Workshop on “Quivers, Calabi-Yau threefolds and Donaldson-Thomas invariants”

April 11-15, 2022 - Sorbonne Université, Paris

Amphithéâtre Charpak, Tour 22, Rez-de-chaussée
Campus Pierre et Marie Curie, Métro Jussieu (Subway line 7)

Organizers :
Tom Bridgeland (Sheffield),
Michela Petrini (Sorbonne U, LPTHE),
Boris Pioline (CNRS LPTHE),
Olivier Schiffmann (CNRS, Paris Saclay),
Dan Waldram (Imperial College London)
Alberto Zaffaroni (INFN, Milano Biccocca)

The workshop will take place in hybrid mode, the Zoom password will be communicated to registered participants in due time.

Preliminary chedule:

MONDAY, 11/04
(Chair : Alberto Zaffaroni)
10h00–11h00 Ben Davison
11h30–12h30 Michele Del Zotto

(Chair : Ilya Itenberg)
14h00–15h00 Hülya Argüz
15h30–16h30 Pietro Longhi

TUESDAY, 12/04
(Chair : Tom Bridgeland)
10h00–11h00 Dominic Joyce
11h30–12h30 Soheyla Feyzbakhsh

(Chair :Michela Petrini)
14h00–15h00 Pierrick Bousseau
15h30–16h30 Wei Li

WEDNESDAY, 13/04
(Chair : Olivier Schiffmann)
10h00–11h00 Sergey Mozgovoy
11h30–12h30 Maxim Kontsevich

14h00–15h00 David Tennyson
(Chair : Tom Bridgeland, Dan Waldram, Alberto Zaffaroni)
15h30–16h30 Q&A

THURSDAY, 14/04
(Chair : Amir Kashani-Poor)
10h00–11h00 Alessandro Tomasiello

11h30–12h30 Hendrik Suess

(Chair : Dan Waldram)
14h00–15h00 Pierre Descombes
15h30–16h30 Murad Alim

FRIDAY, 15/04
(Chair :Kimyeong Lee)
10h00–11h00 Sakura Schaefer-Nameki
11h30–12h30 Alessandro Tanzini

(Chair: TBA)
14h00–15h00 Julius Grimminger
15h30–16h30 Albrecht Klemm

Titles and Abstracts:

Murad Alim (Hamburg) : **Non-perturbative quantum geometry, resurgence and BPS structures**
BPS invariants of certain physical theories correspond to Donaldson-Thomas (DT) invariants of an associated Calabi-Yau geometry. BPS structures refer to the data of the DT invariants together with their wall-crossing structure. On the same Calabi-Yau geometry another set of invariants are the Gromov-Witten (GW) invariants. These are organized in the GW potential, which is an asymptotic series in a formal parameter and can be obtained from topological string theory. A further asymptotic series in two parameters is obtained from refined topological string theory which contains the Nekrasov-Shatashvili (NS) limit when one of the two parameters is sent to zero. I will discuss in the case of the resolved conifold how all these asymptotic series lead to difference equations which admit analytic solutions in the expansion parameters. A detailed study of Borel resummation allows one to identify these solutions as Borel sums in a distinguished region in parameter space. The Stokes jumps between different Borel sums encode the BPS invariants of the underlying geometry and are captured in turn by another set of difference equations. I will further show how the Borel analysis of the NS limit connects to the exact WKB study of quantum curves. This is based on various joint works with Lotte Hollands, Arpan Saha, Iván Tulli and Jörg Teschner.

Hülya Argüz (IST Vienna): **Donaldson--Thomas invariants of quivers with potentials from the flow tree formula**
A categorical notion of stability for objects in a triangulated category was introduced by Bridgeland. Donaldson--Thomas (DT) invariants are then defined as virtual counts of semistable objects. We will focus attention on a natural class of triangulated categories defined via the representation theory of quivers with potentials, and explain how to compute DT invariants in this case from a smaller subset of ``attractor invariants'' which are known in many cases. For this, we investigate wall-crossing in the space of stability conditions and prove a flow tree formula conjectured by Alexandrov-Pioline in this setup. This is joint work with Pierrick Bousseau.

Pierrick Bousseau (Orsay and ETH Zürich) : **D-branes on local P2 revisited**
I will describe a one-parameter family of scattering diagrams computing Donaldson-Thomas invariants of local P2 at any point of the physical slice in the space of Bridgeland stability conditions. The scattering diagrams are made of attractor flow trees and are also projections of special Lagrangian submanifolds in the universal family of mirror curves. I

will also present a connection with the scattering diagram describing Donaldson-Thomas invariants of the McKay quiver associated to local P2. This is joint work with Pierre Descombes, Bruno Le Floch and Boris Pioline.

Ben Davison (Edinburgh): **BPS Lie algebras, from mathematics, for physicists**
Ten years ago, Kontsevich and Soibelman provided a mathematical construction, utilising quiver representations, vanishing cycles, and Hall algebras, of the algebra of BPS states associated to certain 4d N=2 supersymmetric gauge theories. The starting data for the construction is a quiver with potential. In the context of quantum groups, these algebras should be analogues/generalisations of (half of) Yangians associated to semisimple Lie algebras: for very special choices of quivers with potentials this is exactly what their construction produces. In particular, although the algebra is very "large", it is controlled in a precise way by something "small" a graded-finite-dimensional subspace that is endowed with a Lie bracket via the commutator in the ambient algebra.
In this talk I will explain how to define a similar BPS Lie algebra controlling the algebra of BPS states for general quivers with potential, using perverse filtrations, as well as explaining some links with nonabelian Hodge theory. This is meant as a guide to the mathematical construction of the BPS Lie algebra, for physicists.

Pierre Descombes (Sorbonne U): **Donaldson-Thomas invariants of toric quivers**
The category of sheaves on a toric threefold is derived equivalent to the category of representation of a quiver with potential obtained from a brane tiling of the torus. On this class of examples, the numerical DT invariants can be computed by toric localization with respect to an action scaling the arrows of the quiver, by enumerating the fixed points, described combinatorially by pyramid partitions.
A toric localization formula was proposed by the K-theoretic DT formalism in order to compute refined DT invariants, but these computations were shown not to agree with computations of the cohomological refinement provided by the vanishing cycles even in the simplest cases. We present here how to interpret and solve this mismatch, using an equivalent of the Bialynicky Birula decomposition for critical locus of potential. DT invariants for toric threefolds without compact divisors are completely classified, but those of toric threefolds with compact divisors are generally expected to be wild. We present a conjectural formula for attractor invariants of any toric quivers, elementary bricks from which all the DT invariants can be built, corresponding to initial data of the stability scattering diagram of the quiver. We will prove this formula for P^2, the simplest example with compact divisor.

Michele del Zotto (Uppsala): **Remarks on Correspondences from Geometric Engineering**
I will briefly review a strategy to obtain correspondences between supersymmetric quantum field theories in various dimensions building upon geometric engineering techniques. Several new applications and examples will be presented, highlighting the interconnections with the enumerative geometry of backgrounds with special holonomy. In particular, we will include some results about the higher Donaldson-Thomas theory for Calabi-Yau three-folds and 5d BPS quivers, some applications in the context of certain classes of G2 manifolds, and also some ideas in the context of generalizations of level/rank dualities for Vafa-Witten partition functions on ALE spaces.

Soheyla Feyzbakhsh (Imperial College) : **Rank r DT theory from rank 1**
Fix a Calabi-Yau 3-fold X satisfying the Bogomolov-Gieseker conjecture of Bayer-Macr\`i-Toda, such as the quintic 3-fold. After a brief introduction of weak Bridgeland stability conditions, I will describe work with Richard Thomas which expresses Joyce's generalised DT invariants counting Gieseker semistable sheaves of any rank r on X in terms of those

counting sheaves of rank 1. By the MNOP conjecture, the latter are determined by the Gromov-Witten invariants of X. Finally, I will show our result gives an explicit formula for rank r=0 or 2 when X is of Picard rank one.

Julius Grimminger (Imperial College): **3d Coulomb Branch and Magnetic Quivers**
Three-dimensional N=4 theories admit a rich moduli space of vacua, with two distinct hyper-Kähler (HK) subspaces. While the Higgs branch is long understood as a HK quotient (1980s), the Coulomb branch (CB) has proven a much tougher nut to crack (2010s). It turns out it is a new HK construction in its own right.
Understanding CBs is still an active area of research, and we will review the constant progress which was made over the last few years. We will focus on the Hilbert series approach to the CB - the monopole formula - while also mentioning other approaches. CBs have been applied successfully to solve HK problems in both physics and mathematics, mostly under the guise of 'magnetic quivers'. Using examples we will introduce the notion of magnetic quiver, hoping to convey its power.

Dominic Joyce (Oxford): **A universal theory of enumerative invariants and wall-crossing formulae**.
I outline a (very long and complicated, sorry) programme which gives a common universal structure to many theories of enumerative invariants counting semistable objects in abelian or derived categories in Algebraic Geometry, for example, counting coherent sheaves on curves, surfaces, Fano 3-folds, Calabi-Yau 3- or 4-folds, or representations of quivers (with relations). Write A for your abelian or derived category, K(A) for its numerical Grothendieck group, t for the stability condition, M for the usual moduli stack of objects in A, and M^{pl} for the 'projective linear' moduli stack of objects modulo "projective linear" isomorphisms (quotient by multiples of identity morphisms). Then (oversimplifying a bit):
(i) the homology H_*(M,Q) has the structure of a graded vertex algebra (or a graded vertex Lie algebra in the 3-Calabi-Yau case).
(ii) We have H_*(M^{pl},Q) = H_*(M,Q) / D(H_*(M,Q)), where D is the translation operator in the vertex algebra. Therefore H_*(M^{pl},Q) has the structure of a graded Lie algebra. It seems very difficult to understand this Lie bracket without going via the vertex algebra.
(iii) For each class a in K(A) we have a moduli stack M_a^{ss}(t) of t-semistable object in A in class K(A). We can define invariants [M_a^{ss}(t)]_{inv} in H_*(M^{pl}_a,Q). If there are no semistables in class a, this is just the virtual class of M_a^{ss}(t), and is defined over Z. If there are semistables, it is defined over Q, and has a complicated definition involving auxiliary pair invariants.
(iv) If t, t* are two stability conditions, there is a universal wall-crossing formula which writes [M_a^{ss}(t*)]_{inv} as a Q-linear combination of repeated Lie brackets of invariants [M_b^{ss}(t)]_{inv}, using the Lie bracket on H_*(M^{pl},Q) from (ii).
The programme above is proved for invariants defined using Behrend-Fantechi perfect obstruction theories and virtual classes. I expect to extend it to Calabi-Yau 4-fold obstruction theories and virtual classes, a la Borisov-Joyce / Oh-Thomas. The programme includes “reduced” invariants (for example, counting coherent sheaves on surfaces with p_g > 0), for which the wall-crossing formula is modified.The wall-crossing formulae are effective computational tools in examples. I am currently using them to compute invariants counting semistable sheaves on projective surfaces (algebraic Donaldson invariants) from Seiberg-Witten invariants.The appearance of the vertex algebras in (i), in relation to enumerative invariants, is a complete mystery (at least to me). I invite String Theorists to explain it. Based on: arXiv:2005.05637 (joint with Jacob Gross and Yuuji Tanaka), arXiv:2111.04694, and work in progress.

Albrecht Klemm (Bonn): **Topological String on Non-Commutative Resolutions**
Based on a project with Sheldon Katz, Thorsten Schimannek and Eric Sharpe we describe a simple example of a non-commutative resolution namely the one of a singular double cover of P^3. This

exhibits 84 nodes whose small blow ups give rise to torsion classes in $H_2(\hat M,Z)$. The torsion classes support a non-trivial B-field and can be described in terms of non-commutative geometry. We argue that this geometry corresponds to the Landau-Ginzburg phase of the complete intersection of four quadrics in P^7. Like the mirror of the double cover of P^3 the mirror of the latter has a one parameter hyper-geometric Picard-Fuchs equation, albeit with a second point of maximal unipotent monodromy. It is this second MUM point that yields the B-model description of the non-commutative resolution and allows detailed studies of the higher genus BPS invariants on the non-commutative resolution which are obtained from the wave function transform of the standard string partition function Z at the first MUM point. We provide some geometric checks for the BPS states and some implications for the arithmetic understanding of one parameter families that is recently developed.

Maxim Kontsevich (IHES) : **Stability Structures in Holomorphic Morse-Novikov Theory**
I will talk about an elementary example illustrating the general program (by Y.Soibelman and myself) relating Floer theory for complex symplectic manifolds, quantization and resurgence. The specific question is about the space of morphisms between two specific branes in the cotangent bundle to a complex manifold. The first brane is the zero section endowed with a generic rank 1 local system, while the second brane is the graph of a closed holomorphic 1-form. In this case the whole story is reduced to Morse-Novikov theory, DT invariants count gradient lines connecting zeroes of 1-form. The graded Lie algebra in this case is the algebra of matrix-valued functions on an algebraic torus, which is simpler than the usual Lie algebra of Hamiltonian vector fields as in WKB analysis. The wall-crossing structure can be completely characterized by a quadruple of explicit rational matrix-valued functions.

Wei Li (CAS Beijing): **From BPS crystals to BPS algebras: constructions, representations, and applications**
I will explain how to construct BPS algebras for string theory on general toric Calabi-Yau threefolds, based on the crystal melting description of the BPS sectors. The resulting quiver Yangians, together with their trigonometric and elliptic versions, unify various known results and generalize them to a much larger class. I will then explain how to describe their representations using subcrystals and how they can be translated to the framings of the quivers. Time permitting, I will also discuss some applications.

Pietro Longhi (Uppsala): **Counting Lagrangian A-branes with networks**
The framework of spectral networks was introduced in physics as a way to compute BPS states of 4d N=2 gauge theories. In this talk I will review a generalization, known as exponential networks, which produces enumerative invariants associated to special Lagrangians in certain Calabi-Yau threefolds. Applications include the computation of the exact spectrum for the mirror of the local Hirzebruch surface. I will also sketch a new derivation of this framework, which elucidates the geometric meaning of the invariants in terms of elementary data of A-branes.

Sergey Mozgovoy (Trinity College Dublin): **Wall-crossing structures arising from surfaces**
Families of Bridgeland stability conditions induce families of stability data (DT invariants), wall-crossing structures and scattering diagrams on the motivic Hall algebra. These structures can be transferred to the quantum torus if the stability conditions of the family have global dimension at most 2. I will discuss geometric stability conditions on a surface with nef anticanonical bundle. These stability conditions have global dimension 2, hence induce a family of stability data. I will also discuss the relationship of this family to the

family of stability data associated to a quiver with potential, with an emphasis on the projective plane.

Sakura Schaefer-Nameki (Oxford): **Generalized Symmetries in String Compactifications**
Generalized symmetries, such as higher-form and higher-group symmetries, will be discussed in Quantum Field Theories constructed in geometric engineering. The talk will provide an overview of recent, very lively, developments on this topic.

Hendrik Suess (Jena) :**Three-dimensional Calabi-Yau cones with 2-torus action**
There are two main constructions of Calabi-Yau cones in dimension 3. Firstly, the anti canonical cones over (log) del Pezzo surfaces and secondly via Gorenstein toric singularities. The toric construction automatically comes with the action of a 3-dimensional torus and the Calabi-Yau condition is automatically fulfilled. For the construction from del Pezzo surfaces we only obtain a 1-dimensional torus action and the Kähler-Einstein condition for the del Pezzo surfaces is crucial to obtain a Calabi-Yau cone metric. In my talk I will address the remaining cases with 2-torus action by discussing a combinatorial approach which interpolates between the two previous constructions and also explain how the Calabi-Yau property is reflected in this combinatorial language.

Alessandro Tanzini (SISSA): **Surface defects, tt* Toda equations and BPS spectra**
We show that the partition functions of 4d supersymmetric gauge theories with 8 supercharges in presence of surface defects obey tt equations for a suitable isomonodromic deformation problem, and we comment on its M-theory origin. The solution to these equations provides new recursion relations for instanton counting for all simple groups from A to E. The uplift to 5d is a discrete flow generated by automorphisms of the associated BPS quiver. We show that for a class of theories, the 4d reduction of these discrete flows displays an intriguing new relation with Argyres-Douglas SCFTs.*

David Tennyson **Exceptional Complex Structures and K-stability**
I will introduce a new geometric object called the Exceptional Complex Structure (ECS). This is an extension of the notion of complex structure to include all of the degrees of freedom of string backgrounds, much like the Generalised Complex Structure of Hitchin and Gualtieri was an extension that naturally included the B-field. In the first half of the talk I will define the ECS and provide some classification results, showing that they provide a unified framework to describe many supersymmetric geometries in various dimensions including complex structures, G_2, and their flux deformed counterparts. In the second half of my talk, I will discuss an intriguing observation that one may be able to use the ECS to extend the notions of K-stability to any supersymmetric background, possibly providing new tools to define and study G2 stability.

Alessandro Tomasiello (Milano Biccoca): **Holography, Quivers and Geometry**
In string theory, a spacetime with an anti-de Sitter (AdS) factor is related to a conformal field theory (CFT) by a famous correspondence. Many such spacetimes are obtained by considering D-branes on conical Calabi–Yau singularities, which can in turn be obtained by K-stability techniques. The corresponding CFT is often, but not always, suggested by matrix factorization techniques.