Typical correlations and entanglement in random tensor network states

Mostly based on joint work with David Pérez-García, available at arXiv:1906.11682

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Outline

1. Background and motivations
2. Random MPS and PEPS: construction and statement of the main questions & results
3. Typical correlation length in a random TNS through typical spectral gap of its transfer operator
4. Typical correlation length in a random TNS through spectral gap of its parent Hamiltonian
5. Open questions and perspectives

The curse of dimensionality when dealing with many-body quantum systems

A quantum system composed of N d-dimensional subsystems has dimension d^N.
$\longrightarrow$ Exponential growth of the dimension with the number of subsystems.
However, *'physically relevant' states* of many-body quantum systems are often well approximated by so-called *tensor network states (TNS)*, which form a small subset of the global state space.

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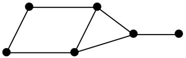
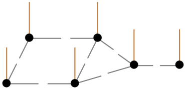
Tensor network state construction on $(\mathbf{C}^d)^{\otimes N}$:
Take a graph G with N vertices and L edges. (degree of v)
Put at each vertex v a tensor $|\chi_v\rangle \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes \delta(v)}$ to get a tensor $|\tilde{\chi}_G\rangle \in (\mathbf{C}^d)^{\otimes N} \otimes (\mathbf{C}^q)^{\otimes 2L}$.
Contract together the indices of $|\tilde{\chi}_G\rangle$ associated to a same edge to get a tensor $|\chi_G\rangle \in (\mathbf{C}^d)^{\otimes N}$.
pure state on $(\mathbf{C}^d)^{\otimes N}$ (up to normalization)

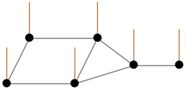
$\longrightarrow$ If $\delta(v) \leqslant r$ for all v, then such state is described by at most $Nq^r d$ parameters (linear in N).



G with 6 vertices and 7 edges



$|\tilde{\chi}_G\rangle \in (\mathbf{C}^d)^{\otimes 6} \otimes (\mathbf{C}^q)^{\otimes 14}$



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d-dimensional indices: *physical* indices. *q*-dimensional indices: *bond* indices.

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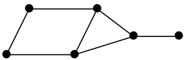

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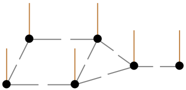

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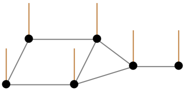

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In this talk: The underlying graph G is a regular lattice in dimension 1 or 2.
$\longrightarrow$ $|\chi_G\rangle$ is a *matrix product state (MPS)* or a *projected entangled pair state (PEPS)*.

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Motivations behind TNS and related questions

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Example: Being a TNS is (conjectured to be) approximately equivalent to being a ground state of a gapped local Hamiltonian (Hastings, Landau/Vazirani/Vidick...)

↳ composed of terms which act non-trivially only on nearby sites

↳ spectral gap lower bounded by a constant independent of N

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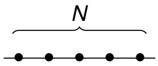
General question: Among the various *qualitative* intuitions about TNS, which ones are at least true generically? And can *quantitative* statements be made about the generic case?

Idea: Sample a TNS at random and study the characteristics that it exhibits with high probability.
In particular, what is its *typical amount of correlations and entanglement*?
Regime we are interested in: large d, q, ideally any N.

Outline

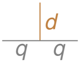
A simple model of random translation-invariant MPS

N particles on a circle

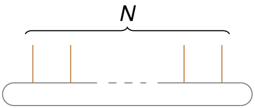


Pick a tensor $|\chi\rangle \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes 2}$ whose entries are independent complex Gaussians with mean 0 and variance $1/dq$.
Repeat it on all sites and contract neighboring q-dimensional indices.
$\longrightarrow$ Obtained tensor $|\chi_N\rangle \in (\mathbf{C}^d)^{\otimes N}$: *random translation-invariant MPS with periodic boundary conditions* (typically almost normalized).

1-site tensor



$$|\chi\rangle = \sum_{x=1}^{d} \sum_{i,j=1}^{q} g_{xij} |xij\rangle$$



$$|\chi_N\rangle = \sum_{x_1,\ldots,x_N=1}^{d} \left(\sum_{i_1,\ldots,i_N=1}^{q} g_{x_1 i_N i_1} \cdots g_{x_N i_{N-1} i_N} \right) |x_1 \cdots x_N\rangle$$

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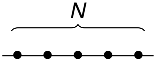

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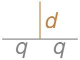
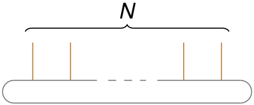
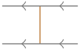




$$|\chi\rangle = \sum_{x=1}^{d} \sum_{i,j=1}^{q} g_{xij} |xij\rangle \qquad |\chi_N\rangle = \sum_{x_1,\ldots,x_N=1}^{d} \left(\sum_{i_1,\ldots,i_N=1}^{q} g_{x_1 i_N i_1} \cdots g_{x_N i_{N-1} i_N} \right) |x_1 \cdots x_N\rangle$$

Associated *transfer operator*: $T : \mathbf{C}^q \otimes \mathbf{C}^q \to \mathbf{C}^q \otimes \mathbf{C}^q$, obtained by contracting the d-dimensional indices of $|\chi\rangle$ and $|\bar{\chi}\rangle$.

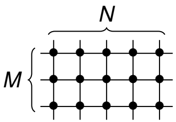


$$T = \sum_{x=1}^{d} \left(\sum_{i,j,k,l=1}^{q} g_{xij} \bar{g}_{xkl} |ik\rangle\langle jl| \right) = \frac{1}{d} \sum_{x=1}^{d} G_x \otimes \bar{G}_x$$

The G_x's are independent $q \times q$ matrices whose entries are independent complex Gaussians with mean 0 and variance $1/q$.

A simple model of random translation-invariant PEPS

MN particles on a torus

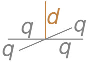


Pick a tensor $|\chi\rangle \in \mathbf{C}^d \otimes (\mathbf{C}^q)^{\otimes 4}$ whose entries are independent complex Gaussians with mean 0 and variance $1/dq^2$.

Repeat it on all sites and contract neighboring q-dimensional indices.

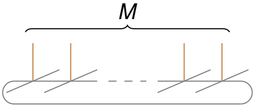
$\longrightarrow$ Obtained tensor $|\chi_{MN}\rangle \in (\mathbf{C}^d)^{\otimes MN}$: *random translation-invariant PEPS with periodic boundary conditions* (typically almost normalized).

1-site tensor



$$|\chi\rangle = \sum_{x=1}^{d} \sum_{i,j,i',j'=1}^{q} g_{xiji'j'} |xiji'j'\rangle$$

1-column tensor



$$|\chi_M\rangle \in (\mathbf{C}^d \otimes \mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M}$$

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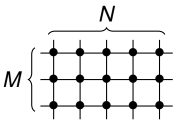

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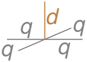

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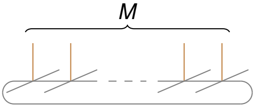

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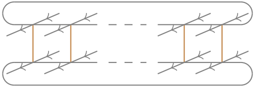
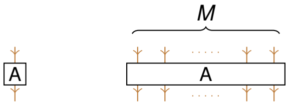


$$T_M = \frac{1}{d^M q^M} \sum_{x_1,\ldots,x_M=1}^{d} \sum_{i_1,j_1,\ldots,i_M,j_M=1}^{q} G_{x_1 i_M i_1} \otimes \bar{G}_{x_1 j_M j_1} \otimes \cdots \otimes G_{x_M i_{M-1} i_M} \otimes \bar{G}_{x_M j_{M-1} j_M}$$

The G_{xij}'s are independent $q \times q$ matrices whose entries are independent complex Gaussians with mean 0 and variance $1/q$.

Correlations in a TNS

MPS: $M=1$. PEPS: $M>1$.
A, B observables on $(\mathbf{C}^d)^{\otimes M}$, i.e. on 1 site for an MPS and on 1 column of M sites for a PEPS.
Goal: Quantify the correlations between the outcomes of A and B, when performed on 'distant' sites or columns.



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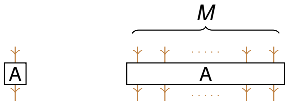

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Compute the value on the TNS $|\chi_{MN}\rangle$ of the observable $A \otimes \mathrm{I}^{\otimes k} \otimes B \otimes \mathrm{I}^{\otimes N-k-2}$, i.e.

$$v_\chi(A,B,k) := \langle \chi_{MN} | A \otimes \mathrm{I}^{\otimes k} \otimes B \otimes \mathrm{I}^{\otimes N-k-2} | \chi_{MN} \rangle.$$

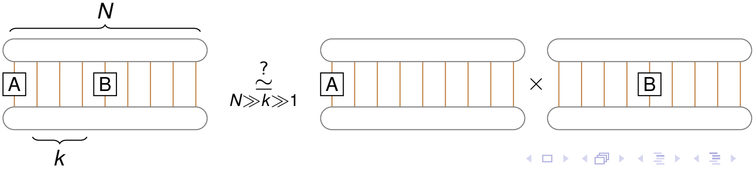
Compare it to the product of the values on $|\chi_{MN}\rangle$ of $A \otimes \mathrm{I}^{\otimes N-1}$ and $\mathrm{I}^{\otimes k+1} \otimes B \otimes \mathrm{I}^{\otimes N-k-2}$, i.e.

$$v_\chi(A)v_\chi(B) := \langle \chi_{MN} | A \otimes \mathrm{I}^{\otimes N-1} | \chi_{MN} \rangle \langle \chi_{MN} | B \otimes \mathrm{I}^{\otimes N-1} | \chi_{MN} \rangle.$$

↳ by translation-invariance of $|\chi_{MN}\rangle$

Correlations in the TNS $|\chi_{MN}\rangle$: $\gamma_\chi(A,B,k) := \left| v_\chi(A,B,k) - v_\chi(A)v_\chi(B) \right|$.

Question: Do we have $\gamma_\chi(A,B,k) \underset{k \ll N \to \infty}{\longrightarrow} 0$? And if so, at which speed?



Intuition: For any TNS $|\chi_{MN}\rangle \in (\mathbf{C}^d)^{\otimes MN}$, the correlations between two 1-site or 1-column observables decay exponentially with the distance separating the two sites or columns, i.e. there exist $C(\chi), \tau(\chi) > 0$ such that, for any $k \ll N$ and any observables A, B on $(\mathbf{C}^d)^{\otimes M}$,

$$\gamma_\chi(A, B, k) \leqslant C(\chi) e^{-\tau(\chi)k} \|A\|_\infty \|B\|_\infty .$$

Correlation length in the TNS $|\chi_{MN}\rangle$: $\xi(\chi) := 1/\tau(\chi)$.

Main result: random TNS typically exhibit fast exponential decay of correlations

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Our main result (informal)

For random translation-invariant MPS and PEPS with periodic boundary conditions this intuition is generically true, with a short correlation length.

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Two ways of proving this: (suited to different dimensional regimes)

- Show that the *transfer operator* associated to the random TNS generically has a *large (upper) spectral gap*. going to 1 as d, q grow ↵
- Show that the *parent Hamiltonian* of the random TNS generically has a *large (lower) spectral gap*. going to 1 as d, q grow ↵

Outline

Correlation length in a TNS and spectrum of its transfer operator

$|\chi_{MN}\rangle \in (\mathbf{C}^d)^{\otimes MN}$ an MPS ($M=1$) or a PEPS ($M>1$).
T its associated transfer operator on $(\mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M}$.
By reading diagrams representing $v_\chi(A,B,k)$, $v_\chi(A)$, $v_\chi(B)$ 'horizontally' instead of 'vertically', its correlation function can be re-written as:

$$\gamma_\chi(A,B,k) = \left|\mathrm{Tr}\left(\tilde{A}T^k\tilde{B}T^{N-k-2}\right) - \mathrm{Tr}\left(\tilde{A}T^{N-1}\right)\mathrm{Tr}\left(\tilde{B}T^{N-1}\right)\right|.$$

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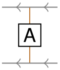

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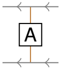

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Important consequence:
Denote by $\lambda_1(T)$ and $\lambda_2(T)$ the largest and second largest eigenvalues of T (with multiplicities).
There exists $C(T)>0$ such that, setting $\varepsilon(T) = |\lambda_2(T)|/|\lambda_1(T)|$, we have

$$\gamma_\chi(A,B,k) \leqslant C(T)\varepsilon(T)^k \|A\|_\infty \|B\|_\infty .$$

$\longrightarrow$ Correlations between two 1-site or 1-column observables decay exponentially with the distance separating the two sites or columns, at a rate $|\log\varepsilon(T)|$, i.e. $\xi(\chi) = 1/|\log\varepsilon(T)|$.

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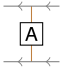

$\tilde{A} : \mathbf{C}^q \otimes \mathbf{C}^q \to \mathbf{C}^q \otimes \mathbf{C}^q$

Important consequence:
Denote by $\lambda_1(T)$ and $\lambda_2(T)$ the largest and second largest eigenvalues of T (with multiplicities).
There exists $C(T) > 0$ such that, setting $\varepsilon(T) = |\lambda_2(T)|/|\lambda_1(T)|$, we have

$$\gamma_\chi(A,B,k) \leqslant C(T)\varepsilon(T)^k \|A\|_\infty \|B\|_\infty.$$

$\longrightarrow$ Correlations between two 1-site or 1-column observables decay exponentially with the distance separating the two sites or columns, at a rate $|\log \varepsilon(T)|$, i.e. $\xi(\chi) = 1/|\log \varepsilon(T)|$.

Question: For a random transfer operator T what are the typical values of $|\lambda_1(T)|$ and $|\lambda_2(T)|$?

Typical spectral gap of the transfer operator of a random MPS

Theorem [Typical spectral gap of the random MPS transfer operator T]

There exist constants $c, C > 0$ such that, for any $d, q \in \mathbf{N}$,

$$\mathbf{P}\left(|\lambda_1(T)| \geqslant 1 - \frac{C}{\sqrt{d}} \text{ and } |\lambda_2(T)| \leqslant \frac{C}{\sqrt{d}} \right) \geqslant 1 - e^{-cq}.$$

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Corollary [Typical correlation length in the random MPS $|\chi_N\rangle$]

There exist constants $c, C' > 0$ such that, for any $d, q, N \in \mathbf{N}$,

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Remark: Similar results obtained on slightly different models, with T composed of independent Haar unitaries (Hastings, Pisier) or blocks of a Haar unitary (González-Guillén/Junge/Nechita).
Motivation: Construction of *quantum expanders*. This Gaussian model provides a new example.

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Proof idea: Approximate first eigenvector for T: maximally entangled state $|\psi\rangle \in \mathbf{C}^q \otimes \mathbf{C}^q$.
Observing that $|\lambda_1(T)| \geqslant |\langle\psi|T|\psi\rangle|$ and $|\lambda_2(T)| \leqslant \|T(\mathrm{I} - |\psi\rangle\langle\psi|)\|_\infty$, we show that:
↳ by minimax principle for singular values, doing as if T were Hermitian

1. $\mathbf{E}|\langle\psi|T|\psi\rangle| = 1$ and $\mathbf{E}\|T(\mathrm{I} - |\psi\rangle\langle\psi|)\|_\infty \leqslant C/\sqrt{d}$ (Gaussian moment computations).
2. These two quantities have a small probability of deviating from their average (local version of Gaussian concentration inequality).

Typical spectral gap of the transfer operator of a random PEPS

Assumption $(\star)$**:** The dimensions d and q grow polynomially with the number of sites M.
More precisely: d, q satisfy $d \simeq M^{\alpha}$ and $q \simeq M^{\beta}$ with $\alpha > 8$ and $(\alpha+1)/3 < \beta < (\alpha-2)/2$.

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Proof idea: Some kind of recursion procedure that uses the MPS results as building blocks.
Approximate first eigenvector for T_M: $|\psi\rangle^{\otimes M} \in (\mathbf{C}^q \otimes \mathbf{C}^q)^{\otimes M}$.

Outline

Parent Hamiltonian of a TNS

Parent Hamiltonian of a TNS: local Hamiltonian which has the TNS as ground state. The latter is unique in the *injectivity regime* (Cirac/Pérez-García/Verstraete/Wolf).

↳ the 1-site tensor, viewed as a map from bond space to physical space, is injective

MPS

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PEPS

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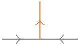

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Fact: If the parent Hamiltonian of a TNS is gapped above its ground state energy then the correlations between two observables on subregions decay exponentially with the distance separating the two subregions. This follows from the *Lieb-Robinson (LR) bound* (Hastings/Koma...) or the *detectability lemma* (Aharonov/Arad/Landau/Vazirani...).

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Parent Hamiltonian of our random TNS:

The random MPS $|\chi_N\rangle$, resp. PEPS $|\chi_{MN}\rangle$, is almost surely injective for $d \geqslant q^2$, resp. $d \geqslant q^4$. In this regime, its parent Hamiltonian H is a 2-local frustration-free Hamiltonian whose unique ground state is $|\chi_N\rangle$, resp. $|\chi_{MN}\rangle$.

↳ the global minimizer minimizes each term

↳ each term acts non-trivially on 2 neighboring sites

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Theorem [Typical spectral gap of the random MPS or PEPS parent Hamiltonian H]

There exist constants $c, C > 0$ such that, for any $N \in \mathbf{N}$ and any $d, q \in \mathbf{N}$ satisfying $d \geqslant q^{a+\varepsilon}$, for some $\varepsilon > 0$,

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Outline

Open questions

- **Typical spectral gap of random transfer operators:** Could the results in the PEPS case be improved so that d, q growing polynomially with M is not needed?
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- Instead of constructing a ground state at random and then studying the spectral properties of the corresponding local translation-invariant Hamiltonian, what about directly constructing the Hamiltonian at random (Movassagh, Lemm)?
- What about other models of random TNS? For instance:
  - Different distribution of the 1-site random tensor: not unitarily-invariant, with some symmetries...
    $\longrightarrow$ Typical properties of TNS having a local symmetry.
  - Different geometry of the graph: higher-dimensional regular lattice, tree...
    $\longrightarrow$ AdS/CFT correspondence: random TNS on hyperbolic graphs reproduce conjectured formulas (Hayden/Nezami/Qi/Thomas/Walter/Yang + work in progress with Michael Walter).

Perspective: what about estimating the typical amount of entanglement?

Observation: The amount of bipartite entanglement in a TNS can be upper bounded in terms of the bond dimension. Indeed, for any subregion having L boundary edges and V bulk vertices, its *entanglement entropy* is at most $L \log q$, which is usually much smaller than $V \log d$.

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Question: What is the generic amount of GME in a random TNS?
MPS case: $E(\chi_N)$ is typically of order $(N-1)\log\min(d,q)$ (work in progress with Ion Nechita).

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