Probing entanglement in quantum technologies

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(LPMMC Grenoble and IQOQI Innsbruck)

CPTGA Meeting Sept 27 2021 (Annecy)

- Quantum technologies and entanglement
- Measuring entanglement
- Current efforts

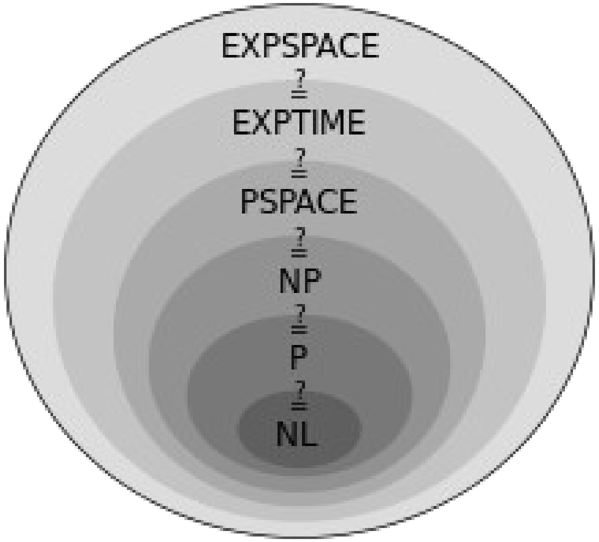
Computational complexity in computer science

The complexity hierarchy of decision problems

- Decision problems have **a yes/no answer**
- **Complexity:** scaling of resources with respect problem size (for a deterministic Turing machine)

Important classes

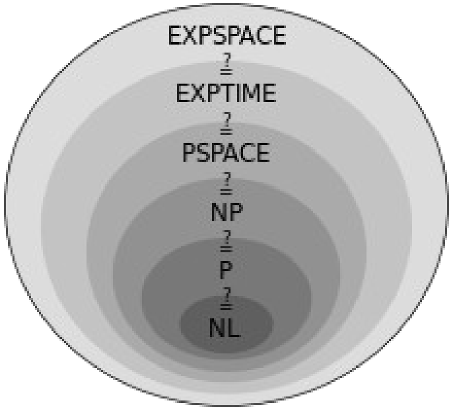
- **P:** Problem solved in polynomial time (w.r.t size of the problem)
- **NP:** A yes answer can be verified in polynomial time
- **NP-HARD:** Every problem in NP can be transformed into this problem in polynomial time
- **NP-COMPLETE:** A problem that is both NP and NP-HARD



Computational complexity in computer science

Example:

- Factorization decision problem (F) : can a given number be factorized?



- No polynomial time algorithm known $\rightarrow$ We do not know if F is in P
- Solutions can be checked `easily' $\rightarrow$ F is in NP

Can we use a quantum machine to solve such `hard' problems?

Why can a quantum computer be powerful?

1 classical bit

$$|\psi\rangle = |0\rangle$$
$$|\psi\rangle = |1\rangle$$

1 quantum bit (qubit)

$$|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle$$

N classical bits

$$|\psi\rangle = |00000000\rangle$$

$$|\psi\rangle = |11111111\rangle$$

2^N configurations

N qubits

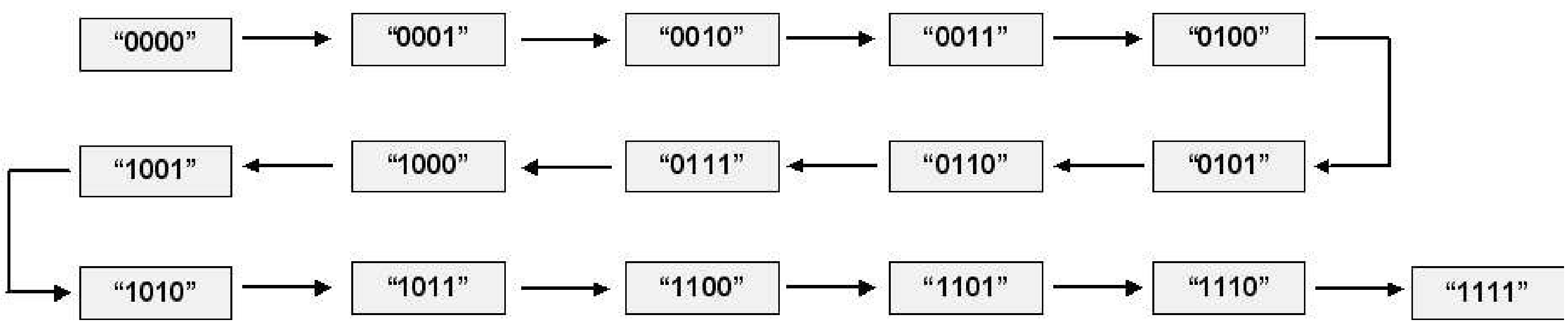
$$|\psi\rangle = c_0\,|00000000\rangle + \cdots + c_{2^N-1}\,|11111111\rangle$$

2^N configurations **'simultaneously'**

The power of quantum parallelism

Example: Data search on 4 bit entries with a classical computer

Goal: find unique x such that f(x)=1

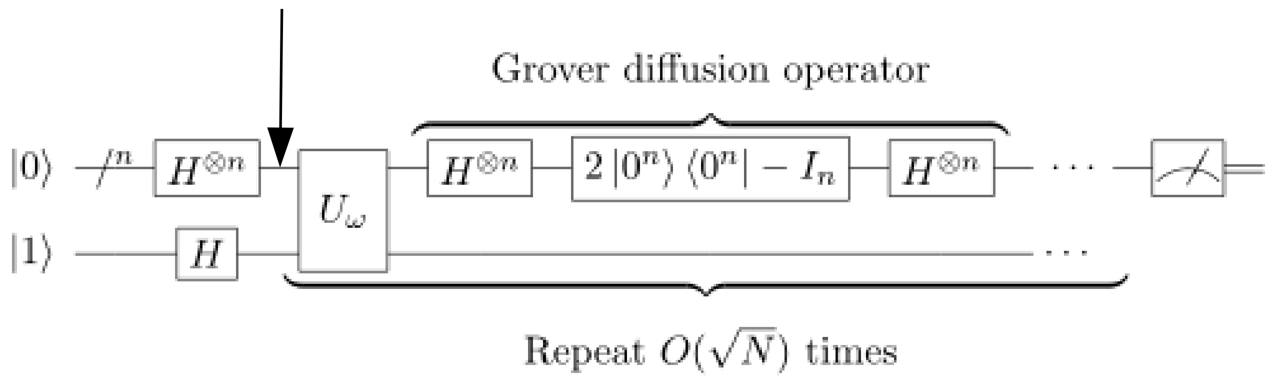


Exponential complexity : $O(N = 2^n)$

The power of quantum parallelism

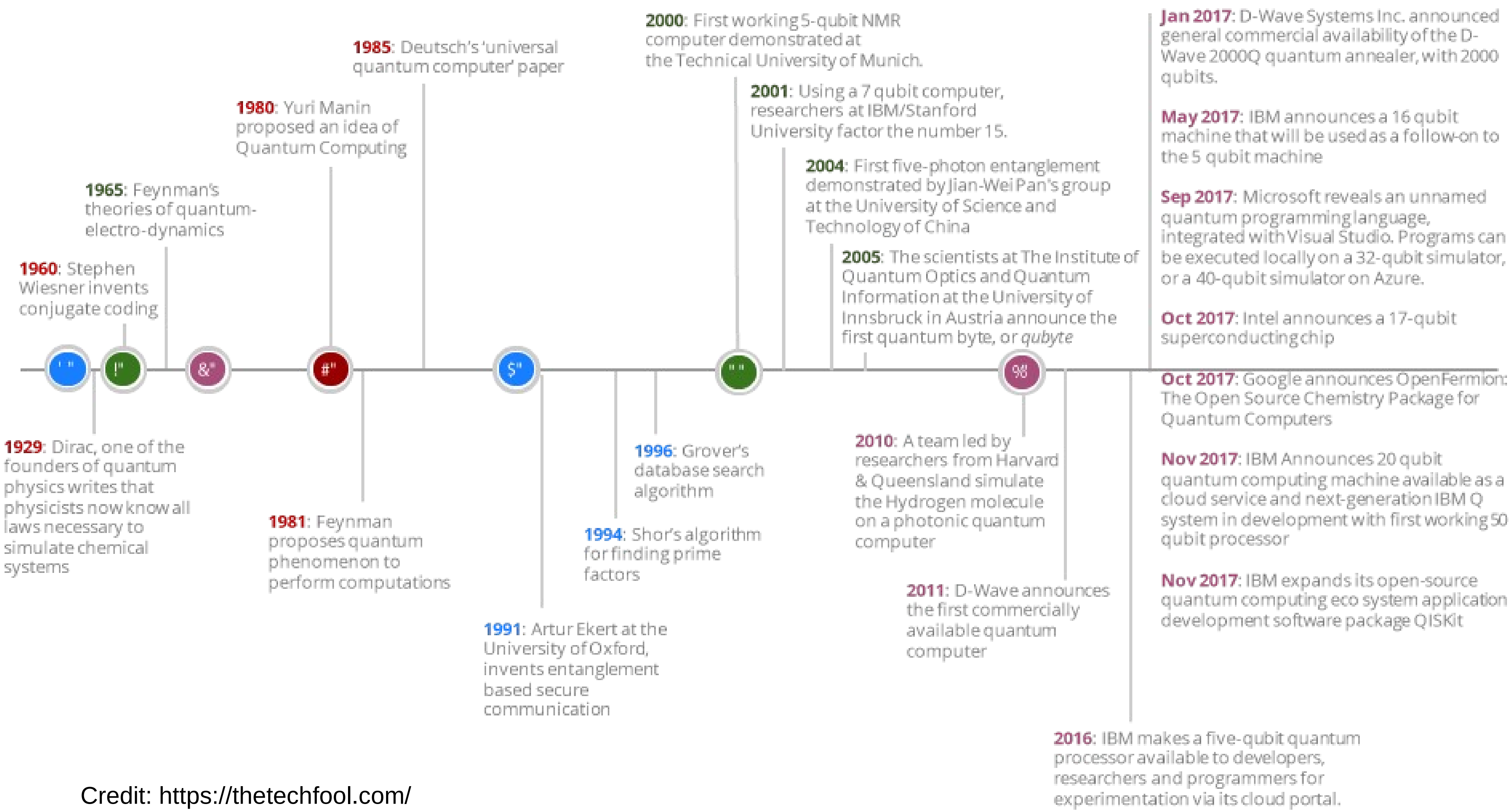
The quantum way (Grover 1996)

$$|s\rangle = \sqrt{\frac{1}{2^N}}\,(|0000\rangle + \cdots + |1111\rangle)$$ We test all states simultaneously



- Exponentially less iterations with a quantum computer
- Require entangling operations (interactions between qubits)
- Bad news: the scaling is still exponential
- If the scaling would have been polynomial, I could have solved any NP problem in polynomial time..

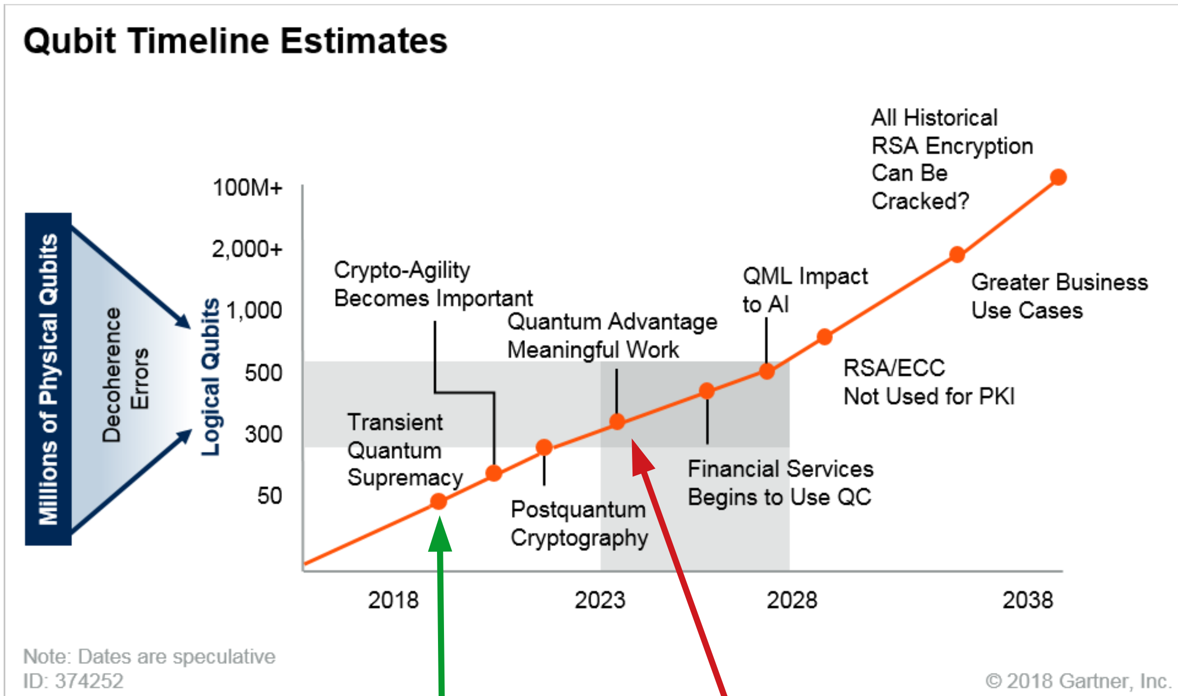
The first era of quantum computing



Credit: https://thetechfool.com/

The NISQ Era and beyond (2018-)

NISQ : noisy intermediate scale quantum



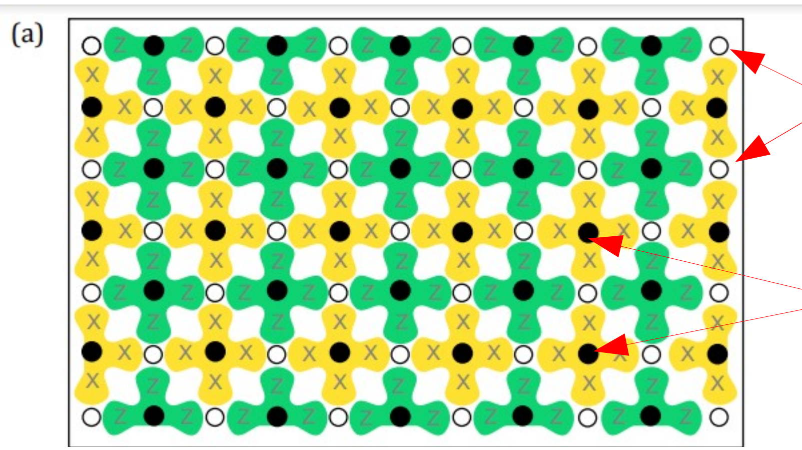
Is this fantasy?

This really happened...

Quantum computing versus errors

- With **quantum error correction,** we need more qubits (>1000) to perform faithful and useful computation

The Surface Code (arxiv:1208.0928)



Many physical qubits

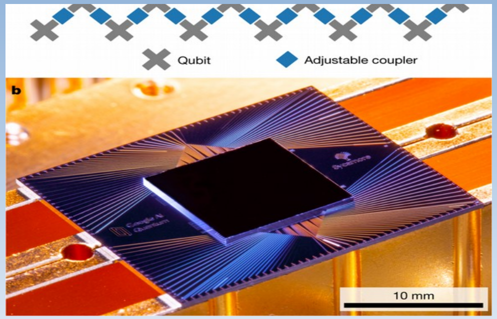
One protected logical qubit

Ancilla qubits
(used for measurements)

*J. Preskill « I've already emphasized repeatedly that it will probably be a long time before we have **fault-tolerant** quantum computers solving hard problems. »*

Current efforts in quantum technologies

Quantum computers

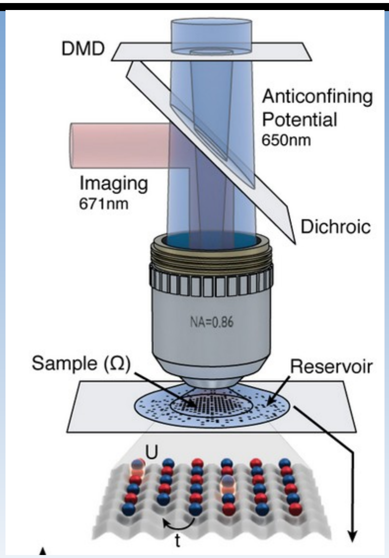


Google Sycamore chip

→ Solve classical problems

→ Requires quantum error correction

Quantum simulators

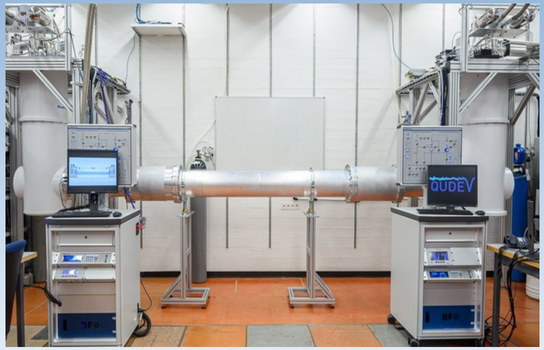


M. Greiner's lab (Harvard)

→ Understand quantum problems: strongly correlated electrons, topological materials, disordered systems, quantum gravity, quantum chemistry

(c.f Talk by Michele)

Quantum networks



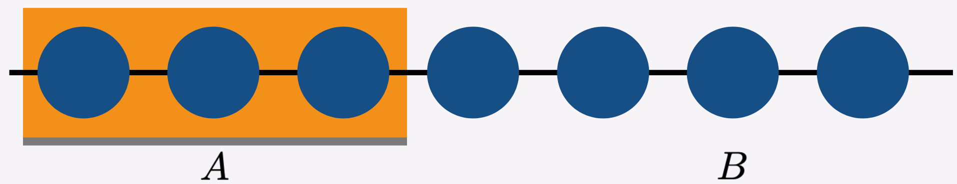
A. Walraff's lab (Zurich)

Quantum communication
Quantum cryptography

Quantum metrology

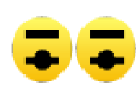
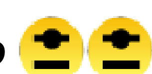
The key physical concept in quantum technologies is many-body entanglement

Defining entanglement



Two subsystems A and B are **entangled** iff $|\Psi\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$

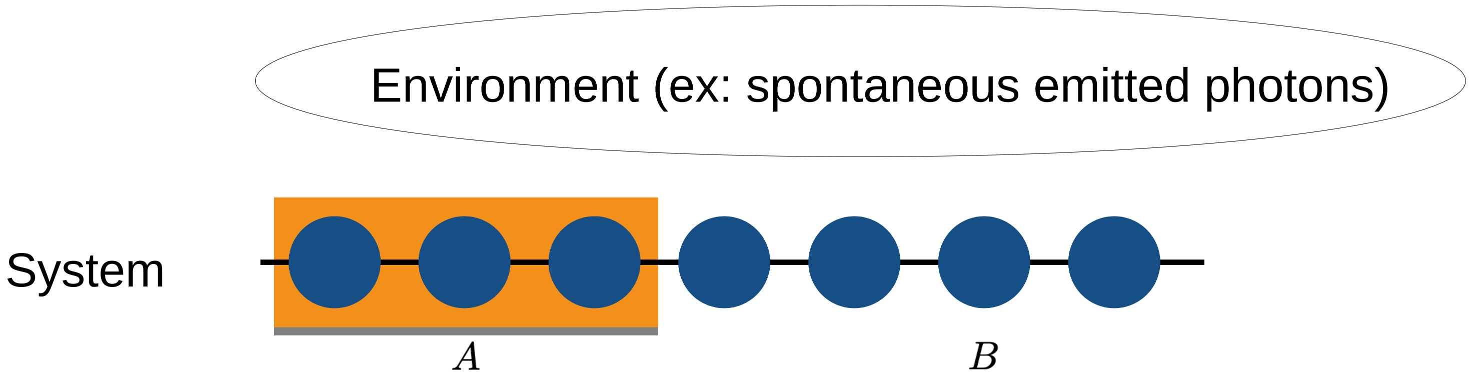
Example: Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$

The state is  ***AND*** 

Entanglement is the key concept/resource in quantum information theory

Defining entanglement

For `noisy' quantum states



Density matrix $\rho = \mathrm{Tr}_E(|\psi_{ABE}\rangle\langle\psi_{ABE}|)$ Describes all system properties as a positive semi-definite matrix

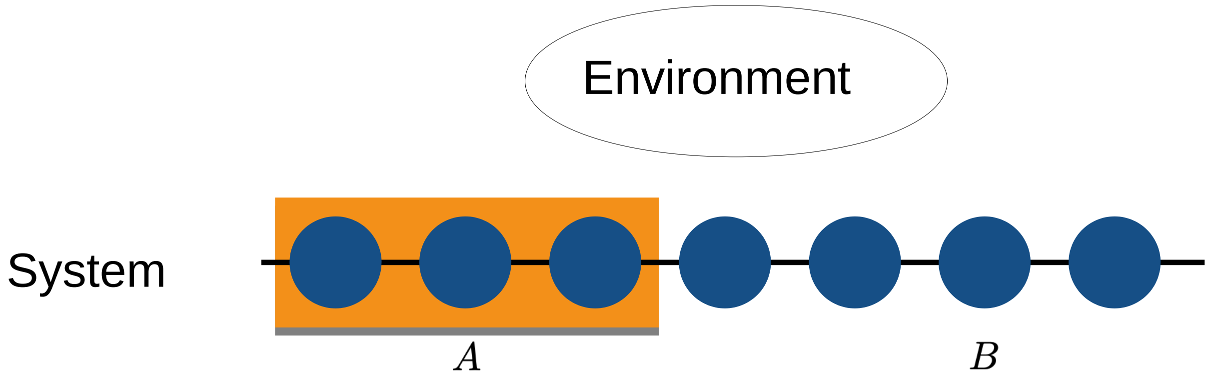
Example: Noisy Bell state

$$|\psi_{ABE}\rangle = \frac{\sqrt{1-p}}{\sqrt{2}}(|01\rangle + |10\rangle)|0_{\mathrm{photon}}\rangle + \sqrt{p}\,|00\rangle|1_{\mathrm{photon}}\rangle$$

$$\rho = (1-p)|\psi_B\rangle\langle\psi_B| + p\,|00\rangle\langle 00|$$

Two subsystems A and B are **entangled** iff $\rho \neq \sum_j p_j \rho_j^{(A)} \otimes \rho_j^{(B)} \qquad (p_j \geq 0)$

Detecting entanglement



- Reduced density matrix $\rho_A = \mathrm{Tr}_B(\rho)$

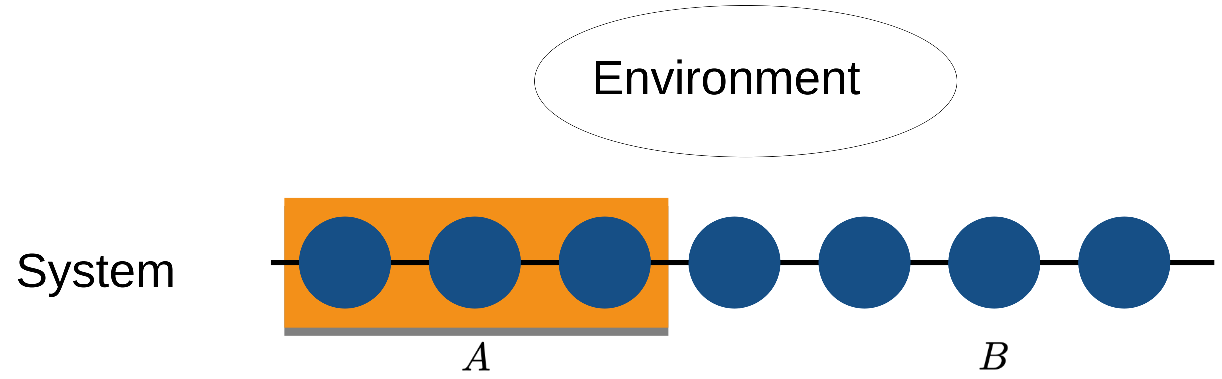
- Entanglement condition (Horodecki 1996) $\mathrm{Tr}\left[\overset{\text{purity}}{\rho_A^2}\right], \mathrm{Tr}\left[\rho_B^2\right] < \mathrm{Tr}\left[\rho^2\right]$

Example: Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ $\rho = |\psi\rangle\langle\psi|$

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \qquad \mathrm{Tr}(\rho_A^2) = \frac{1}{2} < 1$$

States which are quasi pure globally, but not pure (mixed) locally, are entangled

Quantifying entanglement



- Entanglement measures (pure states)

von-Neumann entropy $$S_A = -\mathrm{Tr}_A\left[\rho_A \log \rho_A\right]$$

Rényi entropy $$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}_A\left[\rho_A^n\right] \quad \leq S_A$$

- Entanglement entropies are entanglement monotones for pure states (they cannot increase under local operation)

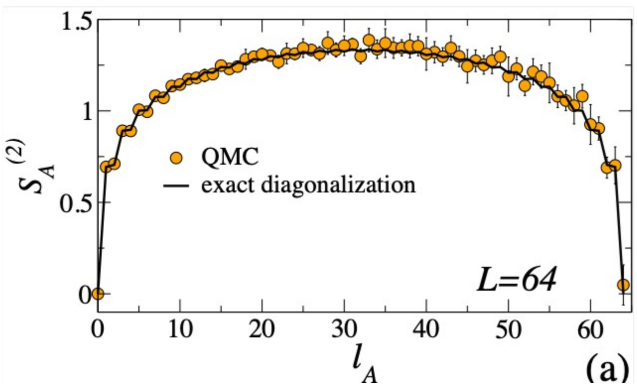
Measuring entanglement entropies

Quantum computing:

Purity and Entanglement Verification

Quantum simulation (understanding quantum problems in an experiment):

Quantum Phase transitions

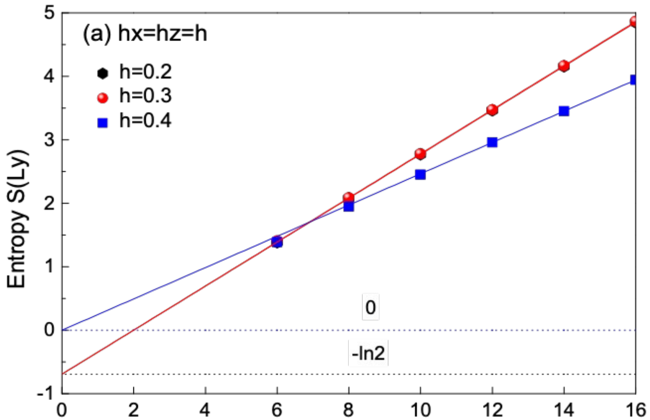


P. Calabrese and J. Cardy, J. Stat. Mech (2004).
Humeniuk, Roscilde PRB (2012)

$$S_A^{(2)} \approx (c/4)\log(L_A)$$

Central charge

Topological order



Kitaev, Preskill, PRL 2006
Levin, Wen, PRL 2006
Jian et al, NP 2012

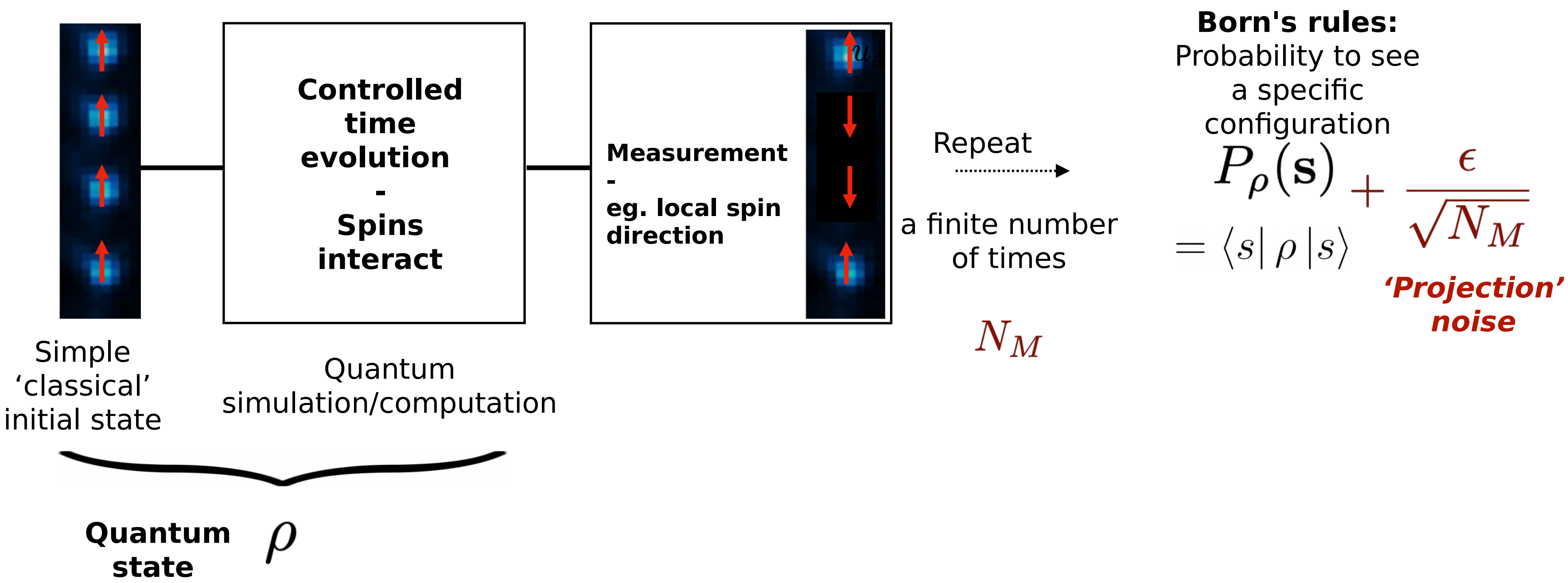
$$S_A^{(n)} \approx \alpha_n L_A - \gamma$$

Topological entanglement entropy

How to measure the purity/entanglement entropies in such many-body quantum systems?

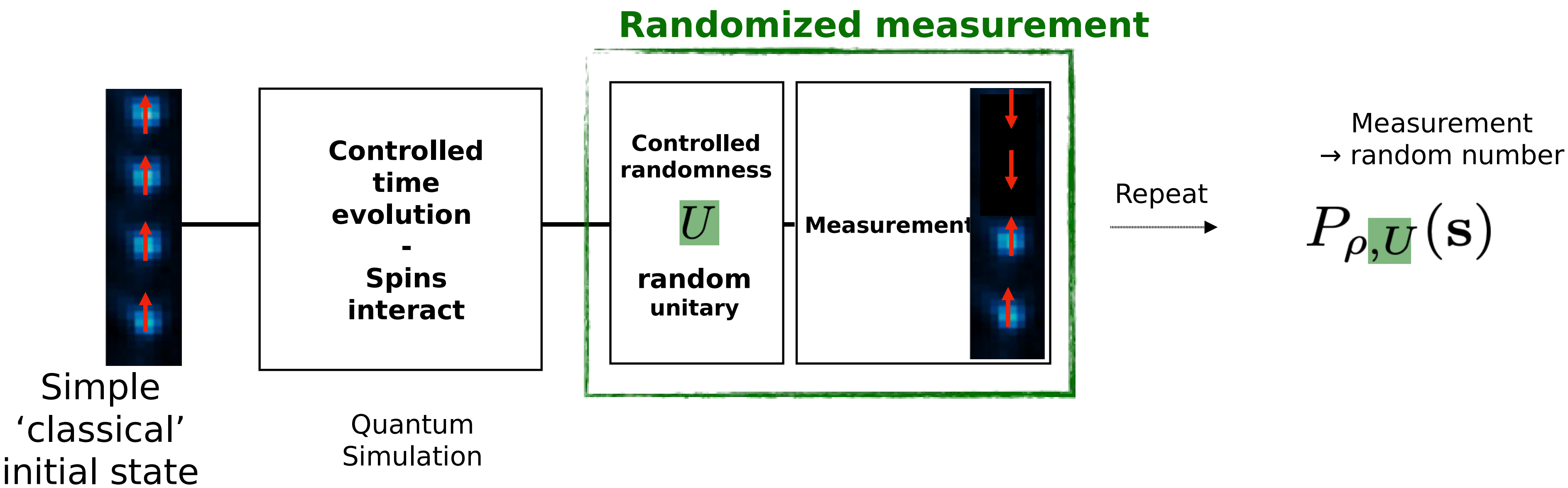
- Quantum technologies and entanglement
- **Measuring entanglement**
- Current efforts

A standard measurement protocol



- **Limited to `observables',** correlation functions, etc
- Not directly applicable to nonlinear functions w.r.t the density matrix $\mathrm{tr}(\rho^2)$
- It is possible to measure the density matrix with 3^N measurement settings (Tomography) with a measurement budget 4^N-8^N

A new tool: randomized measurement protocols



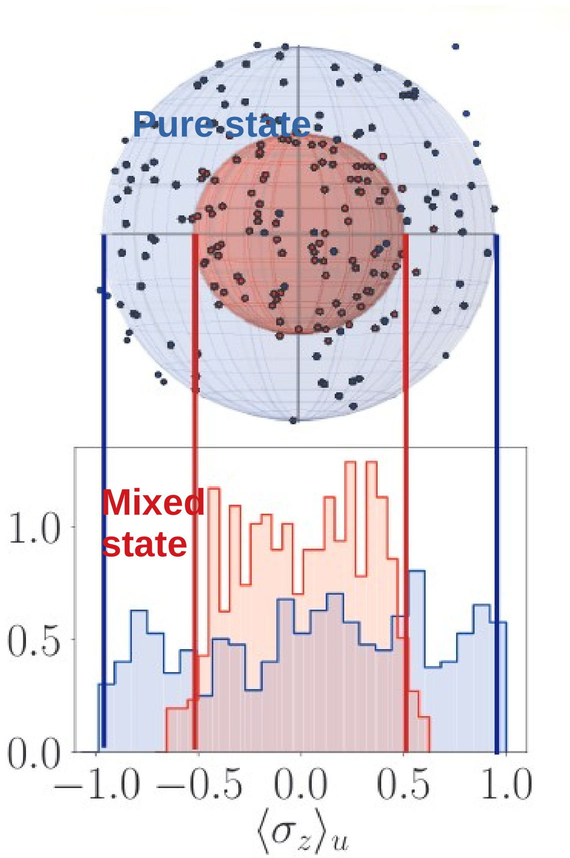
Ensemble average over random unitaries

Correlations of probabilities $\overline{P_{\rho,U}(\mathbf{s}_1)P_{\rho,U}(\mathbf{s}_2)}$

Measuring a single qubit purity

van Enk, Beenakker PRL 2012

- **Randomized measurement** $\longrightarrow \rho \longrightarrow P_u(s) = \langle s| u\rho u^\dagger |s\rangle$



✗ Pure state |0> $P_u(s) = |\langle s|u|0\rangle|^2$

→ fluctuates in [0,1]

✗ Completely mixed state $P_u(s) = \langle s|u\frac{1}{2}u^\dagger|s\rangle = \frac{1}{2}$

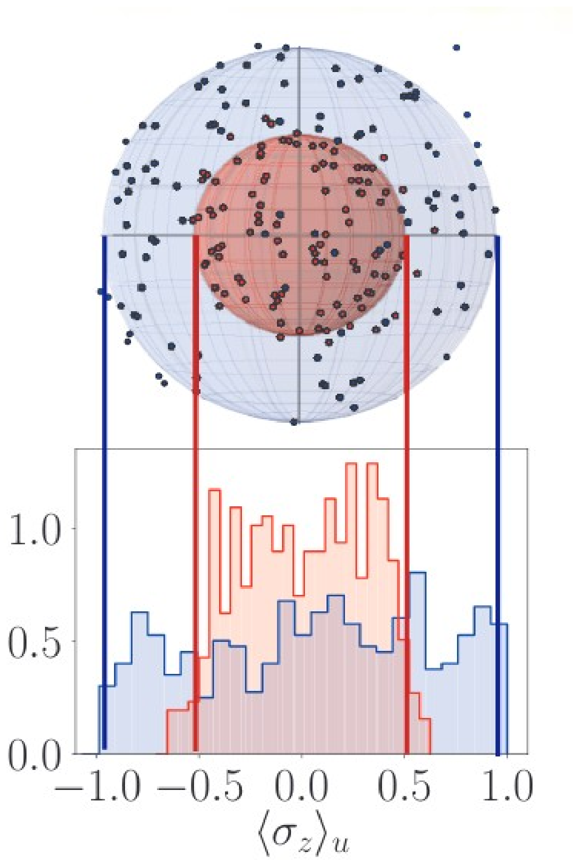
→ does not fluctuate!

- The purity can be understood as statistical fluctuations over randomized measurements

Measuring a single qubit purity

van Enk, Beenakker PRL 2012

- **Statistics of randomized measurements equals purity:**



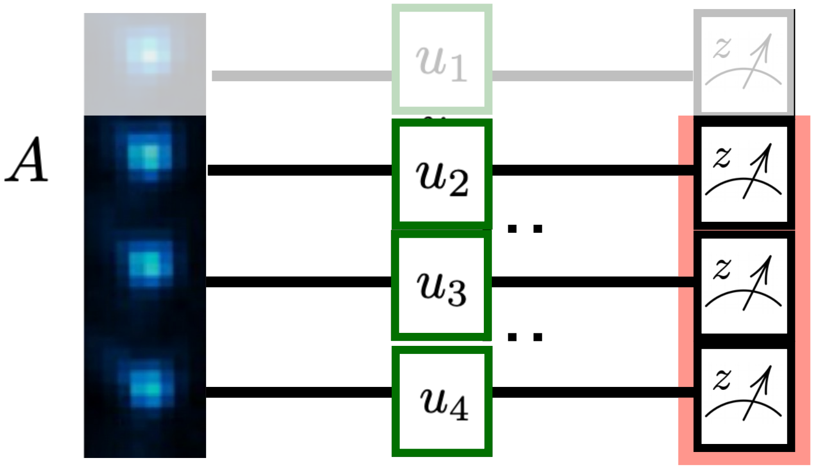
variance over the **circular unitary ensemble (CUE)**

$$\mathrm{tr}(\rho^2) = (d+1)\sum_s \mathrm{var}(P_u(s)) + \frac{1}{d}$$

- **Limitation:** Requires ``global random unitaries'' for a many-body system

Protocol for qubit systems with local random unitaries

Elben, BV et al. (PRL 2018, PRA 2019)



$u_i \in \mathrm{CUE}(d)$

Local random unitaries from **CUE**

$U_A = \bigotimes u_i$

Hamming distance

$\mathrm{Tr}\left[\rho_A^2\right] = \overline{X_U}$ with $X_U = 2^{N_A} \sum_{s_A, s'_A} (-2)^{-D[s_A, s'_A]} P_U(s_A) P_U(s'_A)$

Cross correlations

Number of measurements to overcome stat. errors : ~ $2^{N[A]}$ (Compared to tomography: ~ >$4^{N[A]}$)

Proof

Elben, BV et al. (PRL 2018, PRA 2019)

- \`Replica' trick $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(S\rho_A \otimes \rho_A)$ Swap operator $S = \sum_s |s', s\rangle \langle s, s'|$
- 2-design properties of CUE local unitaries (Replica+\`Twirling')

$$\mathbb{E}_U \left[(U \otimes U)(O)(U^\dagger \otimes U^\dagger) \right] = \sum_{\sigma,\tau} c_{\sigma,\tau} \mathrm{Tr}(\tau O) \sigma$$

Permutation Operators $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_N$, $\tau = \tau_1 \otimes \cdots \otimes \tau_N$, $\sigma_i, \tau_i = 1, S_i$

- Our goal is to find a \`measurable' O such we keep only the term $\sigma = S$

Randomized measurements protocol with local unitaries

Proof

Elben, BV et al. (PRL 2018, PRA 2019)

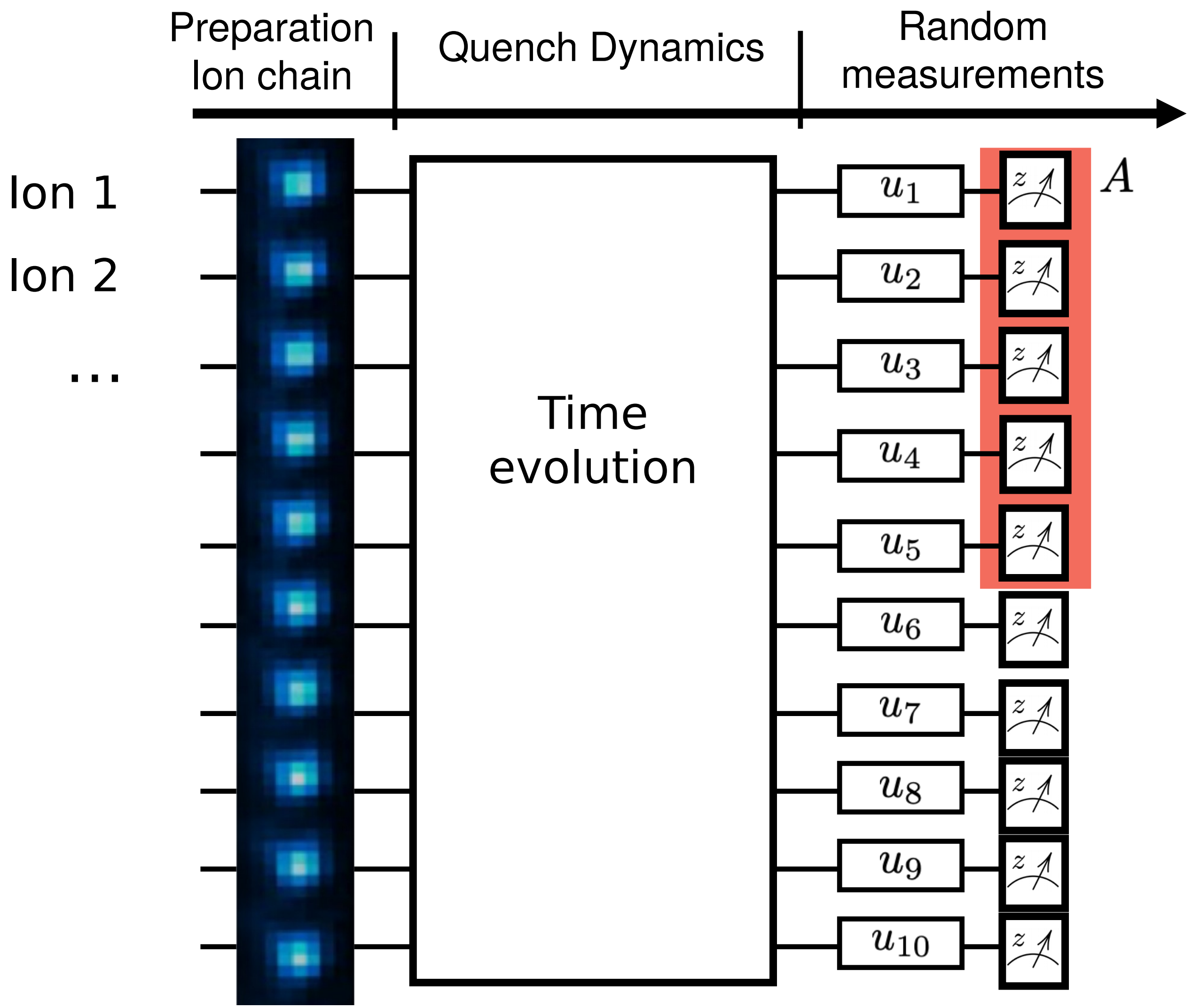
- For $O = 2^N \sum_{s,s'} (-2)^{D[s,s']} |s,s'\rangle \langle s,s'|$ $\quad \mathbb{E}_U \left[(U \otimes U)(O)(U^\dagger \otimes U^\dagger) \right] = S$

- Replacing S in our replica trick expression $\quad \mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(S \rho_A \otimes \rho_A)$

$\mathrm{Tr}\left[\rho_A^2\right] = \overline{X_U}$ with $X_U = 2^{N_A} \sum_{s_A, s'_A} (-2)^{-D[s_A, s'_A]} P_U(s_A) P_U(s'_A)$

Experimental demonstration with trapped ions

Brydges et al, Science 2019



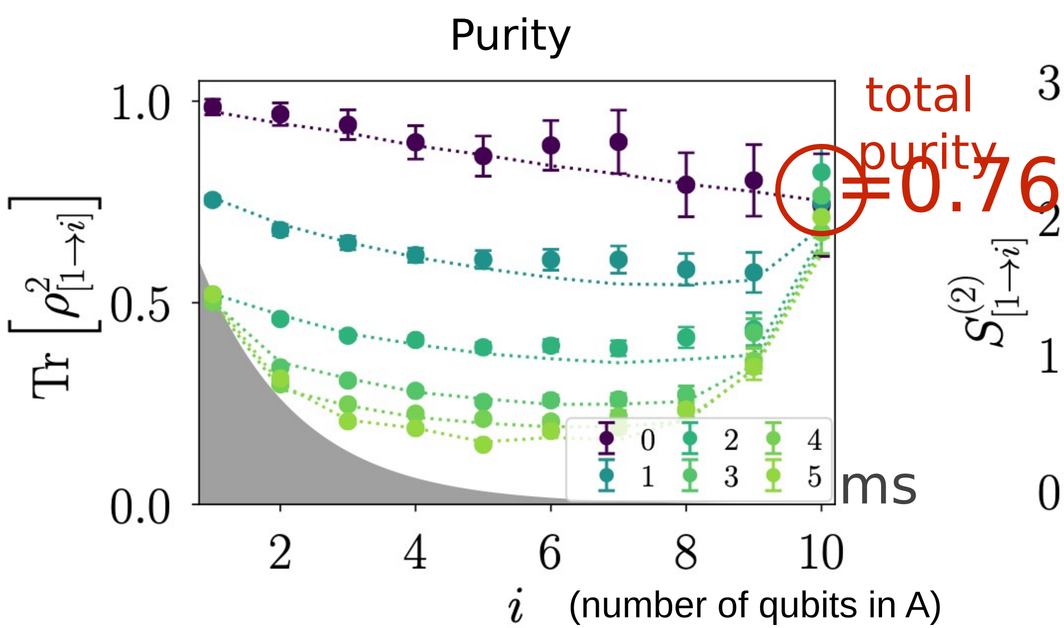
$$|\psi(t)\rangle = e^{-iH_{XY}t}|01\ldots01\rangle$$

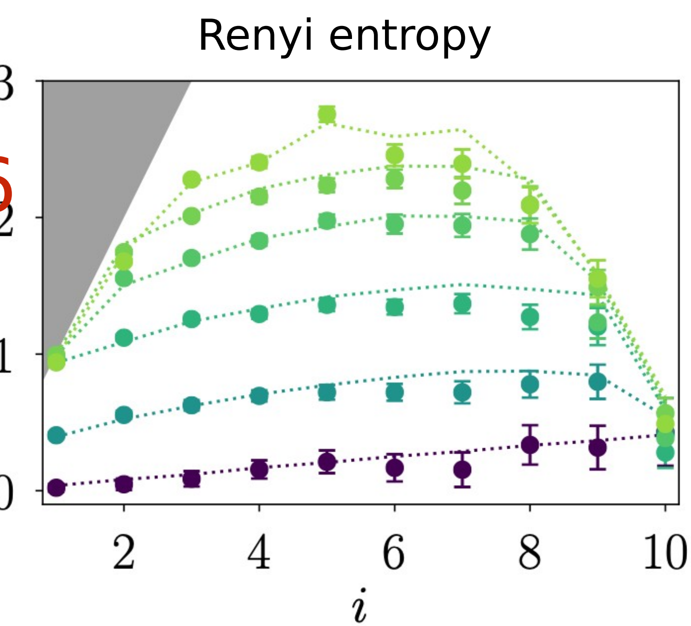
$$H_{XY} = \hbar \sum_{i<j} J_{ij}(\sigma_i^+\sigma_j^- + \sigma_i^-\sigma_j^+) + \hbar \sum_j (B + b_j)\sigma_j^z$$

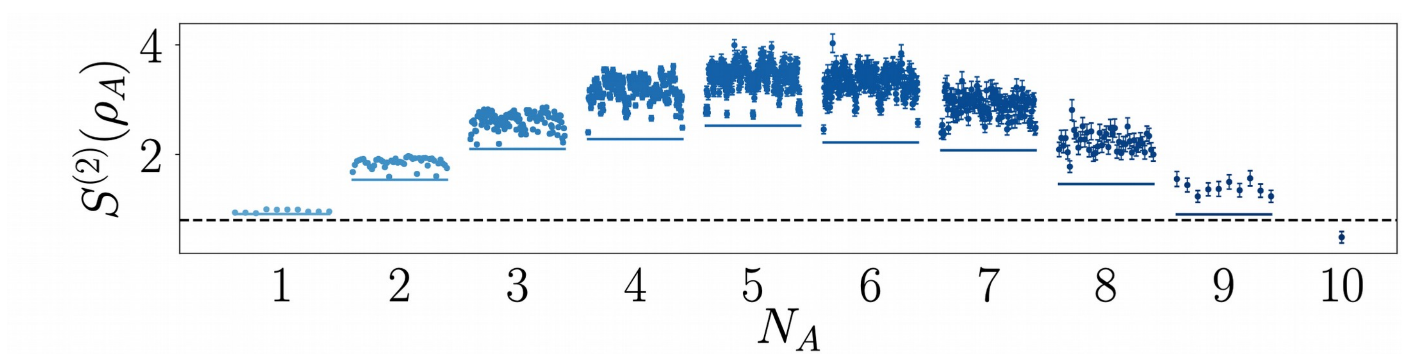
Experimental demonstration with trapped ions

Following the growth of entanglement as a function of time

Brydges et al, Science 2019



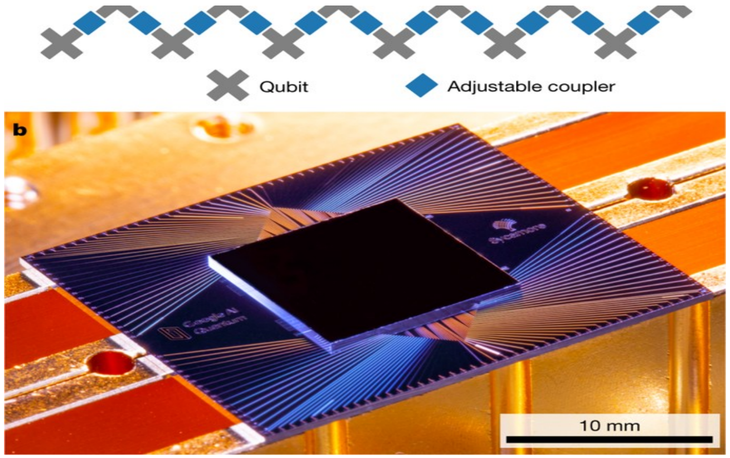




all sub partitions ... entangled

Purity measurements with superconducting qubits

arXiv:2104.01180

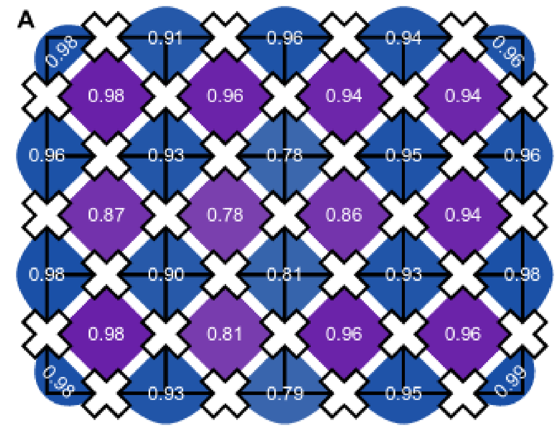


Google's Sycamore

cf. talk by Michele

Toric code:

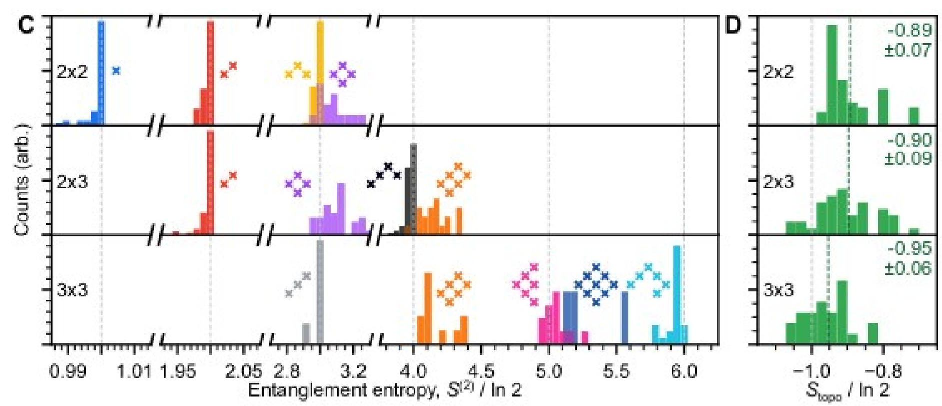
→ Toy model for interacting topological phases
→ Quantum error correction code



The topological entanglement entropy

$$S_{\text{topo}} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$$

is quantized (Levin, Wen, Preskill, Kitaev)

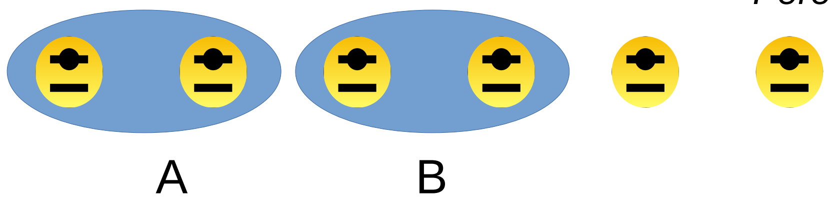


- Quantum technologies and entanglement
- Measuring entanglement
- **Current efforts**

Mixed-state entanglement

Positive-Partial-Transpose (PPT) Condition for mixed state entanglement

Peres, Horodecki, Horodecki, Horodecki 1996



- We consider the Partial-Transpose (PT) `map'

$$\rho = \sum_j \rho_{i,j,k,l} \left|i,j\right\rangle \left\langle k,l\right| \rightarrow \rho^{\Gamma} = \sum_j \rho_{i,j,k,l} \left|k,j\right\rangle \left\langle i,l\right|$$

- If the state is separable (not entangled)

$$\rho = \sum_j p_j \rho_A^{(j)} \otimes \rho_B^{(j)} \longrightarrow \rho^{\Gamma} = \sum_j p_j (\rho_A^{(j)})^T \otimes \rho_B^{(j)}$$ is positive semi-definite

PPT Condition: If the PT density matrix is not semi-definite, then the state is entangled

How to probe this `concept' in quantum computers?

Mixed-state entanglement

Our approach: Measuring PT moments via randomized measurements

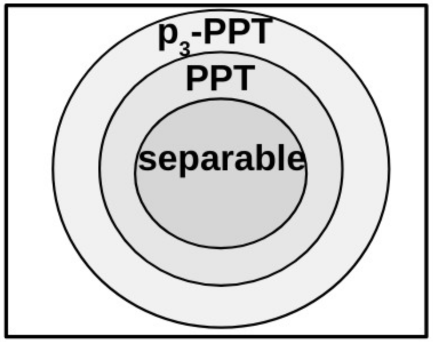
A. Elben (Innsbruck→Caltech) R. Kueng (Caltech→Linz), R. Huang (Caltech), R. van Bijnen (Innsbruck), C. Kokail (Innsbruck) , M. Dalmonte (Trieste), P. Calabrese (Trieste), B. Kraus, (Innsbruck) John Preskill (Caltech), Peter Zoller (Innsbruck), and BV, PRL 2020

$$p_n = \mathrm{Tr}[(\rho_{AB}^{T_A})^n] \quad \text{for } n = 1, 2, 3, \ldots.$$

- Quantify mixed-entanglement in quantum-field theories

- A measurable powerful entanglement condition

$$\mathrm{Tr}(\rho^\Gamma(\rho^\Gamma - p_2)^2) = p_3 - p_2^2 \geq 0$$ for non-entangled states



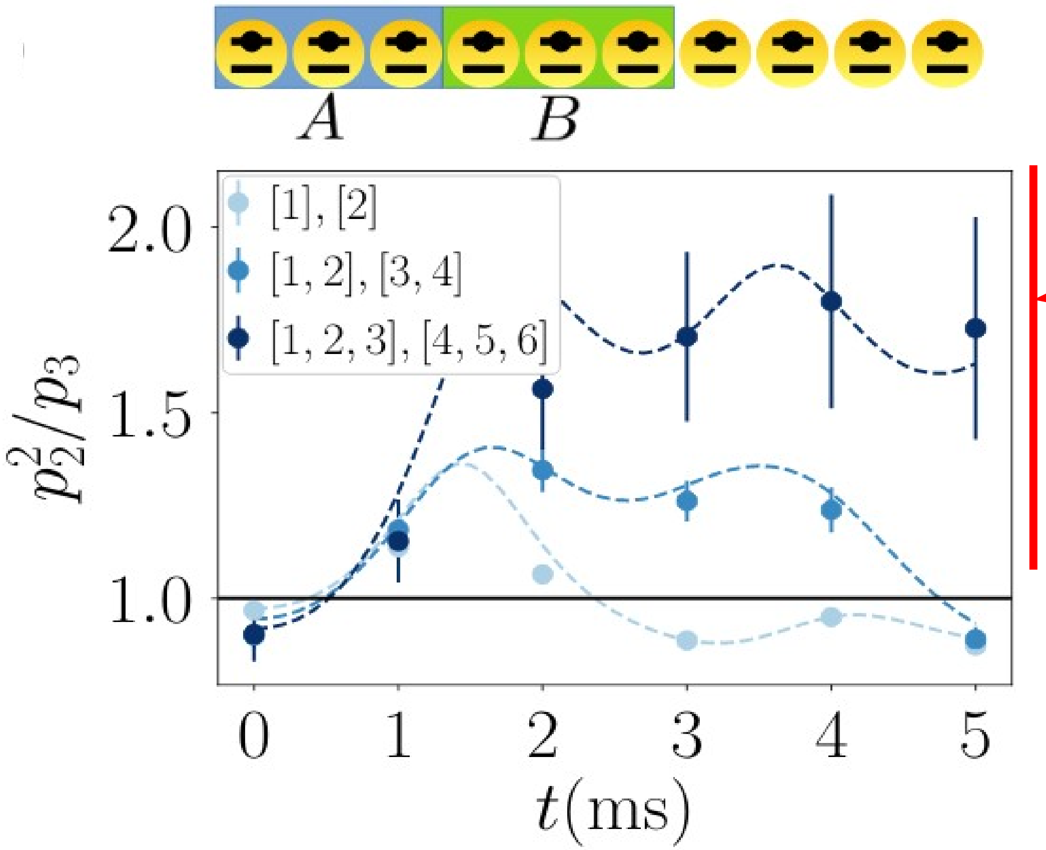
- Accessible via randomized measurements

Mixed-state entanglement

Experimental measurements of PT moments

Elben et al, Phys. Rev. Lett. 125, 200501 (2020)

Data: Brydges , Science 2019 (reanalyzed)



Entanglement detection

Probing many-body quantum chaos in quantum simulators

From quantum states to quantum evolutions

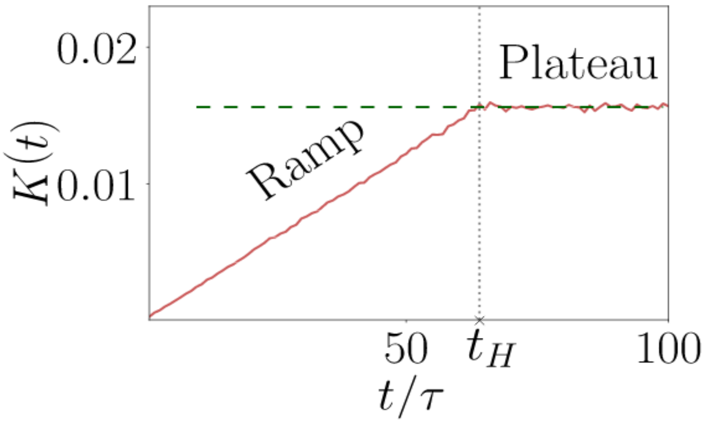
- The time evolution operators summarizes all the property of the time evolution of a closed quantum system

$$|\psi(t)\rangle = T(t)\,|\psi_0\rangle$$

- In quantum chaos theory, **the eigenvalues** of the time evolution operator display universal signatures:

- **Spectral Form Factor:**

$$T(t) = U^t \qquad K(t) \equiv \frac{1}{D^2}\overline{\sum_{i,j} e^{i(E_i - E_j)t}}$$



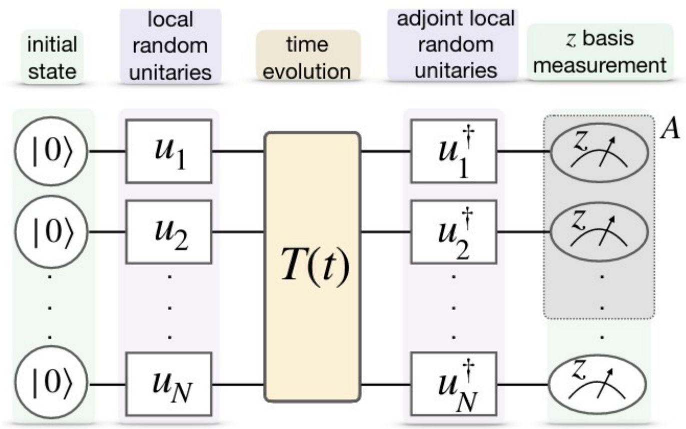
Can we observe these universal features of quantum chaos in a quantum computer?

Probing many-body quantum chaos in quantum simulators

Measurement protocol for Spectral form factors

Lata Joshi ,A. Elben, A, Vikram, BV, V. Galitski, and P. Zoller

arXiv:2106.15530



1) We derive a Bell state `Twirl'

$$\mathbb{E}_U\left[(U^* \otimes U)(O)(U^T \otimes U^\dagger)\right] = 2^{-N} |\Phi_N^+\rangle\langle\Phi_N^+|$$

where $|\Phi_N^+\rangle = 2^{-N/2}\sum_s |s\rangle \otimes |s\rangle$ is a Bell state.

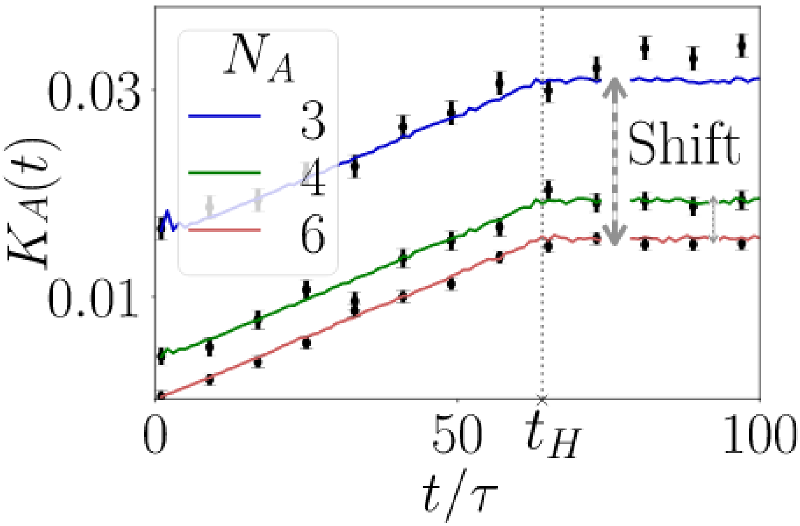
2) We express the SFF as a function of our twirl

$$K(t) = \langle\Phi_N^+|\mathbb{1}\otimes T(t)|\Phi_N^+\rangle\langle\Phi_N^+|\mathbb{1}\otimes T^\dagger(t)|\Phi_N^+\rangle$$

$$K(t) = \mathbb{E}_U\left[K_U(t)\right]$$

$$K_U(t) = \sum_s (-2)^{|s|} |\langle s| U^\dagger T(t) U |0_N\rangle|^2$$

Measurable!



Thank you for your attention

And thanks to my collaborators on randomized measurements

Peter Zoller

Andreas Elben ($\rightarrow$ Caltech)

and M. Dalmonte, I. Cirac, R. Kueng, R. Huang, J. Preskill, B. Kraus. C. Kokail, R, van Bijnen, L. Sieberer, A. Rath, J. Carrasco, A. Neven, F. Pollmann, Z. P Cian, M. Hafezi, G. Zhu, J. Yu, H. Dehghani, M. Barkeshli,N. Yao, M. Joshi, T. Brydges, C.Maier, B. Lanyon, P. Jurcevic, C. Roos, R. Blatt, P. Calabrese, V. Vitale, C. Branciard, A. Minguzzi

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