Exact finite volume expectation values of conserved currents

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$$\begin{cases} \text{arXiv:1903.06990}^{[\text{'19 Bajnok, Smirnov}]} \\ \text{arXiv:1908.07320}^{[\text{'19 Borsi, Pozsgay, Pristyák}]} \end{cases} \Rightarrow \text{arXiv:1911.08525}$$

Generalized Hydrodynamics

A set of conservation laws:

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0$$

Ex.: Navier-Stokes:

$$\begin{array}{ccc} \rho(x,t), v(x,t) & & \rho(x,t), p(x,t) = j_\rho = \rho v \\ & \Rightarrow & \\ \partial_t \rho + \partial_x (v\rho) = 0 & & \partial_t \rho + \partial_x j_\rho = 0 \\ \rho(\partial_t v + v\partial_x v) = -\partial_x P(\rho) + \zeta \partial_x^2 v & & \partial_t p + \partial_x j_p = 0 \end{array}$$

where $j_p = P + v^2\rho - \zeta\partial_x v$. Generally:

$$j_i = F_i(\underline{q}).$$

GHD - GGE

Generalized Gibbs ensemble:

$$Z = \mathrm{Tr}\left(e^{-\sum_j \beta^j \hat{Q}_j}\right) \quad \Rightarrow \quad \langle \hat{\mathcal{O}} \rangle_{\underline{\beta}}$$

$$\{\beta^j\} \leftrightarrow \{q_i\}$$

Local entropy maximisation:

$$q_i(x,t) = \langle \hat{q}_i \rangle_{\underline{\beta}(x,t)}$$
$$j_i(x,t) = \langle \hat{j}_i \rangle_{\underline{\beta}(x,t)}$$

Flux Jacobian:

$$\frac{\partial j_i}{\partial q_j} = A_i^j(\underline{q}) \quad \Rightarrow \quad RAR^{-1} = \mathrm{diag}(v_1^{\mathrm{eff}}, v_2^{\mathrm{eff}} \ldots, \ldots)$$

Normal modes:

$$\partial_t n_i + v_i^{\mathrm{eff}}(\underline{n}(x,t))\partial_x n_i = 0$$

GHD - TBA ['16 Castro-Alvaredo, Doyon, Yoshimura]

$$q_i = \int \mathrm{d}\theta \, \rho(\theta) h_i(\theta) = -\frac{\partial F}{\partial \beta^i}, \quad \text{eg.} \begin{cases} h_1(\theta) = E(\theta) = m \cosh \theta \\ h_2(\theta) = p(\theta) = m \sinh \theta \\ \dots \end{cases}$$

$$2\pi(\rho(\theta) + \rho_h(\theta)) = p'(\theta) + \int \mathrm{d}u \, \overbrace{\varphi}^{-i\partial \ln S}(\theta - u)\rho(u)$$

$$\Rightarrow \quad \boxed{2\pi\rho(\theta) = n(\theta)(p')^{\mathrm{dr}}(\theta)}$$

$$n(\theta) = \frac{\rho(\theta)}{\rho(\theta) + \rho_h(\theta)} = \frac{1}{1 + e^{\epsilon(\theta)}}$$

$$\epsilon(\theta) = \overbrace{\sum_j \beta^j h_j(\theta)}^{w(\theta)} - \int \frac{\mathrm{d}u}{2\pi} \varphi(\theta - u) \ln\left(1 + e^{-\epsilon(u)}\right)$$

Main formulae

Pairing:

$$\boxed{f(\theta)\circ g(\theta) = \int \frac{\mathrm{d}\theta\ n(\theta)}{2\pi} f(\theta) g(\theta)}$$

Dressing:

$$\boxed{g^{\mathrm{dr}}(\theta) = g(\theta) + \varphi(\theta - u)\circ g^{\mathrm{dr}}(u)}$$

$$\Rightarrow \boxed{q_i = (p')^{\mathrm{dr}}(\theta)\circ h_i(\theta), \quad j_i = (E')^{\mathrm{dr}}(\theta)\circ h_i(\theta)}$$

Properties:

$$f(\theta)\circ g(\theta) = g(\theta)\circ f(\theta)$$
$$f(\theta)\circ g^{\mathrm{dr}}(\theta) = f^{\mathrm{dr}}(\theta)\circ g(\theta)$$

Connection to Leclair-Mussardo formula

$$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \int \frac{n(\theta_k)\mathrm{d}\theta_k}{2\pi} F_{2n,c}(\theta_1, \dots, \theta_n)$$

$$F^{q_i}_{2n,c}(\theta_1, \dots, \theta_n) = E(\theta_1)\varphi(\theta_1 - \theta_2)\dots\varphi(\theta_{n-1} - \theta_n)h_i(\theta_n) + \mathrm{perm.}$$
$$F^{j_i}_{2n,c}(\theta_1, \dots, \theta_n) = p(\theta_1)\varphi(\theta_1 - \theta_2)\dots\varphi(\theta_{n-1} - \theta_n)h_i(\theta_n) + \mathrm{perm.}$$

Ex.:

$$\langle q_i \rangle = \sum_{n=0}^{\infty} E(\theta_1) \circ \varphi(\theta_1 - \theta_2) \circ \dots \circ \varphi(\theta_{n-1} - \theta_n) \circ h_i(\theta_n)$$

Finally:

$$\langle q_i \rangle = E(\theta) \circ h_i^{\mathrm{dr}}(\theta)$$
$$\langle j_i \rangle = p(\theta) \circ h_i^{\mathrm{dr}}(\theta)$$

Continuation to the finite volume channel

Transformation

$$
\begin{aligned}
\underline{\beta} &\rightarrow (L, 0, 0, 0, \ldots) \\
\theta &\rightarrow \quad \theta^{\gamma} \equiv \theta + \frac{i\pi}{2} \\
(x, t) &\rightarrow (it, ix) \\
(j, q) &\rightarrow (iq, ij) \\
(p, E) &\rightarrow (iE, ip)
\end{aligned}
$$

LM formula

$$
\langle 0|\mathcal{O}|0\rangle_L = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} \int \frac{n(\theta_k) d\theta_k}{2\pi} F^{\mathcal{O}}_{2n,c}(\theta_1^{\gamma}, \ldots, \theta_n^{\gamma})
$$

VEVs

$$
\begin{aligned}
\langle 0|q_k|0\rangle_L &= i(E')^{\mathrm{dr}}(\theta) \circ h_k(\theta^{\gamma}) = i\langle j_k^{\gamma}\rangle \\
\langle 0|j_k|0\rangle_L &= i(p')^{\mathrm{dr}}(\theta) \circ h_k(\theta^{\gamma}) = i\langle q_k^{\gamma}\rangle
\end{aligned}
$$

Simplification for the charges

Useful relation

$$\partial_\theta \epsilon(\theta) = \sum_j \beta^j (h_j')^{\mathrm{dr}}(\theta) \qquad \overset{\underline{\beta}=(L,0,0,0\ldots)}{\Rightarrow} \qquad \partial_\theta \epsilon(\theta) = L(E')^{\mathrm{dr}}(\theta)$$

$$\langle j_i \rangle|_{\underline{\beta}=(L,0,0,0\ldots)} = (E')^{\mathrm{dr}}(\theta) \circ h_i(\theta) = \frac{1}{L}\partial_\theta \epsilon(\theta) \circ h_i(\theta)$$
$$= \frac{1}{L}\int \frac{\mathrm{d}\theta}{2\pi} h_i'(\theta) \ln\left(1 + e^{-\epsilon(\theta)}\right)$$

Finite volume charge density

$$\langle 0|q_k|0\rangle_L = i\langle j_k^\gamma \rangle = \frac{i}{L}\int \frac{d\theta}{2\pi} h_k'(\theta^\gamma)\log(1+e^{-\epsilon(\theta)})$$

Energy density

$$h_1(\theta) = E(\theta) \qquad \Rightarrow \qquad ih_1'(\theta^\gamma) = -E(\theta)$$

$$E_L = L\langle 0|q_1|0\rangle_L = -\int \frac{d\theta}{2\pi} E(\theta)\log(1+e^{-\epsilon(\theta)}) \checkmark$$

Excited states and re-defined pairing

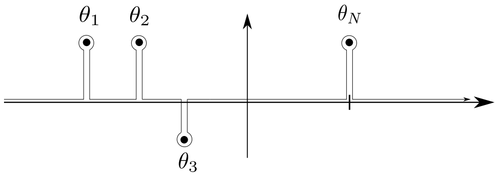
$$\epsilon(\theta) = \overbrace{\sum_i \beta_i h_i(\theta) + \sum_k \eta_k \log S(\theta - \theta_k)}^{w(\theta)} - \int \frac{du}{2\pi} \varphi(\theta - u) \ln(1 + e^{-\epsilon(u)})$$

Active singularities:

$$\epsilon(\theta_j) = i\pi(2n_j + 1)$$

Pairing and dressing for excited states: ['19 Bajnok, Smirnov]

$$\boxed{f(\theta) \bullet g(\theta) = \sum_j \frac{\eta_j i f(\theta_j) g(\theta_j)}{\partial_\theta \epsilon(\theta)|_{\theta=\theta_j}} + f(\theta) \circ g(\theta)}$$



Conjecture

New dressing:

$$\boxed{g^{\mathbf{dr}}(\theta) = g(\theta) + \varphi(\theta - u) \bullet g^{\mathbf{dr}}(u)}$$

Keeping original formulae:

$$\langle q_k \rangle = (p')^{\mathbf{dr}}(\theta) \bullet h_k(\theta) = p'(\theta) \bullet h_k^{\mathbf{dr}}(\theta) = ?$$
$$\langle j_j \rangle = (E')^{\mathbf{dr}}(\theta) \bullet h_k(\theta) = E'(\theta) \bullet h_k^{\mathbf{dr}}(\theta) = ?$$

Simplification for the charges

$$\partial_\theta \epsilon(\theta) = L(E')^{\mathbf{dr}}(\theta) \quad \Rightarrow \quad \langle j_k \rangle = \frac{1}{L}\partial_\theta \epsilon(\theta) \bullet h_k(\theta) =$$

$$= \frac{1}{L}\left(i \sum_j \eta_j h_k(\theta_j) + \int \frac{\mathrm{d}\theta}{2\pi} h'_k(\theta) \ln(1 + e^{-\epsilon(\theta)}) \right)$$

Finite volume channel:

$$\langle \theta_1, \ldots, \theta_n | q_k | \theta_1, \ldots, \theta_n \rangle_L = i \langle j_k^\gamma \rangle =$$

$$= \frac{1}{L}\left(- \sum_j \eta_j h_k(\theta_j^\gamma) + \int \frac{\mathrm{d}\theta}{2\pi} i h'_k(\theta^\gamma) \ln(1 + e^{-\epsilon(\theta)}) \right)$$

i.e. in sinh-Gordon: $\theta_j \equiv \bar{\theta}_j + \frac{i\pi}{2}$, $\eta_j = 1$:

$$E_L(\bar{\theta}_1, \ldots, \bar{\theta}_n) = \sum_j E(\bar{\theta}_j) - \int \frac{\mathrm{d}\theta}{2\pi} E(\theta) \ln(1 + e^{-\epsilon(\theta)})$$

Gaudin-matrix

$$\epsilon(\theta) = \overbrace{\sum_i \beta_i h_i(\theta) + \sum_k \eta_k \log S(\theta - \theta_k)}^{w(\theta)} - i \log S(\theta - u) \circ \partial_u \epsilon(u)$$

$$\partial_\theta \epsilon(\theta) = \overbrace{w'(\theta)}^{iD(\theta)} + \varphi(\theta - u) \circ \partial_u \epsilon(u) = iD^{\mathrm{dr}}(\theta)$$

$$\partial_{\theta_j} \epsilon(\theta) = \overbrace{-\eta_j i \varphi(\theta - \theta_j)}^{-i\varphi_j(\theta)} + \varphi(\theta - u) \circ \partial_{\theta_j} \epsilon(u) = -i\varphi_j^{\mathrm{dr}}(\theta)$$

Gaudin-matrix:

$$G_{jk} = -i\partial_{\theta_j} \epsilon(\theta_k) = -i \left(\delta_{jk} \partial_\theta \epsilon(\theta) + \partial_{\theta_j} \epsilon(\theta)\right) |_{\theta=\theta_k} = \delta_{jk} D^{\mathrm{dr}}(\theta_k) - \varphi_j^{\mathrm{dr}}(\theta_k)$$

Relation between dressings

$$\boxed{g^{\mathbf{dr}}(\theta) = g^{\mathrm{dr}}(\theta) + \sum_j \varphi_j^{\mathrm{dr}}(\theta) G_{jk}^{-1} g^{\mathrm{dr}}(\theta_k)}$$

$$\boxed{f(\theta) \bullet g^{\mathbf{dr}}(\theta) = \sum_{j,k} \eta_j f^{\mathrm{dr}}(\theta_j) G_{jk}^{-1} g^{\mathrm{dr}}(\theta_k) + f(\theta) \circ g^{\mathrm{dr}}(\theta)}$$

Expression via Gaudin matrix

$$\langle q_i \rangle = (p')^{\mathbf{dr}}(\theta) \bullet h_i(\theta) = \sum_{j,k} \eta_j (p')^{\mathrm{dr}}(\theta_j) G^{-1}_{jk} h_i^{\mathrm{dr}}(\theta_k) + p'(\theta) \circ h_i^{\mathrm{dr}}(\theta)$$

Finite volume channel:

$$\boxed{\langle \{\theta\} | j_i | \{\theta\} \rangle_L = i \sum_{j,k} \eta_j (p')^{\mathrm{dr}}(\theta_j) G^{-1}_{jk} h_i(\theta_k^{\gamma})^{\mathrm{dr}} + i p'(\theta) \circ h_i(\theta^{\gamma})^{\mathrm{dr}}}$$

Consistency in the Bethe Ansatz regime

sinh-Gordon: $\theta_j \equiv \bar{\theta}_j + \frac{i\pi}{2}$, $\eta_j = 1$:

$$\epsilon(\bar{\theta}_j + \frac{i\pi}{2}) = i\pi(2n_j + 1)$$

Dropping all e^{-mL}-type corrections: ['19 Borsi, Pozsgay, Pristyák]

$$\langle \bar{\theta}_1, \ldots, \bar{\theta}_n | j_i | \bar{\theta}_1, \ldots, \bar{\theta}_n \rangle_L = \sum_{j,k} E'(\bar{\theta}_j) G^{-1}_{jk} h_i(\bar{\theta}_k)$$

Consistency in the Lüscher regime

Excited state formula ['14 Pozsgay, Szécsényi, Takács]

$$\langle\theta_1,\ldots,\theta_n|\mathcal{O}|\theta_1,\ldots,\theta_n\rangle_L = \frac{\sum_{\{\theta_+\}\cup\{\theta_-\}} \mathcal{D}_\epsilon^{\mathcal{O}}(\{\theta_+\})\rho(\{\theta_-\}|\{\theta_+\})}{\rho(\theta_1,\ldots,\theta_n)}$$

$$\rho(\theta_1,\ldots,\theta_n) = \det G \qquad \text{(Gaudin-matrix)}$$

$$\rho(\{\theta_-\}|\{\theta_+\}) = \det G_- \qquad \text{(submatrix of } \{\theta_-\} \text{ particles)}$$

$$\mathcal{D}_\epsilon^{\mathcal{O}}(\{\theta\}) = \sum_{n=0}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\int\frac{n(u_k)\mathrm{d}u_k}{2\pi}F_{2(m+n),c}^{\mathcal{O}}(\theta_1,\ldots,\theta_m,u_1,\ldots,u_n)$$

For 1 particle states, up to e^{-mL} order in the large volume expansion (Lüscher-order), this gives the same result for the current!

Outlook

- Non-diagonal matrix elements
- Generalisation for non-diagonally scattering theories
- Cumulatns of charges and currents ['19 Vu] in excited states