Remodeling the Effective One-Body Formalism in Post-Minkowskian Gravity

Pierre Vanhove

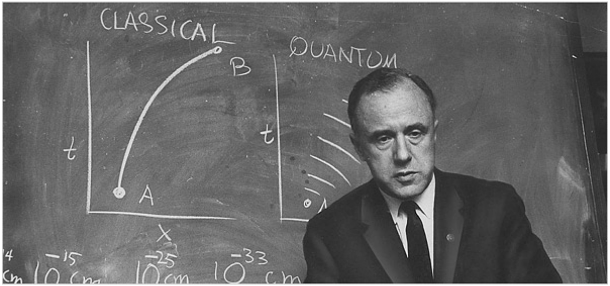
IPhT Saclay & HSE

5ème assemblée GDR ondes gravitationelles
LAPP Annecy, France

based on 2104.04510, 2105.05218, 2107.12891, 2108.11248,
N.E.J. Bjerrum-Bohr, Poul Damgaard, Ludovic Planté

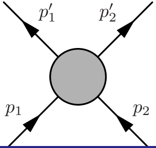
  

Classical Gravity from quantum scattering



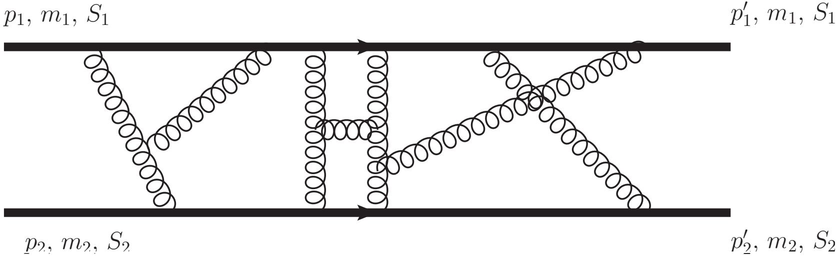
One important **new** insight is that the **classical** gravitational two-body interactions (conservative and radiation) can be extracted from **quantum scattering amplitudes**

$$\mathcal{M}(p_1\cdot p_2, q^2) =$$



$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}^{\text{L-loop}}$$

Classical Gravity from quantum scattering

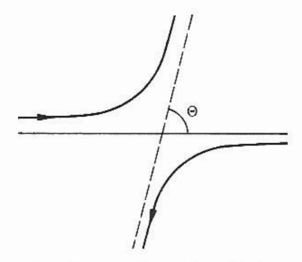


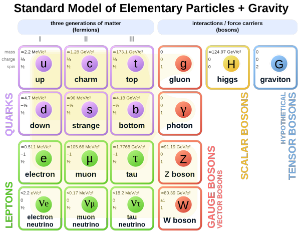
For $\hbar, q^2 \to 0$ with $\underline{q} = q/\hbar$ fixed at each loop order the classical contribution is of order $1/\hbar$ ($\gamma = p_1 \cdot p_2/(m_1 m_2)$)

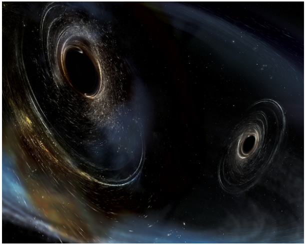
$$\mathcal{M}_L(\gamma, \underline{q}, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(D-4)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(D-4)}{2}+2-L}} + O(\hbar^0)$$

In this approach the **classical gravity physics contributions** are determined by the **unitarity** of the quantum scattering amplitudes

Nove sed non nova: classical observables







- Classical scattering: scattering angle χ : a lot of physical information for bound orbits
- Quantum scattering for generic EFT of gravity: probability amplitude $\mathcal{M}$
- Spinning black holes as higher-spin massive particles

The Effective One-Body (EOB) formalism (adapted from Post-Newtonian to Post-Minkowskian formulations) connects the scattering regime to the bound-state regime [Buonano, Damour; Damour]

Effective EOB metric

A general parametrization of the effective metric $g^{\text{eff}}_{\mu\nu}$ in isotropic coordinates

$$ds^2_{\text{eff}} = A(r)dt^2 - B(r)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

The scattering angle in such an external metric is derived using the principal function

$$\mathcal{S} = \mathcal{E}_{\text{eff}}t + J_{\text{eff}}\varphi + W(r)$$

of the associated Hamilton-Jacobi equation

$$g^{\alpha\beta}_{\text{eff}}\partial_\alpha\mathcal{S}\partial_\beta\mathcal{S} = \mu^2$$

to obtain

$$\frac{\mathcal{E}^2_{\text{eff}}}{A(r)} - \frac{J^2_{\text{eff}}}{B(r)r^2} - \frac{1}{B(r)}\left(\frac{dW(r)}{dr}\right)^2 = \mu^2$$

Effective EOB metric

The scattering angle $d(\chi/2)/dr \equiv \partial(dW(r)/dr)/\partial J_{\rm eff}$

$$\frac{\chi}{2} = J_{\rm eff}\int_{r_m}^{\infty}\frac{dr}{r^2}\frac{1}{\sqrt{\frac{B(r)}{A(r)}\mathcal{E}_{\rm eff}^2 - \frac{J_{\rm eff}^2}{r^2} - B(r)\mu^2}} - \frac{\pi}{2}$$

comparing with the expression from the radial action

$$p_r^2 = p_\infty^2 - \mathcal{V}_{\rm eff}(r,E) - \frac{J^2}{r^2}$$

we get

$$\frac{\chi}{2} = -\int_{\hat r_m}^{\infty} dr\frac{\partial p_r}{\partial J} - \frac{\pi}{2} = b\int_{\hat r_m}^{\infty}\frac{dr}{r^2}\frac{1}{\sqrt{1 - \frac{b^2}{r^2} - \frac{\mathcal{V}_{\rm eff}(r,E)}{p_\infty^2}}} - \frac{\pi}{2}$$

For identifying the two expressions under the square root we need the EOB maps for the effective energy, momentum and angular momentum

EOB energy, momentum and angular momentum maps

The effective-one-body formalism is based on the following maps

- The *energy map* $E = (m_1 + m_2)\sqrt{1 + 2\frac{m_1 m_2}{(m_1+m_2)^2}\left(\frac{m_1+m_2}{m_1 m_2}\mathcal{E}_{\rm eff} - 1\right)}$
- The *momentum map* $p_\infty^2 = \frac{(E^2-(m_1+m_2)^2)(E^2-(m_1-m_2)^2)}{4E^2}$, $\frac{p_{\rm eff}}{\mu} = \frac{p_\infty E}{m_1 m_2}$
- An *angular momentum map*

$$b = \frac{J}{p_\infty} = \frac{J_{\rm eff}}{p_{\rm eff}} \implies J_{\rm eff} = J\frac{p_{\rm eff}}{p_\infty} = J\,\frac{E}{M}$$

⚠ This maps differs from the one used by Damour $J_{\rm eff} = J$

The metric coefficients are then fully determined by the effective potential

$$1 - \frac{\mathcal{V}_{\rm eff}(r,E)}{p_\infty^2} = \frac{B(r)}{\gamma^2 - 1}\left(\frac{\gamma^2}{A(r)} - 1\right)$$

☺ With these maps we **never need any non-metric contributions** to the contrary to the “standard” EOB approach of [Buonano, Damour]

Effective EOB metric

In order to fix the parametrisation ambiguity we parameterise the metric coefficient using the *Ansatz*

$$A(r) = \left(\frac{1-h(r)}{1+h(r)}\right)^2; \qquad B(r) = (1+h(r))^4 .$$

to get

$$\left(h(r) + \frac{\gamma-1}{\gamma+1}\right)\left(h(r) + \frac{\gamma+1}{\gamma-1}\right)(1+h(r))^4 = (1-h(r))^2\left(1 + \frac{E^2}{(\gamma^2-1)M^2}\frac{\mathcal{V}_{\text{eff}}(r,E)}{\nu^2M^2}\right).$$

This equation can **always** be solved in perturbation theory with $h(r) = \sum_{n\geqslant 1} h_n(GM/r)^n$ for any perturbatively expanded effective potential $\mathcal{V}_{\text{eff}} = -\sum_{n\geqslant 1} f_n(G_NM/r)^n$.

The effective potential from the scattering amplitudes I

We then need to determine the effective potential from the scattering amplitudes

In the isotropic coordinates, there exists a very simple relationship between centre-of-mass momentum p and the effective classical potential $\mathcal{V}_{\rm eff}(r,p)$ of the form

$$p^2 = p_\infty^2 - \mathcal{V}_{\rm eff}(r,E); \qquad \mathcal{V}_{\rm eff}(r,E) = -\sum_{n\geqslant 1} f_n \left(\frac{G_N(m_1+m_2)}{r}\right)^n$$

the coefficients f_n are directly extracted from the scattering angle

$$\chi = \sum_{k\geqslant 1} \frac{2b}{k!} \int_0^\infty du \left(\frac{d}{du^2}\right)^k \left[\frac{1}{u^2+b^2}\left(\frac{\mathcal{V}_{\rm eff}\left(\sqrt{u^2+b^2}\right)(u^2+b^2)}{\gamma^2-1}\right)^k\right].$$

Scattering angle

The scattering angle is obtained by from the classical eikonal phase $\delta(\gamma, b)$

$$\sin\left(\frac{\chi}{2}\right)\Big|_L = -\frac{\sqrt{(p_1+p_2)^2}}{m_1 m_2\sqrt{\gamma^2-1}}\,\frac{\partial \delta_L(\gamma,b)}{\partial b}.$$

The classical eikonal phase $\delta(\gamma, b)$ is defined by the exponentiation of the S-matrix

$$1 + i\mathcal{T} = 1 + i\sum_{L\geqslant 0} G_N^{L+1}\mathcal{M}_L(\gamma,b) = (1+i2\Delta)e^{\frac{2i\delta(\gamma,b)}{\hbar}}.$$

The classical eikonal phase is then connected to the $1/\hbar$ coefficient of the scattering amplitude, i.e. the classical part of the amplitude

Exponentiation of the S-matrix

Using an exponential representation of the $\widehat{S}$ matrix [Damgaard, Planté, Vanhove]

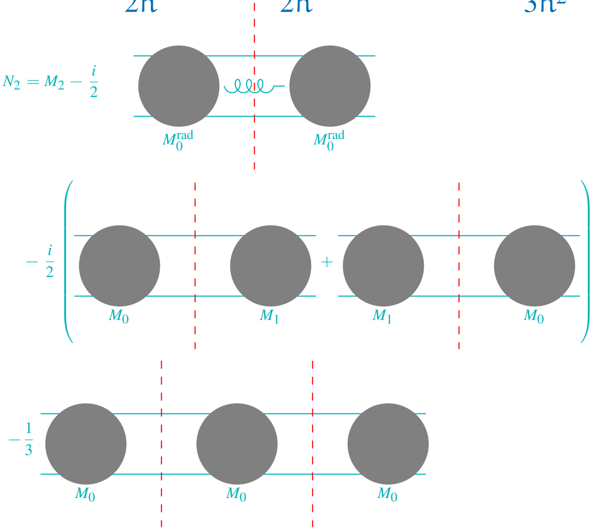
$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

with the completeness relation that includes all the exchange of gravitons for $n \geqslant 1$ entering the radiation-reaction contributions $\hat{N}^{\rm rad}$

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^{(D-1)}k_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_1}} \frac{d^{(D-1)}k_2}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{k_2}}$$
$$\times \frac{d^{(D-1)}\ell_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{(D-1)}\ell_n}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \ldots \ell_n\rangle\langle k_1, k_2; \ell_1, \ldots \ell_n|,$$

Exponentiation of the S-matrix

$$\hat{N}_2 = \hat{T}_2 - \frac{i}{2\hbar}(\hat{T}_0^{\mathrm{rad}})^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_1 + \hat{T}_1\hat{T}_0) - \frac{1}{3\hbar^2}\hat{T}_0^3,$$



Velocity cuts

In practice, we need only evaluate matrix elements in the soft $\underline{q}^2$-expansion, this means that we expand genuine unitarity cuts around the velocity cuts introduced recently [Bjerrum-Boh, Damgaard, Planté, Vanhove]

These velocity cuts seem to provide the most natural way to organise amplitude calculations.

From the complete determination of the two-body amplitude up to 3PM in [Bjerrum-Bohr, Damgaard, Planté, Vanhove] we have a full control of the scattering from the small velocity to the high energy limit

Conclusion

We have given a new way of connecting the scattering amplitude to the EOB effective potential

1. Changing the map between the effective angular momentum J_{eff} and J simplifies drastically the determination of the effective EOB metric from the effective potential with **no need non-metric terms**
2. A new exponentiation formula and the velocity cuts males the relation between the classical part of the scattering amplitude and the effective potential simple and efficient [Bjerrum-Bohr, Damgaard, Planté, Vanhove]
3. From the complete evaluation of the 2-body scattering to 3PM order we have the effective metric including the *radiation-reaction term*
4. We have derived the amplitude in D dimensions to 3PM order and to 5PM order in the probe limit [Bjerrum-Bohr, Planté, Vanhove, to appear]
5. **gravity is richer in higher dimensions!** Since the amplitude approach has been validated in higher-dimensions we can explore various interesting classical gravity physics in higher dimensions