LARGE N MATRIX-MODEL BOOTSTRAP

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Introduction

Generally, the matrix model is defined by the following partition function:

$$Z = \int d^{N^2}A\, d^{N^2}B\, d^{N^2}C \dots \, \mathrm{e}^{-\mathrm{tr}\mathcal{V}(A,B,C,\dots)} \tag{1}$$

Many applications: mesoscopics, 2D QG models, stat. mechanics on planar graphs, QCD ...

This work is about solving large N matrix model by bootstrap method.

Bootstrap

Basically bootstrap method is solving problems in theoretical physics by optimization theory.

- Quadratic programming:

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y = x^2 + 3x + 1 \end{aligned} \tag{2}$$

- Linear programming:

$$\begin{aligned} \max \quad & 300x + 100y \\ \text{s.t.} \quad & 6x + 3y \leq 40 \\ & x - 3y \leq 0 \\ & x + \tfrac{1}{4}y \leq 4 \end{aligned} \tag{3}$$

Semi-definite Programming

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$$\begin{aligned} \min \quad & 2x + 3y \\ \text{s.t.} \quad & \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \end{aligned} \tag{4}$$

- Linear programming and Quadratic programming are special situations of Semi-definite Programming(SDP).
- They all fall into the class of Convex Optimization.
- Generally we cannot solve large-scale non-convex optimization problem.

A Simplest bootstrap model

Consider the single-variable integral:

$$Z = \int_{-\infty}^{\infty} \mathrm{exp}(-\frac{x^2}{2} - g\frac{x^4}{4})\mathrm{d}x, \quad g > 0, \tag{5}$$

We want to compute its k-moment for a given g:

$$\mathcal{W}_k = \frac{1}{Z}\int_{-\infty}^{\infty} x^k \,\mathrm{exp}(-\frac{x^2}{2} - g\frac{x^4}{4})\mathrm{d}x \tag{6}$$

We have a lot of choices to do the integration!

Loop equations and global symmetries

Loop equations are Dyson-Schwinger equations. They can be derived by make the variable translation $x \to x + \epsilon$ or in our model by integration by part:

$$(k+1)\mathcal{W}_k = \mathcal{W}_{k+2} + g\mathcal{W}_{k+4} \tag{7}$$

Global symmetry:

$$\mathcal{W}_k = 0, \text{ for odd } k \tag{8}$$

The conclusion is all the k-moments are linear functions of $\mathcal{W}_2$, so correlation matrix is a linear function of $\mathcal{W}_2$.

Correlation matrix

The bootstrap method is that considering the expectations of square of polynomials are always positive semi-definite:

$$\frac{1}{Z}\int_{-\infty}^{\infty}(\sum \alpha_i x^i)^2 \exp(-\frac{x^2}{2} - g\frac{x^4}{4}) \geq 0,\ \forall \alpha \tag{9}$$

This is a quadratic form in α, its positivity is equivalent to:

$$\mathbb{W} = \begin{pmatrix} \mathcal{W}_0 & \mathcal{W}_1 & \mathcal{W}_2 & \dots \\ \mathcal{W}_1 & \mathcal{W}_2 & \mathcal{W}_3 & \dots \\ \mathcal{W}_2 & \mathcal{W}_3 & \mathcal{W}_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0 \tag{10}$$

This condition will be referred as the positivity of **correlation matrix**.

Bootstrap

We can solve the Semi-Definite Programming(SDP) maximizing or minimizing $\mathcal{W}_2$ constrained by a truncation of the positivity of correlation matrix:

$$\min \text{ or } \max \ \mathcal{W}_2 \tag{11}$$

$$\mathbb{W}_\Lambda \succeq 0 \tag{12}$$

Here $\mathbb{W}_\Lambda$ is the top $(\Lambda + 1) \times (\Lambda + 1)$ sub-matrix of $\mathbb{W}$.

Result

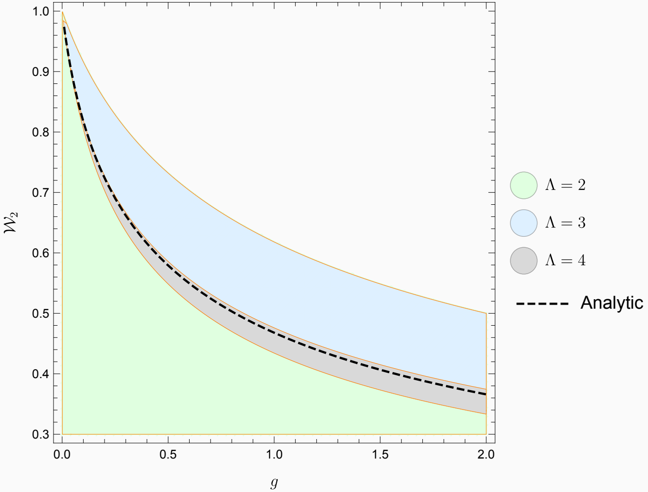
Analytic result:

$$\mathcal{W}_2 = \frac{\pi\left(-I_{-\frac{1}{4}}\left(\frac{1}{8g}\right) + (4g+1) I_{\frac{1}{4}}\left(\frac{1}{8g}\right) - I_{\frac{3}{4}}\left(\frac{1}{8g}\right) + I_{\frac{5}{4}}\left(\frac{1}{8g}\right)\right)}{2\sqrt{2} g K_{\frac{1}{4}}\left(\frac{1}{8g}\right)} \tag{13}$$

For $g = 1$, $\Lambda = 10$, we can get the numerical bootstrap result:

$$0.4679137 \leq \mathcal{W}_2 = 0.4679199170 \leq 0.4679214 \tag{14}$$

Result



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
$\mathcal{W}_2$
0.0
0.5
1.0
1.5
2.0
g
$\Lambda = 2$
$\Lambda = 3$
$\Lambda = 4$
Analytic

Bootstrap in general

A lot of problems in physics concerns with the expectation values of some operators under a probability measure:

$$\langle \mathcal{O} \rangle = \int \mathcal{O} \mathrm{d}\mu \tag{15}$$

- Choose the basis of operators such that they are in some sense complete(or their linear span is dense in the space of operators).
- Impose some known equality and positivity conditions on these operators, as well as the assumption of global symmetry.
- Set a numerical truncation of the basis, equalities and inequalities. Then we can get rigorous upper bounds and lower bounds on the expectation values.

Advantages

1. We don't necessarily know or truly understand the probability measure.
    - We can actually bootstrap the probability measure. For example, the recent works on quantum mechanics bootstrap.[Han et al., 2020, Berenstein and Hulsey, 2021, Bhattacharya et al., 2021]
    - We always have functional integrals in QFT. Sometimes they are not mathematically well-defined or they are not UV-complete, but hopefully we can still get some rigorous bounds on the dynamical quantities.
2. The symmetry of the problem is easier to implement.
3. The results will be rigorous. Increasing the cutoff will only improve the bound.

One Matrix model

We consider the following one matrix model defined on integral over Hermitian matrix:

$$Z_N = \int d^{N^2} M \, \mathrm{e}^{-N \mathrm{tr} V(M)}, \quad V(x) = \frac{1}{2}\mu x^2 + \frac{1}{4} g x^4, \tag{16}$$

Large N Matrix model

- Large N factorization: expectation values of a double trace operator is dominated by the product of the expectations of the single trace operators.
- Tunnelling effect is exponentially suppressed. We only need a deep enough local minima in the potential to make the model sensible.

$$Z = \int_{-\infty}^{\infty} \exp(-\frac{x^2}{2} - g\frac{x^4}{4})\mathrm{d}x, \;\; g \geq 0, \tag{17}$$

$$Z = \lim_{N\to\infty} \int d^{N^2}M\, \mathrm{e}^{-N\mathrm{tr}V(M)}, \quad V(x) = \frac{1}{2}x^2 + \frac{1}{4}gx^4,\; g \geq -\frac{1}{12} \tag{18}$$

Bootstrapping Large N one-matrix model

This is a repetition of Lin's work[Lin, 2020]. The partition function is chosen to be:

$$Z = \lim_{N\to\infty} Z_N = \lim_{N\to\infty} \int d^{N^2}M\, \mathrm{e}^{-N\mathrm{tr}V(M)}, \quad V(x) = \frac{1}{2}\mu x^2 + \frac{1}{4}gx^4, \tag{19}$$

The integration is over Hermitian matrix.

The basis of operators are:

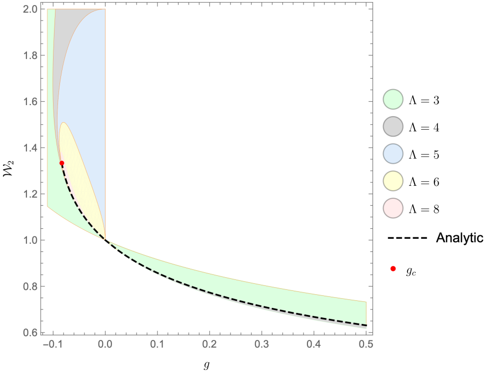
$$\mathcal{W}_k = \langle \mathrm{Tr} M^k \rangle = \lim_{N\to\infty} \int \frac{d^{N^2}M}{Z_N} \frac{1}{N} \mathrm{tr} M^k \mathrm{e}^{-N\mathrm{tr}V(M)}. \tag{20}$$

And the loop equations:

$$\mu \mathcal{W}_{k+1} + g\mathcal{W}_{k+3} = \sum_{l=0}^{k-1} \mathcal{W}_l\ \mathcal{W}_{k-l+1},\ k = 1, 2, 3, ... \tag{21}$$

The positivity of correlation matrix is the same.

This is the result of bootstrapping $\mu = 1$ and $\mathbb{Z}_2$ symmetry preserving solution $\mathcal{W}_1 = 0$. From the loop equation and symmetry assumption, all moments are polynomial functions of $\mathcal{W}_2$.



A remark

The loop equations are quadratic equations. So this is a non-linear SDP, i.e. a kind of non-convex optimization problems. In general we cannot solve large-scale non-convex optimization problems directly.

Due to the simplicity of one-matrix model, we don't really have trouble with this non-convexity here. But we will have trouble when dealing with multi-matrix model.

Multi-matrix bootstrap: an example

Here we propose to study the following two-matrix model:

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\, \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2+A^2/2+gA^4/4+B^2/2+gB^4/4\right)} \tag{22}$$

The integration is over Hermitian matrix. To the best of our knowledge, this model with general g and h value, is not solvable!

The cutoff $\Lambda = 4$ is the lowest order where the quadratic loop equations appear.

At this cutoff, the longest operator has length 8.

This model has a $\mathbb{Z}_2^3$ symmetry:

$$\begin{cases} A \to -A \\ B \to -B \\ A \leftrightarrow B \end{cases} \tag{23}$$

and a $A \to A^{\mathrm{T}}$, $B \to B^{\mathrm{T}}$ symmetry which makes all the expectations real. Up to these global symmetry, with cutoff=4, we have 20 operators:

$$\mathrm{Tr}A^2,\ \mathrm{Tr}A^4,\ \mathrm{Tr}A^2B^2,\ \mathrm{Tr}ABAB,\ \mathrm{Tr}A^6,\ \mathrm{Tr}A^4B^2,\ \mathrm{Tr}A^3BAB,\ \mathrm{Tr}A^2BA^2B,\ \mathrm{Tr}A^8,\ \mathrm{Tr}A^6B^2,\ \mathrm{Tr}A^5BAB,\ \mathrm{Tr}A^4BA^2B,\ \mathrm{Tr}A^4B^4,\ \mathrm{Tr}A^3BA^3B,\ \mathrm{Tr}A^3BAB^3,\ \mathrm{Tr}A^3B^2AB^2,\ \mathrm{Tr}A^2BABAB^2,\ \mathrm{Tr}A^2BAB^2AB,\ \mathrm{Tr}A^2B^2A^2B^2,\ \mathrm{Tr}ABABABAB\,. \tag{24}$$

CUTOFF=4: LOOP EQUATIONS

We first write down the loop equations by:

$$\int d^{N^2}A\, d^{N^2}B\ \mathrm{tr}(\partial_M(\mathrm{Word} \times \mathrm{e}^{-N\mathrm{tr}\, V(A,B)}) = 0, \qquad M = \{A, B\} \tag{25}$$

and introduce the notation:

$$\beta = (\mathrm{Tr}A^2)^2 = (\mathrm{Tr}B^2)^2 = \mathrm{Tr}A^2\mathrm{Tr}B^2. \tag{26}$$

We have 14 loop equations, they are algebraically independent.

$$
\begin{aligned}
1 &= \mathrm{Tr}A^2 + g\mathrm{Tr}A^4 - h(-2\mathrm{Tr}A^2B^2 + 2\mathrm{Tr}ABAB)\\
0 &= -2\mathrm{Tr}A^2 + \mathrm{Tr}A^4 - h(2\mathrm{Tr}A^3BAB - 2\mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^6\\
0 &= -\mathrm{Tr}A^2 + \mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2BA^2B + 2\mathrm{Tr}A^3BAB - \mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^4B^2\\
0 &= -h(2\mathrm{Tr}A^2BA^2B - 2\mathrm{Tr}A^3BAB) + g\mathrm{Tr}A^3BAB + \mathrm{Tr}ABAB\\
\beta &= -2\mathrm{Tr}A^4 + \mathrm{Tr}A^6 - h(2\mathrm{Tr}A^5BAB - 2\mathrm{Tr}A^6B^2) + g\mathrm{Tr}A^8\\
\beta &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^4B^2 - h(-\mathrm{Tr}A^3B^2AB^2 + 2\mathrm{Tr}A^3BAB^3 - \mathrm{Tr}A^4B^4) + g\mathrm{Tr}A^6B^2\\
0 &= -2\mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2B^2A^2B^2 + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^6B^2\\
0 &= -\mathrm{Tr}A^4 + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^4B^4 - h(-\mathrm{Tr}A^4BA^2B + 2\mathrm{Tr}A^5BAB - \mathrm{Tr}A^6B^2)\\
0 &= \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^2BAB^2AB - \mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3BAB^3) + g\mathrm{Tr}A^5BAB - \mathrm{Tr}ABAB\\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^5BAB - 2\mathrm{Tr}ABAB - h(-2\mathrm{Tr}A^2BABAB^2 + 2\mathrm{Tr}ABABABAB)\\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^3BAB^3 - h(-\mathrm{Tr}A^3BA^3B + 2\mathrm{Tr}A^4BA^2B - \mathrm{Tr}A^5BAB)\\
0 &= g\mathrm{Tr}A^3BA^3B + \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^3B^2AB^2 - 2\mathrm{Tr}A^3BAB^3)\\
0 &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^2BA^2B - h(-\mathrm{Tr}A^2BAB^2AB + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + g\mathrm{Tr}A^4BA^2B\\
\beta &= \mathrm{Tr}A^2BA^2B + g\mathrm{Tr}A^3B^2AB^2 - h(2\mathrm{Tr}A^3BA^3B - 2\mathrm{Tr}A^4BA^2B).
\end{aligned}
\tag{27}
$$

CUTOFF=4: CORRELATION MATRIX

Under the $\mathbb{Z}_2^3$ symmetry our correlation matrix decouples into a block-diagonal matrix with three blocks. They are, respectively, the inner product[1] matrix of even-even words:

$$I, AA, BB, AAAA, AABB, ABAB, ABBA, BAAB, BABA, BBAA, BBBB \tag{28}$$

odd-odd words:

$$AB, BA, AAAB, AABA, ABAA, ABBB, BAAA, BABB, BBAB, BBBA \tag{29}$$

and even-odd words:

$$B, AAB, ABA, BAA, BBB\,. \tag{30}$$

[1]Here inner product of $\mathcal{O}_1$ and $\mathcal{O}_2$ is defined to be $\langle \mathrm{Tr}\mathcal{O}_1^\dagger \mathcal{O}_2 \rangle$.

CUTOFF=4: CORRELATION MATRIX

For example, the block for the even-odd words reads:

$$\begin{pmatrix} \mathrm{Tr}A^2 & \mathrm{Tr}A^4 & \mathrm{Tr}A^2B^2 & \mathrm{Tr}ABAB & \mathrm{Tr}A^2B^2 \\ \mathrm{Tr}A^4 & \mathrm{Tr}A^6 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^4B^2 \\ \mathrm{Tr}A^2B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^2BA^2B \\ \mathrm{Tr}ABAB & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^2BA^2B & \mathrm{Tr}A^3BAB \\ \mathrm{Tr}A^2B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^2BA^2B & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^4B^2 \end{pmatrix} \tag{31}$$

All the constraints are convex except the quadratic loop equations!

Relaxation

Most naively we treat all the quadratic terms as independent variables, and eliminate them from the loop equations. Actually we have better choice.

Relaxation

Suppose we have only three quadratic "loop equations":

$$\begin{cases} x^2 = T_1 \\ y^2 = T_2 \\ xy = T_3 \end{cases} \tag{32}$$

Here $T_i = \sum_j q_i^j w_j$, $(i = 1, 2, 3)$ denote linear combinations of some other variables $w_1, w_2, \ldots$. We can relax them to make them convex by replacing $x^2 = T_1$ with $x^2 \leq T_1$ or, in the positive semi-definite matrix form,

$$\begin{pmatrix} 1 & x \\ x & T_1 \end{pmatrix} \succeq 0\,. \tag{33}$$

But the same operation cannot be reproduced for equation $xy = T_3$, since neither $xy \leq T_3$ nor $xy \geq T_3$ is convex [2]. It is tempting to consider the positive semi-definite combinations:

$$(x + \alpha y)^2 \leq T_1 + \alpha^2 T_2 + 2\alpha T_3, \ \forall \alpha \in \mathbb{R} \,. \tag{34}$$

In its turn, it is equivalent to:

$$\mathrm{Det} \begin{pmatrix} 1 & x & y \\ x & T_1 & T_3 \\ y & T_3 & T_2 \end{pmatrix} \geq 0. \tag{35}$$

[2] Because the bilinear form xy is not positive semi-definite.

We come to the conclusion that:

$$\begin{pmatrix} 1 & x & y \\ x & T_1 & T_3 \\ y & T_3 & T_2 \end{pmatrix} \succeq 0. \tag{36}$$

Our general strategy is, we treat the quadratic terms in the loop equations as independent variable, and replace the algebraic equality by the convex inequality:

$$X = xx^{\mathrm{T}} \tag{37}$$

to:

$$\mathcal{R} = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & X \end{pmatrix} \succeq 0. \tag{38}$$

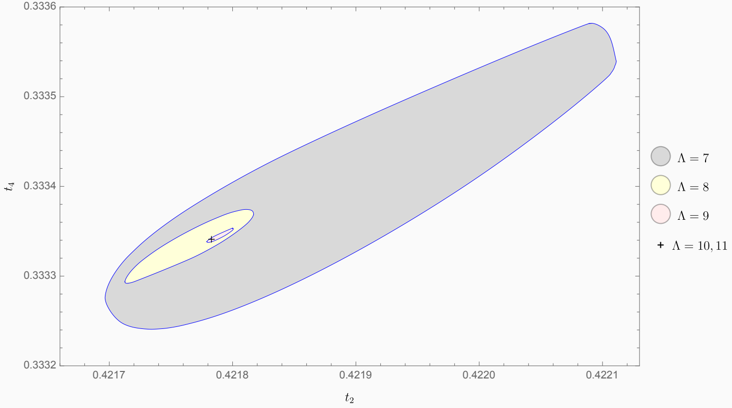
In our example, at cutoff=4, the relaxation matrix is:

$$\begin{pmatrix} 1 & \mathrm{Tr}A^2 \\ \mathrm{Tr}A^2 & \beta \end{pmatrix} \succeq 0. \tag{39}$$

After setting $g = 1, h = 1$

$$0.393566 \leq \mathrm{Tr}A^2 \leq 0.431148\,. \tag{40}$$

Result



$$\Lambda = 11 : \begin{cases} 0.421783612 \leq t_2 \leq 0.421784687 \\ 0.333341358 \leq t_4 \leq 0.333342131 \end{cases} \quad \text{6 digits precision} \qquad (41)$$

Conclusions and prospects

- Matrix bootstrap- an interesting alternative to Monte Carlo for computations of functional integrals. Gives exact inequalities on physical quantities. Efficient solution of large N loop equations.
- To overcome the non-convexity of nonlinear loop equations, we proposed to use relaxation.
- Important target-large N lattice Yang-Mills theory via Migdal-Makeenko loop equations. Pioneering paper of Anderson-Kruzcenski[Anderson and Kruczenski, 2017], but the tools need to be improved. We are still far from the universality and precision of Monte Carlo, but at least it is an alternative.

Questions?

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