DYNAMICAL FRICTION IN SCALAR FIELD DARK MATTER SCENARIOS

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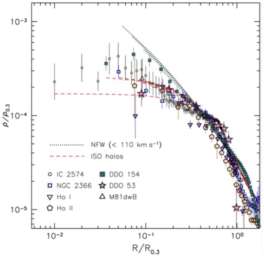
CDM model conflicts

- Conflicts at small scales like the core cusp problem, the missing satellite problem, the too big to fail etc.



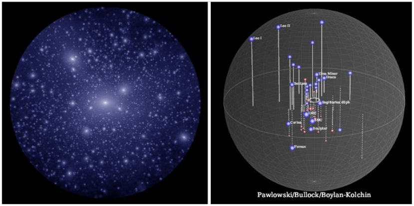


FIGURE – **Left Figure** : Dark matter density profiles of the seven THINGS dwarf galaxies. From Se-Heon Oh et al. (2011) **Right figure** : From Robles et al. (2019)

Scalar Field Dark Matter Model

- DM is composed by bosons (spin - 0) with masses from $10^{-22}eV < m < eV$
- Can form stable equilibrium configurations : Solitons/Boson stars, Bose condensates
- Smooth density profile at the origin solving one of the CDM tensions at galactic scales
- Recovers the successes of ΛCDM at large scale

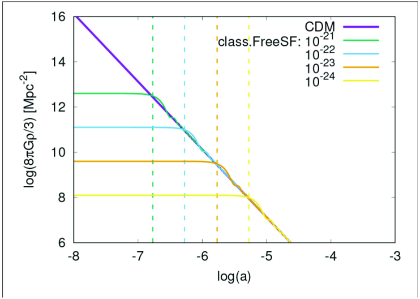


FIGURE – The background solution of the SFDM density, for different values of the boson mass m a , in comparison with the case of CDM. From Urena-Lopez (2019)

SFDM

- SFDM Action : $S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$ where
  $V(\phi) = \frac{m^2}{2}\phi^2 + V_{\mathrm{I}}(\phi)\,, \qquad V_{\mathrm{I}}(\phi) = \frac{\lambda_4}{4}\phi^4$ (obey relation $\rho \propto a^{-3}$)
- In non-relativistic limit $\phi = \frac{1}{\sqrt{2m}}\left(\psi \exp^{-imt} + \psi^* \exp^{imt}\right)$ where
  we separated slow modes (ψ) from fast modes ($e^{|imt|}$)
- In this same limit, the Madelung transformation : $\psi = \sqrt{\frac{\rho}{m}}e^{is}$, ρ plays the role
  of a density, s a phase defining a curl-free field $\vec{v} = \frac{\vec{\nabla}s}{m}$

Dynamical friction

- Dynamical friction/Gravitational drag :
  Loss of momentum of moving objects through gravitational interactions
- Here :
  Loss of momentum of a Schwarzschild Black Hole (BH) in motion in SFDM sea
  Initial conditions : $\vec{v}_0 = v_0 \hat{z}, \quad z_0 \to -\infty, \quad \theta_0 = \pi$
- Can help solve some cosmological problems (as globular clusters timing problem) while constraining SFDM mass
- The dynamical friction formula [from Hui et al (2017)] :

$$\boxed{\vec{F} = -\oint dS_j T_{jz} \vec{e_z} = -\oint dxdy T_{zz} \vec{e_z}} \quad \text{since} \quad \vec{dS} = dS_z \vec{e_z} = dxdy \vec{e_z}$$

Sketch of the system

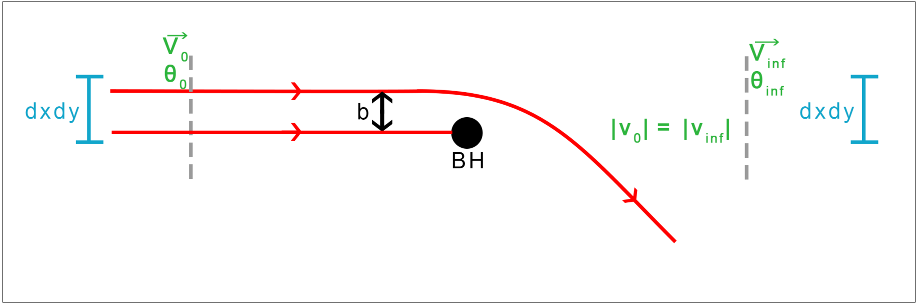


FIGURE – Sketch of two particles coming from infinity with a velocity $\vec{v_0}$: One being deviated by a BH and the other being absorbed by this same BH

Euler-Lagrange equations of motion

- We can use the decomposition as for the non-relativistic case, which gives the Euler-Lagrange equation of motion [from Brax et al. (2020)] :

$$i\dot{\psi} = -\frac{1}{2m}\sqrt{f}h^3\vec{\nabla}\cdot\left(\sqrt{fh}\vec{\nabla}\psi\right) + m\frac{f-1}{2}\psi$$

- Using the Madelung transformation :

$$\dot{\rho} + \sqrt{f}h^3\vec{\nabla}\cdot\left(\sqrt{fh}\rho\frac{\vec{\nabla}s}{m}\right) = 0\,,$$

$$\frac{\dot{s}}{m} + \frac{f}{m}\frac{\left(\vec{\nabla}s\right)^2}{2m^2} = \frac{1-f}{2}$$

Momentum components

- Considering the Euler equation on s as an Hamilton-Jacobi equation :

$$s = -Et + \int_{t_0}^{t} dt \left(\frac{\vec{p}^2}{2m} + \frac{h}{f} \left(E + m \frac{1-f}{2} \right) \right) , E = k^2/(2m) \, , k = mv_0 \, ,$$

and a central force potential, the momentum components :

$$p_\theta = L = kb \, , \quad |p_r| = \sqrt{- \left(\frac{L^2}{m^2 r^2} - \frac{2}{m} \frac{h}{f} \left(E + m \frac{1-f}{2} \right) \right)}$$

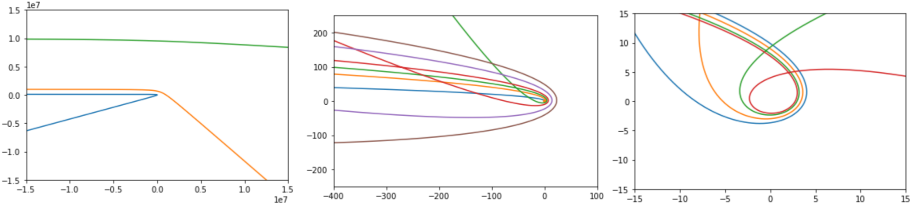


FIGURE – Different trajectories for DM particles

Density Yield

- Mass conservation implies $\int dV\rho = \int dV_0\rho_0$ which gives :

$$\frac{\rho}{\rho_0} = \frac{r_0^2 \sin(\theta_0)}{r^2 \sin(\theta)} J^{-1} = \frac{1}{r^2 \sin(\theta)\widetilde{J}} ,$$

with the Jacobian $J = \frac{\partial r}{\partial r_0}\frac{\partial \theta}{\partial \theta_0} - \frac{\partial r}{\partial \theta_0}\frac{\partial \theta}{\partial r_0}$

- The Jacobian is :

$$b\widetilde{J} = \frac{|p_r|}{L}(\theta_{def} - \theta) + \frac{|p_r|}{|p_{r_0}|}\frac{\partial \theta_{def}}{\partial b} + L^2|p_r| \int_r^\infty \frac{dr}{r^4|p_r|^3} ,$$

where we used the deflection angle $\theta_{def} = \pi + 2L\int_{r_{min}}^\infty \frac{dr}{r^2|p_r|}$

Kepler approximation

- In Kepler approximation :

$$\boxed{F = 2\pi\rho_0 v_0^2 \left[\left(\frac{GM}{v_0^2}\right)^2 \ln\left(\frac{(b^+)^2 + \left(\frac{GM}{v_0^2}\right)^2}{(b^-)^2 + \left(\frac{GM}{v_0^2}\right)^2} \right) + \frac{(b^-)^2}{2} \right]}$$

which is m-independent, as expected.

- This result is the same as the one obtained by Chandrasekhar (1943).
- Chandrasekhar formula : $F \approx \dfrac{G^2 M^2 \rho}{v^2} C$

Klein-Gordon equation

- The Klein-Gordon equation from Brax et al. (2019) :

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} - \sqrt{\frac{f}{h^3}} \vec{\nabla} \cdot \left(\sqrt{fh} \vec{\nabla} \phi \right) + f \frac{\partial V(\phi)}{\partial \phi} = 0}$$

- If the radial and angular derivatives are discarded (local approximation), we can recognize the Duffing equation, and so :

$$\boxed{\phi = \phi_0(r,\theta) cn[\omega(r,\theta)t - \mathbf{K}(k)\beta(r,\theta), k(r,\theta)]}$$

where ϕ_0 is the amplitude, $cn[u,k]$ is the Jacobi elliptic function of argument u and modulus k, $\omega = 2\mathbf{K}\omega_0/\pi$ is the angular frequency, $\mathbf{K}$ is the complete elliptic integral of the first kind and β is an unknown function to determine

Conservation equation

- At leading order in the large-m limit and using $cn'' = (2k^2-1)cn - 2k^2 cn^3$, we obtain the system :

$$(\nabla\beta)^2 = \frac{h}{f}\left(\frac{2\omega_0}{\pi}\right)^2 - \frac{hm^2}{(1-2k^2)\mathbf{K}^2}\,,$$
$$\frac{\lambda_4\phi_0^2}{m^2} = \frac{2k^2}{1-2k^2}$$

- At large distances ($r \to \infty$) : $\;k \to k_0\;$ with $\;k_0^2 \simeq \dfrac{\lambda_4\phi_0^2}{2m^2} = \dfrac{\lambda_4\rho_0}{m^4}$

- From the conservation equation $\langle\nabla_\mu T_0^\mu\rangle = 0$ (where $\langle\ldots\rangle$ is the average over the oscillations, to ensure steady state) :

$$\nabla\cdot(\rho_{\rm eff}\nabla\beta) = 0\,, \qquad \rho_{\rm eff} = \sqrt{fh}\,\phi_0^2\omega\mathbf{K}\langle {\rm cn}'^2\rangle$$

Low and high velocity branches

- In the case of radial accretion, the effective continuity equation can be integrated at once, since only depending on radial derivatives
- In this case, that we will not develop here, we get 2 solutions for k such that $k_1 < k_{\max} < k_2$ where $k_{\max}$ is the value of k corresponding to a maximum value of the flux $\mathcal{F}_{max}$
  - k_1 : high-velocity branch (close to free fall)
  - k_2 : low-velocity branch (supported by the pressure built by the self-interactions)
- we expect to have k_1 at $r < r_c$ and k_2 at $r > r_c$, r_c being the radius at which we have the minimum $\mathcal{F}_{max}$
- In our case,
  - We expect to have the same k near the BH since the regime will be similar
  - We only need to solve at large-radii along the low-velocity branch

Low-k regime

- In this regime, we can simplify some of our quantities :

$$\phi_0^2 = \frac{2m^2k^2}{\lambda_4}\,, \quad \rho_{\text{eff}} = \frac{\pi m^2 k^2}{2\lambda_4}\omega_0 \propto k^2,$$

- Using the rescaled quantities $\hat{r} = \frac{r}{r_s}$ and $\hat{\beta} = \frac{\pi}{2mr_s}\beta$, we obtain :

$$(\hat{\nabla}\hat{\beta})^2 = \frac{3}{2}k_0^2 + v_0^2 + \frac{1}{\hat{r}} - \frac{3}{2}k^2 = \frac{3}{2}\left[k_+(\hat{r})^2 - k^2\right]\,,$$

where we introduced $k_+(\hat{r})^2 = k_0^2 + \frac{2}{3}v_0^2 + \frac{2}{3\hat{r}}$

- With this, we can re-express the conservation equation as :

$$\boxed{\hat{\nabla}\cdot\left[\left(k_+(x)^2 - \frac{2}{3}(\hat{\nabla}\hat{\beta})^2\right)\hat{\nabla}\hat{\beta}\right] = 0}$$

Linear flow (1)

- At small radii (but far from BH), we are in low-velocity radial accretion regime and so $(\hat{\nabla}\hat{\beta})^2 \ll k_+^2$:

$$\hat{\nabla} \cdot \left[k_+(\hat{r})^2 \hat{\nabla}\hat{\beta} \right] = 0$$

- The spherical symmetry of k_+^2 implies that the angular part of the linear modes can be expanded over spherical harmonics

- We only need $Y_l^0(\theta, \phi)$ since we have axisymmetric solutions and so the Green function $G_\ell(\hat{r}, \theta) = G_\ell(\hat{r})\, P_\ell(\cos\theta)$ with :

$$\frac{d}{d\hat{r}} \left(\hat{r}^2 k_+^2 \frac{dG_\ell}{d\hat{r}} \right) - \ell(\ell+1) k_+^2 G_\ell = 0$$

Linear flow (2)

- The boundary condition at large radius, $\hat{r} \to \infty$: $\quad \hat{\beta} = v_0 \hat{r} \cos\theta$
- The inner boundary condition, $\hat{r} = \hat{r}_{\rm m}$: $\quad \dfrac{\partial \hat{\beta}}{\partial \hat{r}} \simeq v_r^{\rm m}, \quad \dfrac{\partial \hat{\beta}}{\partial \theta} \simeq 0$
- Thus, at linear level we only generate the monopole and the dipole :

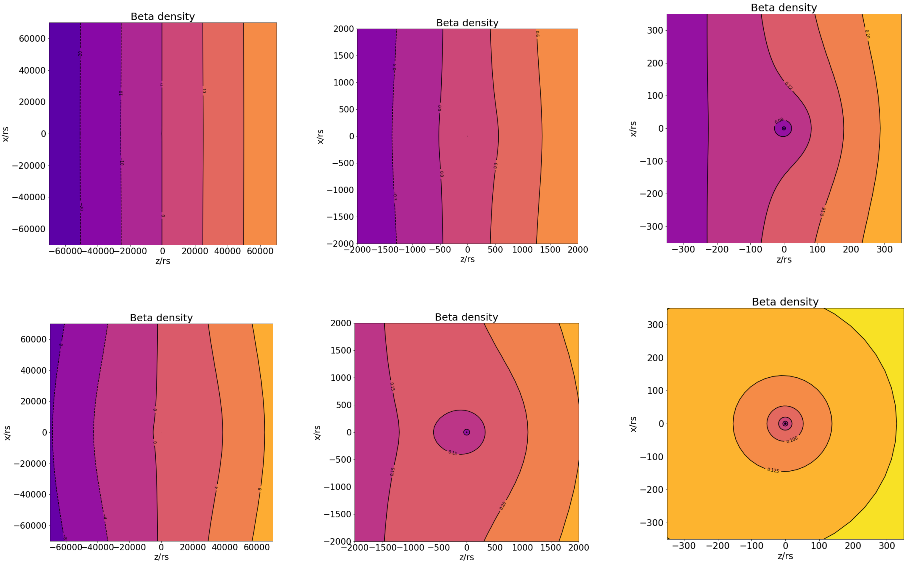
$$\hat{\beta}^L = \hat{\beta}_0^L(\hat{r}) + \hat{\beta}_1^L(\hat{r}) \, \cos(\theta) \, ,$$

where :

$$\hat{\beta}_0^L(\hat{r}) = -v_r^{\rm m} \left(\gamma \hat{r}_{\rm m}^2 + \hat{r}_{\rm m}\right) \ln\left(\gamma + \frac{1}{\hat{r}}\right) \, ,$$

$$\hat{\beta}_1^L(\hat{r}) = \frac{v_0}{\gamma} \, (\gamma \hat{r})^{\sqrt{2}} \, \frac{\Gamma(-1+\sqrt{2})\,\Gamma(2+\sqrt{2})}{\sqrt{2}\,\Gamma(1+2\sqrt{2})} \times \, {}_2F_1(2+\sqrt{2}, -1+\sqrt{2}; 1+2\sqrt{2}; -\gamma\hat{r})$$

Linear flow (3) - Preliminary results



Beyond linear flow

- To go beyond linear flow, we can consider the second term in the conservation equation as a source term :

$$\nabla \cdot (k_+^2 \nabla \beta) = S, \quad S = \frac{2}{3} \nabla \cdot [(\nabla \beta)^2 \nabla \beta]$$

- In this case, using $\nabla \cdot (k_+^2 \nabla G) = \delta_D(\vec{r} - \vec{r}\,')$ and developing G and β in spherical harmonics :

$$G(\vec{r}, \vec{r}\,') = \sum_{\ell,m} G_\ell(r, r') Y_\ell^m(\theta', \varphi')^* Y_\ell^m(\theta, \varphi) ,$$

$$\beta(r, \theta) = \sum_{\ell} \beta_\ell(r) P_\ell(\cos\theta) ,$$

we obtain :

$$\boxed{\beta_\ell = \beta_\ell^L + \int_{r_\mathrm{m}}^{\infty} dr' \; r'^2 G_\ell(r, r') S_\ell(r')}$$

Conclusions and prospects

- Considering the system of a Schwarzschild BH moving through a DM sea :
  - The results obtained for the dynamical friction with a free scalar field are consistent with what is currently known.
  - For future work, we hope that we will obtain good results with a self-interacting scalar field.
- We will extend to the case of a Kerr black hole, which will induce a loss of angular symmetry.

Qu'est-ce que la physique ?

« ~~L'huître~~ La physique [...] est un monde opiniâtrement clos. Pourtant on peut l'ouvrir [...] c'est un travail grossier. [...] A l'intérieur, l'on trouve tout un monde, à boire et à manger [...]. Parfois très rare une formule perle à leur gosier de nacre, d'oû l'on trouve aussitôt à s'orner. »

Francis Ponge - *Le parti pris des choses (1942)*

« ~~L'huître~~ La physique, elle aussi, à des ennemis. »

Alexis Tolstoï


THANK YOU

History and evidence

- Hypotheses at the end of the 19th century
- First evidence in the 1930s :
  - Application of the virial theorem - Coma Cluster (F. Zwicky)
    Large amount of invisible matter (*dunkle materie*)
  - Measurements of galaxy rotation curve - Andromeda nebula (H. Babcock)
    Mass-to-light ratio increases radially
- Lots of evidence in the following decades :
  - Gravitational lensing
  - Cosmic Microwave Background (CMB)
  - Baryon Acoustic Oscillations (BAO)
  - etc.

History and evidence

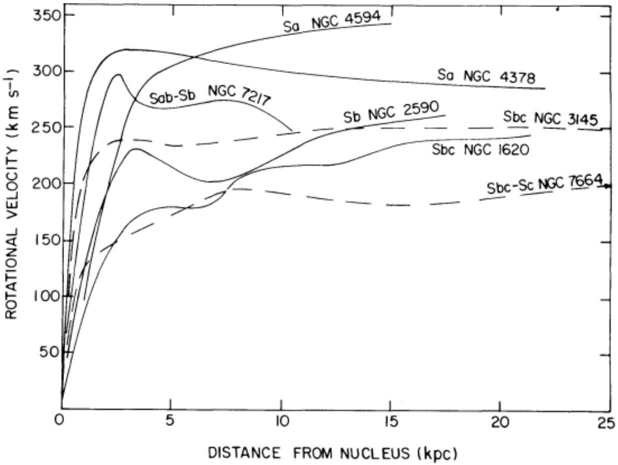


FIGURE – Rotation curves of several spiral galaxies. From Rubin et al. (1978)

Properties

- Roughly 27% of the energy density of the universe
- Approximate dark matter-visible matter ratio of six to one
- **Few interactions** with ordinary matter, mostly via gravity
- Dominated by **non-baryonic** component
- **“Non relativistic”**
- Until now : **unknown nature** but many hypotheses

A large range of masses

- Huge uncertainty on candidates

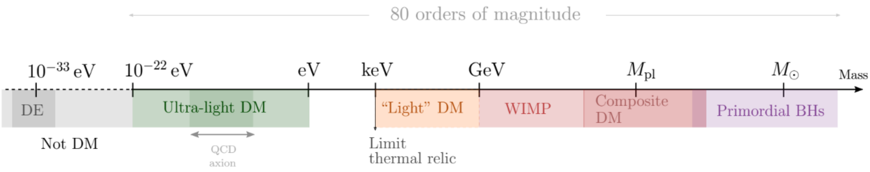


FIGURE – Sketch of some possible DM models that have been conceived, and their typical energy. From Elisa G. M. Ferreira (2020)

CDM model & WIMPs

- Perfect fluid, massive, cold (non-relativistic at the time of structure formation)
- **WIMP Miracle** : Extensions of the Standard Model of particle physics predict a new particle with these properties
- Detection experiments is also an important motivation to search for this candidates

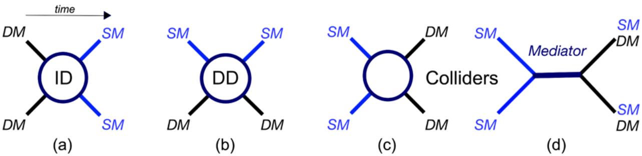


FIGURE – Diagram showing DM interactions. (a) Indirect Detection. (b) Direct Detection. (c) Production of DM particles (d) Production through a mediator particle between DM and SM particles. From ATLAS Collaboration

Classification of SFDM

- **Fuzzy Dark Matter :** (wave DM, ψDM)
  - Just gravity
  - One free parameter, the mass of the particle
  - Quantum pressure & gravity
- **Self Interacting Field Dark Matter** (repulsive DM, scalar field DM, fluid dark matter)
  - Two free parameters. Mass of the particle and the strength and sign of the interaction
  - Self interaction (attractive or repulsive) + quantum pressure + gravity

Schwarzschild metric

- $ds^2 = -f(r)\,dt^2 + h(r)\,(dr^2 + r^2\,d\vec{\Omega}^2)$
- Isotropic metric functions :

$$f(r) = \left(\frac{1 - r_s/(4r)}{1 + r_s/(4r)}\right)^2 ,$$
$$h(r) = (1 + r_s/(4r))^4$$

- In this coordinates, the BH horizon is located at $r = \dfrac{r_s}{4}$
- At large radii : $f = 1 + 2\Phi_\mathrm{N}$ and $h = 1 - 2\Phi_\mathrm{N}$

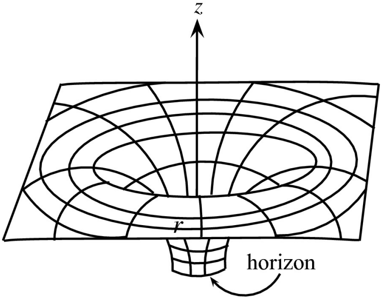


FIGURE – From Manoukian (2020)

Dynamical friction formula

- The dynamical friction formula [from Hui et al (2017)] :

$$\boxed{\vec{F} = -\oint dS_j T_{jz}\vec{e_z} = -\oint dxdy T_{zz}\vec{e_z}} \quad \text{since} \quad \vec{dS} = dS_z\vec{e_z} = dxdy\vec{e_z}$$

- The energy-momentum tensor :

$$T_{\mu\nu} = -g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right) + \partial_\mu\phi\partial_\nu\phi$$

Dynamical friction calculation

- The energy momentum tensor $(\mu\nu) \rightarrow (zz)$ component :

$$T_{zz} = \frac{\rho}{2m^2}\Bigg(-2m\dot{s} + \left[\dot{s}^2 - (\vec{\nabla}s)^2 + 2(\partial_z s)^2\right] + \frac{1}{4\rho^2}\left[\dot{\rho}^2 - (\vec{\nabla}\rho)^2 + 2(\partial_z \rho)^2\right]\Bigg)$$

- We have everything we need to compute the dynamical friction since $\vec{F} = -\oint dS_j T_{jz}\vec{e_z}$

Kepler approximation

- The surface integral can be calculated at $r \to \infty$, before and after the BH, the Kepler approximation gives us : $|P_r| \to \sqrt{k^2 + \frac{2GMm}{r} - \frac{L^2}{r^2}}$
- Therefore : $\theta_{def} \to -2\sin^{-1}\left(1/\sqrt{1+\left(\frac{bv_0^2}{GM}\right)^2}\right)$ and $b\widetilde{J} \to \frac{\partial \theta_{def}}{\partial b}$
- In this regime, $T_{zz} \approx \rho v_z^2$.
- By doing the change of variable $(dx, dy) \to (db, d\phi)$, we obtain :

$$F = -2\pi\rho_0 v_0^2 \left(\int_{b^-}^{b^+} b\cos(\theta_{\mathrm{def}})db - \int_0^{b^+} bdb\right),$$

where b^- is the minimum impact parameter possible for a particle to not be absorbed by the BH and b^+ the chosen maximum impact parameter

Boundary condition at large radii

- At large radii, $k \ll 1$ and the solution takes the form $\phi_0 \cos(\omega_0 - \pi\beta/2)$
- We obtain the boundary conditions :

$$r \to \infty : \quad \phi_0 = \frac{\sqrt{2\rho_0}}{m}, \quad \beta = \frac{2}{\pi} m v_0 z, \\ \omega_0 = (1 + \alpha + v_0^2/2) m,$$

with $\alpha = \Phi_N - \frac{3\lambda_4 \rho_0}{4m^4} - \frac{r_s}{2r}$

- Then, using $\frac{\lambda_4 \phi_0^2}{m^2} = \frac{2k^2}{1 - 2k^2}$:

$$r \to \infty : \quad k \to k_0 \;\; \text{with} \;\; k_0^2 \simeq \frac{\lambda_4 \phi_0^2}{2m^2} = \frac{\lambda_4 \rho_0}{m^4}$$

Linear flow

- By setting $\gamma = \frac{3k_0^2}{2} + v_0^2$ and $\frac{3}{2}k_+^2 = \frac{1}{\hat{r}} + \gamma$:

$$G_0^+(\hat{r}) = 1\,, \qquad G_0^-(\hat{r}) = \ln\left(\gamma + \frac{1}{\hat{r}}\right)\,,$$
$$G_\ell^+(\hat{r}) = \hat{r}^{a-\nu}\ {}_2F_1(a, 1-b; 1-b+a; -\gamma\hat{r})\,,$$
$$G_\ell^-(\hat{r}) = \hat{r}^{-\nu}\ {}_2F_1(a, b; c; -1/(\gamma\hat{r}))\,,$$

with :

$$\nu = \frac{1+\sqrt{1+4\ell(\ell+1)}}{2}\,, \qquad a = \nu + \sqrt{\nu(\nu-1)}\,,$$
$$b = \nu - \sqrt{\nu(\nu-1)}\,, \qquad c = 2\nu$$

Beyond linear flow

- Since β_ℓ^L already matches the boundary conditions, we require the Green function to become negligible at r_m and large radii, and $\frac{\partial G_0}{\partial r}(r_\mathrm{m}) = 0$ to recover the radial velocity :

$$
\begin{aligned}
r < r' : &\quad G_0(r,r') = -3G_0^-(r')/2 \\
r > r' : &\quad G_0(r,r') = -3G_0^-(r)/2 \\
r < r' : &\quad G_\ell(r,r') = A\left[G_\ell^-(r_\mathrm{m})G_\ell^+(r) - G_\ell^+(r_\mathrm{m})G_\ell^-(r)\right] , \\
r > r' : &\quad G_\ell(r,r') = B\, G_\ell^-(r) ,
\end{aligned}
$$

where :

$$
\begin{aligned}
A &= \frac{3G_\ell^-(r')}{2G_\ell^-(r_\mathrm{m})(r' + \gamma r'^2)[G_\ell^+(r')G_\ell^{-\prime}(r') - G_\ell^-(r')G_\ell^{+\prime}(r')]} , \\
B &= A\left[G_\ell^-(r_\mathrm{m})\frac{G_\ell^+(r')}{G_\ell^-(r')} - G_\ell^+(r_\mathrm{m})\right]
\end{aligned}
$$

Numerical simulation

- To obtain numerical results, one solution is to express S in spherical harmonics to :

$$S = \sum_{\ell} S_\ell P_\ell(\cos\theta)$$

- Then, we can solve :

$$\boxed{S_\ell = \int d\vec{\Omega}\frac{2}{3}\nabla\cdot[(\nabla\beta)^2\nabla\beta]P_\ell(\cos\theta)}$$

- And finally, we can calculate the new value of β_ℓ
- This iteration go on until we reach convergence (if there is)