Gravitational-wave mode (2,1) at the 3.5PN order

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System of interest

What we want to model:

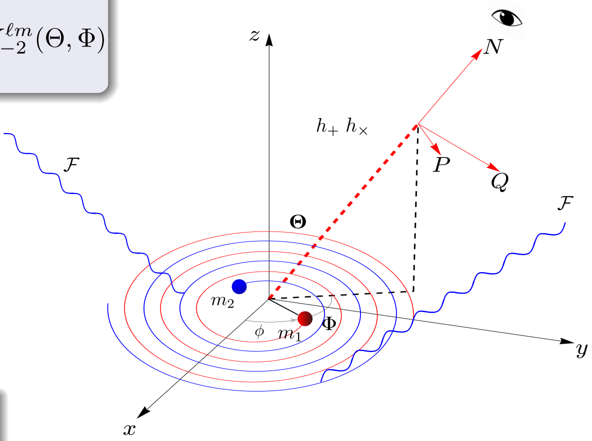
$$\overbrace{h_+}^{h_{PP}} - \mathrm{i} \overbrace{h_\times}^{h_{PQ}} = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi)$$

$$h^{\ell m} = \mathcal{A}^{\ell m} \mathrm{e}^{-\mathrm{i} m \phi}$$

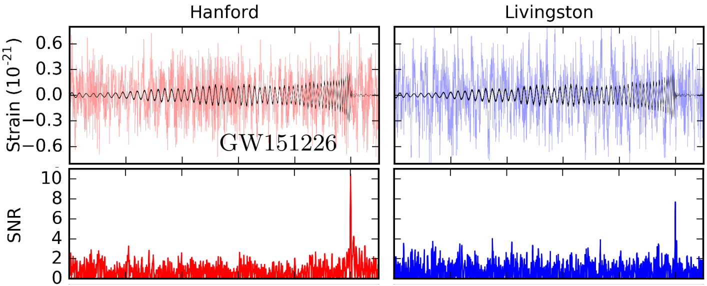
- Amplitudes $\mathcal{A}^{\ell m}$
- Phases $\phi_{\mathrm{GW}} = m\,\phi$ for the mode (ℓ, m)

$\hookleftarrow$ obtained from

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}$$



Why do we need high accuracy?



Credit: LVC

- cross-correlation for detection & parameter estimation
  $\hookleftarrow$ ET, LISA
- reduction of biases when comparing/combining with NR

$\Rightarrow$

computation of ϕ to 4PN
... and beyond

What do we compute?

first term of the $1/R$ expansion of the form

$$h^{\rm rad}_{ij}(\boldsymbol{X},T) = \frac{4G}{c^4 R}\sum_{\ell=2}^{+\infty}\frac{1}{c^{\ell-2}\ell!}\Bigg\{\underbrace{N_{L-2}}_{N_{i_1}(\Theta,\Phi)\dots N_{i_{\ell-2}}(\Theta,\Phi)}\underbrace{U_{ijL-2}}_{i_1\dots i_\ell} - \frac{2\ell}{c(2\ell+1)}N_{aL-2}\,\varepsilon_{ab(i}V_{j)bL-2}\Bigg\}^{\rm TT}(T-R/c)$$

Generalizes the *quadrupole formula* [Einstein (1918)]

$$h^{\rm rad}_{ij}(\boldsymbol{X},T) = \frac{2G}{c^4 R}U^{\rm TT}_{ij}(T-R/c)$$

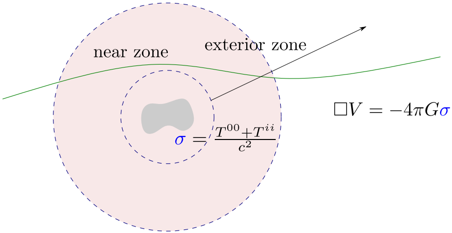
$$\text{with } U_{ij} = I^{(2)}_{ij} + \mathcal{O}(G) = Q^{(2)}_{ij} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

Leading order contributions

$$h^{\rm rad}_{ij}(\boldsymbol{X},T) = \frac{4G}{c^4 R}\left\{\frac{1}{2!}I^{(2)}_{ij} - \frac{4}{5c}N_a\varepsilon_{ab(i}J^{(2)}_{j)b} + \frac{1}{3!c}N_k I^{(3)}_{ijk} + \dots\right\}^{\rm TT}(T-R/c)$$

[Larrouturou *et al.*, (2021) arXiv:2110.02240, 2110.02243] this talk @ 3PN

MPM formalism



$\Box V = -4\pi G\sigma$

Near zone: post-Newtonian (PN) expansion $\qquad \varepsilon = \left(\frac{v}{c}\right)^2 \ll 1$

$$V = G\int \frac{\mathrm{d}^3\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\sigma\left(\boldsymbol{x}', t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right) \quad \frac{r'}{c} \ll T \quad \Rightarrow \quad \frac{L}{\lambda} \ll 1$$

$$\Rightarrow \qquad \overline{V} = G\left\{\int \frac{\mathrm{d}^3\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\sigma(\boldsymbol{x}', t) - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{x}'\sigma(\boldsymbol{x}', t) + \ldots\right\}$$

Exterior zone: multipole expansion + weak field $\qquad \frac{L}{r} \leq 1 \qquad \varepsilon = \frac{GM}{Lc^2} \ll 1$

$$\mathcal{M}(V) = G\int \mathrm{d}^3\boldsymbol{x}'\left\{\frac{1}{r}\sigma\left(\boldsymbol{x}', t - \frac{r}{c}\right) - x'^i \frac{\partial}{\partial x^i}\left[\frac{1}{r}\sigma\left(\boldsymbol{x}', t - \frac{r}{c}\right)\right] + \ldots\right\} \quad \left(\frac{L}{r} \leq 1\right)$$

$$= G\sum_{\ell=0}^{+\infty}\frac{(-1)^\ell}{\ell!}\underbrace{\partial_{i_1}\partial_{i_2}\ldots\partial_{i_\ell}}_{\partial_L}\left\{\frac{1}{r}\underbrace{\int \mathrm{d}^3\boldsymbol{x}' x'^{i_1} x'^{i_2}\ldots x'^{i_\ell}\sigma(\boldsymbol{x}', t - r/c)}_{M_{i_1 i_2 \ldots i_\ell} \equiv M_L \text{ at time } t-r/c}\right\}$$

Multipolar expansion of the gravitational field

Harmonic gauge equations (arbitrary dimension

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(gauge condition on shell)}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \equiv \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu}(\partial h, \partial h)$$

$$\begin{aligned} h^{\mu\nu} &= \Box_{\mathrm{R}}^{-1}\frac{16\pi G \tau^{\mu\nu}}{c^4} \\ &= \mathrm{Reg}\frac{16\pi G}{c^4}\int \mathrm{d}^d\boldsymbol{x}'\mathrm{d}t' G_{\mathrm{R}}(t-t', \boldsymbol{x}-\boldsymbol{x}')\Big[\underbrace{(\overline{\tau^{\mu\nu} - \mathcal{M}(\tau^{\mu\nu})})}_{0\ \text{outside near zone}} + \mathcal{M}(\tau^{\mu\nu})\Big](\boldsymbol{x}', t) \\ &= \mathrm{Reg}\frac{16\pi G}{c^4}\Box_{\mathrm{R}}^{-1}\mathcal{M}(\tau^{\mu\nu}) + \sum_{\ell=0}^{+\infty}\frac{(-1)^\ell}{\ell!}\int \mathrm{d}t'\,\mathrm{Reg}\int \mathrm{d}^d\boldsymbol{x}' G_{\mathrm{R}}(t-t', \boldsymbol{x}) x'^L \Big[\overline{\tau^{\mu\nu}} - \overline{\mathcal{M}(\tau^{\mu\nu})}\Big](\boldsymbol{x}', t) \end{aligned}$$

vanishes if $\mathrm{Reg}\int \mathrm{d}^d\boldsymbol{x}' r'^{\alpha} = 0$

$$\text{homogeneous solution } \mathcal{H}^{\mu\nu} = -\frac{4}{c^4}\sum_{\ell=0}^{+\infty}\partial_L \widetilde{\mathcal{F}}_L^{\mu\nu}$$

convolution with a time kernel

STF

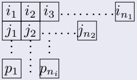
Decomposition into irreducible elements

What we want: $\lambda_1^{K_1} T^{L_1} + \lambda_2^{K_2} T^{L_2} = 0 \Rightarrow \lambda^{K_1} \; \& \; \lambda^{K_2} = 0$

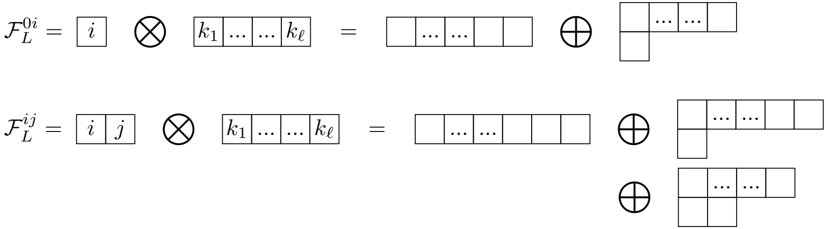
What we don't want: $T^{ij} + \delta^{ij} T = 0$ with $T^{ij} = -\delta^{ij} T$

$$\Rightarrow$$

work with irreducible elements



$+$ traceless



Source moments in d dimensions

- Einstein's equation in vacuum $[h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}]$

$$\Box h^{\mu\nu} = \Lambda^{\mu\nu} \quad \text{(REE)} \qquad\qquad \partial_\nu h^{\mu\nu} = 0 \quad \text{(Gauge)}$$

- Most general retarded solution at linear order (with $\Lambda^{\mu\nu} \to 0$) in d dim

$$h_1^{\mu\nu} = h_{1\,\mathrm{can}}^{\mu\nu} + (\text{gauge transformation})^{\mu\nu}$$

$$h_{1\,\mathrm{can}}^{00} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L \widetilde{I}_L$$

$$h_{1\,\mathrm{can}}^{0i} = \frac{4}{c^3}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\left[\partial_{L-1}\dot{\widetilde{I}}_{iL-1} + \frac{\ell}{\ell+1}\partial_L \widetilde{J}_{i|L}\right]$$

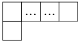
$$h_{1\,\mathrm{can}}^{ij} = -\frac{4}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\left[\partial_{L-2}\ddot{\widetilde{I}}_{ijL-2} + \frac{2\ell}{\ell+1}\partial_{L-1}\dot{\widetilde{J}}_{(i|\underline{L-1}j)} + \frac{\ell-1}{\ell+1}\partial_L \widetilde{K}_{ij|L}\right]$$

Principle of the method

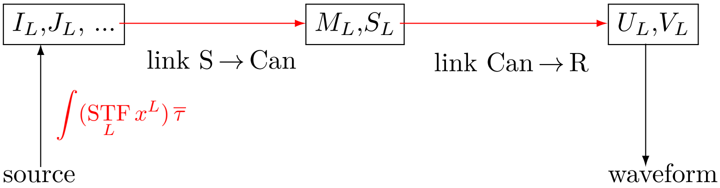
- Expression of $J_{i|L}$

$$J_{i|L} = \underset{i|L}{\mathrm{Sym}} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^d\mathbf{x}\, \tilde{r}^B \left\{ -2\hat{x}^L \frac{\overline{\tau}^i_{[\ell]}}{c} + \frac{2(2\ell+d-2)}{c^2(\ell+d-1)(2\ell+d)} \hat{x}^{aL} \dot{\overline{\tau}}^{ia}_{[\ell+1]} \right\}$$

$\mathrm{Sym}_{i|L} = \mathcal{A}_{ii_\ell}\, \mathrm{TF}_{iL}\, \mathrm{STF}_L$

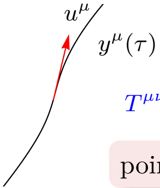
In 3 dimension: $J_L = \varepsilon$ 

- Link between radiative moments and source moments



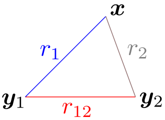
Source modeling

- Modeling of binary systems



$$T^{\mu\nu} = m\,\delta^{\rm cov} u^\mu u^\nu$$

- $y^\mu(\tau)$: position
- $u^\mu = \dfrac{\mathrm{d}y^\mu}{c\,\mathrm{d}\tau}$



point particle divergences cured by dimensional regularization

- Expression of a given integral in FP and dim reg

$$I^{\rm HR}_{\mathcal{R}} = \underset{\alpha=0}{\rm FP}\,\underset{\beta=0}{\rm FP}\int_{r<\mathcal{R}} \mathrm{d}^3x \left(\frac{r_1}{s_1}\right)^\alpha \left(\frac{r_2}{s_2}\right)^\beta F(x) \qquad I^{\rm DR}_{\mathcal{R}} = \int_{r<\mathcal{R}} \frac{\mathrm{d}^d x}{\ell_0^{d-3}} F^{(d)}(x)$$

- Difference between the two regularizations [Blanchet *et al.* (2004)]

$$\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q+1)\varepsilon}}_{\text{IR pole}} + \ln\left(\frac{s_1}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \underset{1}{f}{}^{(\varepsilon)}_{3,q}(\boldsymbol{n}) + 1 \leftrightarrow 2 + \mathcal{O}(\varepsilon)$$

Calculation of J_{ij}

3 types of terms (of 210)

- compact support: 50 terms

$$\mathop{\mathrm{STF}}_{ij}\,\varepsilon_{iab}\int \mathrm{d}^3\boldsymbol{x}\,x^a x^j \sigma_b$$

- non-compact support: 131 terms

$$-\mathop{\mathrm{STF}}_{ij}\frac{4\varepsilon_{iab}}{G\pi c^6}\int \mathrm{d}^3\boldsymbol{x}\left(\frac{r}{r_0}\right)^B x^a x^j \hat{Z}_{kl}\partial_{kl}V_b$$

- surface: 29 terms

$$-\mathop{\mathrm{STF}}_{ij}\frac{\varepsilon_{iab}}{2G\pi c^2}\int \mathrm{d}^3\boldsymbol{x}\left(\frac{r}{r_0}\right)^B x^a x^j \Delta\Big[V V_b\Big]$$

↪ distributional parts of derivatives taken into account

$$\partial_{ij}\left(\frac{1}{r_1}\right)\Big|_{\mathrm{distr}} = -\frac{4\pi}{3}\delta_{ij}\delta^3(\boldsymbol{x}-\boldsymbol{y}_1)$$

Expression of the gravitational-wave mode (2,1)

$$h^{\ell m} = \frac{2G\, m_{\rm tot}\, \nu\, x}{R\, c^2} \sqrt{\frac{16\pi}{5}}\, \hat{H}^{\ell m}\, e^{-im\psi} \quad \text{with}$$

$$x = \left(\frac{G m_{\rm tot}\omega}{c^3}\right)^{2/3}, \qquad \nu = \frac{m_1 m_2}{m_{\rm tot}^2}$$

$$\begin{aligned}
\hat{H}^{21} = \frac{\mathrm{i}}{3}\,\Delta \Bigg[& x^{1/2} + x^{3/2}\left(-\frac{17}{28} + \frac{5\nu}{7}\right) + x^2\left(\pi + \mathrm{i}\left[-\frac{1}{2} - 2\ln 2\right]\right) \\
& + x^{5/2}\left(-\frac{43}{126} - \frac{509\nu}{126} + \frac{79\nu^2}{168}\right) \\
& + x^3\left(\pi\left[-\frac{17}{28} + \frac{3\nu}{14}\right] + \mathrm{i}\left[\frac{17}{56} + \nu\left(-\frac{353}{28} - \frac{3}{7}\ln 2\right) + \frac{17}{14}\ln 2\right]\right) \\
& + x^{7/2}\left(\frac{15223771}{1455300} + \frac{\pi^2}{6} - \frac{214}{105}\gamma_{\rm E} - \frac{107}{105}\ln(4x) - \ln 2 - 2(\ln 2)^2\right. \\
& \left. + \nu\left[-\frac{102119}{2376} + \frac{205}{128}\pi^2\right] - \frac{4211}{8316}\nu^2 + \frac{2263}{8316}\nu^3 + \mathrm{i}\pi\left[\frac{109}{210} - 2\ln 2\right]\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^8}\right)
\end{aligned}$$

Conclusion

- Results
  - We investigated the multipole moments in d dimensions
  - We computed J_{ij} at the 3PN order ← test of constant shift [Damour & Iyer (1991)]
  - We obtained h^{21} at the 3.5PN order ← test particle limit [Tagoshi & Sasaki (1992)]
- Prospects
  - Computation of the 4PN flux
  - Consideration of effects beyond the point-particle model (spins)
  - Generalization of the expression for h^{21} to eccentric orbits