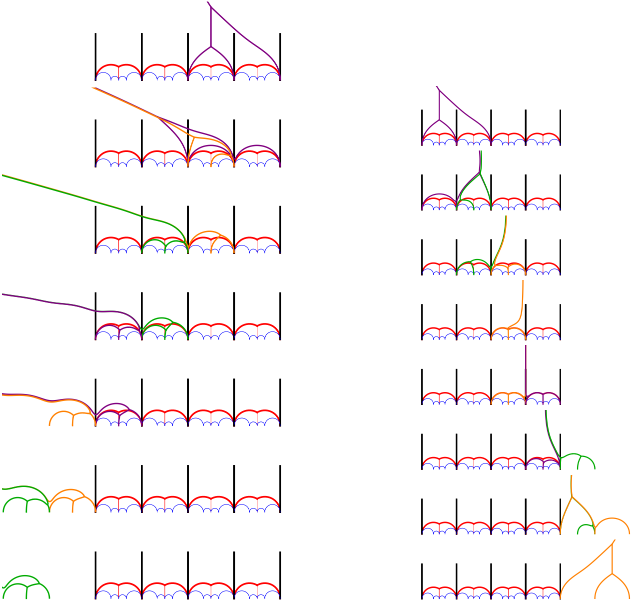
- The global picture of $\mathfrak{D}(\pi/2)$ also give a clear description of the correspondence between normalized $(-1 < \mu \leq 0)$ torsion free Gieseker semi-stable sheaves on $\mathbb{P}^2$ and representations of the Beilinson quiver.
- For these objects $\mathcal{D}(0)$ gives a path from the large volume point to the orbifold point avoiding the walls of marginal stability.

Interpolation between $\psi = 0$ and $\psi = \frac{\pi}{2}$

- Interesting discrete jumps.
- Infinite sequence of critical phases.
- Physics interpretation?



Jumps in $\mathcal{D}(\psi)$ as function of ψ



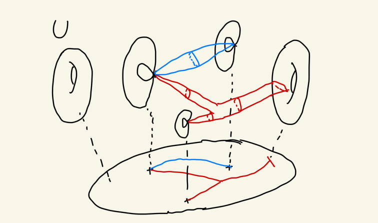
Comparison with spectral networks

- Both (exponential) spectral networks and scattering diagrams compute the same DT/BPS numbers, “orthogonal approaches":
  - Spectral networks drawn on a projection of the mirror curves.
  - Scattering diagrams drawn on the base of the family of mirror curves.
- Analogy: scattering diagrams as “4d spectral network".

Analogy with spectral networks

- "2d spectral newtworks":
  - capture BPS spectrum of a $\mathcal{N} = (2,2)$ 2d theory
  - rays interact according to the 2d Cecotti-Vafa wall-crossing formula, geometrically: special Lagrangians in T^*C with boundary on spectral curve $\Sigma \to C$ (n:1 cover).
  - Wall-crossing structure for $\mathfrak{gl}_n$.
  - Jumps in ψ: BPS spectrum of $\mathcal{N} = 2$ 4d theory containing the $\mathcal{N} = (2,2)$ 2d theory as a surface defect.
- Scattering diagrams = "4d spectral newtworks":
  - capture BPS spectrum of a $\mathcal{N} = 2$ 4d theory
  - rays interact according to the 4d Kontsevich-Soibelman wall-crossing formula, geometrically: special Lagrangians in a torus fibration with boundary on a torus fiber.
  - Wall-crossing structure for the quantum torus Lie algebra.
  - Jumps in ψ?

- U: elliptic fibration with base a 3:1 cover of $\mathcal{M}_K(X)$. HyperKähler 4-manifold.
- Known string duality: IIA on $\mathbb{R}^{1,3} \times X$ dual to IIA on $\mathbb{R}^{1,5} \times U$ with a NS5-brane on $\mathbb{R}^{1,3} \times T^2$, where T^2 is an elliptic fiber of U.
- 4d BPS spectrum: D2-branes/special Lagrangians in U ending on the NS5 brane. Scattering diagrams: projections on the base (string junctions).
- $\mathcal{N} = 2$ 4d theory as a defect in the $\mathcal{N} = (1,1)$ 6d theory obtained by IIA on $\mathbb{R}^{1,5} \times U$. Jumps of the scattering diagrams as a function of ψ: closed $D2$-branes/special Lagrangians in U, i.e. 6d BPS states.
- 4d-6d wall-crossing formula?



Thank you for your attention !