The Quantum Fishchain

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Based on
arXiv:2101.01232 with Nikolay Gromov and Julius

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The Fishnet Model

Double scaling limit of γ-deformed $\mathcal{N} = 4$ SYM:

$$L = N\,\mathrm{tr}(|\partial\phi_1|^2 + |\partial\phi_2|^2 + (4\pi)^2\xi^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2) \tag{1}$$

where N is large and ξ is the fishnet coupling.

Features:

- Interacting but simple
- Conformal
- Not supersymmetric
- Integrable

Consider operators of the $U(1)$ sector:

$$\mathcal{O}_J = \mathrm{tr}\left[\partial^m \phi_1^J \left(\phi_2 \phi_2^\dagger\right)^n \ldots\right] \tag{2}$$

$$\langle \mathcal{O}_J(0)\, \mathcal{O}_J(x)\rangle = \frac{1}{x^{2\Delta_J}} \tag{3}$$

The anomalous dimension of operators $\mathcal{O}_J$ can be mapped to the spectrum of an integrable spin chain, the closed fishchain (Gromov, Sever 2019).

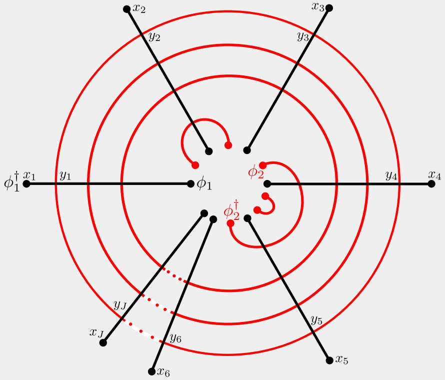
We can compute the following correlator using a loop expansion:

$$\psi(x_0, x_1 \ldots x_J) = \left\langle \mathcal{O}_J\left(x_0\right) \mathrm{tr}\left[\phi_1^\dagger\left(\vec{x}_1\right) \ldots \phi_1^\dagger\left(\vec{x}_J\right)\right]\right\rangle \tag{4}$$

ψ is called CFT wavefunction, and contains the information to compute any correlator of $\mathcal{O}_J$.

Graph-building operator

Feynman diagrams contributing to ψ have iterative structure:



We can define a graph-building operator $\hat{B}$!

$$\psi = \sum_{i=0}^{\infty} (\hat{B})^i = \frac{1}{1-\hat{B}} \tag{5}$$

Physical operators will satisfy:

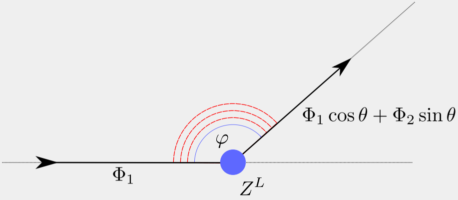
$$\hat{B}|\Psi\rangle = |\Psi\rangle \tag{6}$$

This eigenvalue equation fixes the conformal dimension of $\mathcal{O}_J$ in terms of the conformal charges and the coupling ξ (Gromov, Sever 2019; Gromov, Kazakov, Korchemski 2018)
$\hat{B}$ can be identified with the closed Fishchain Hamiltonian, and the above becomes an eigenvalue equation for its ground state:

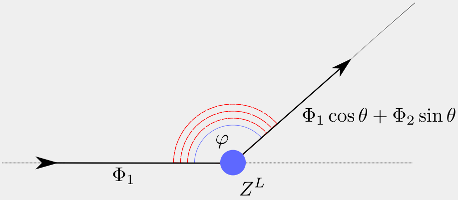
$$(\hat{B}-1)|\Psi\rangle = 0 \tag{7}$$

The Open Fishchain

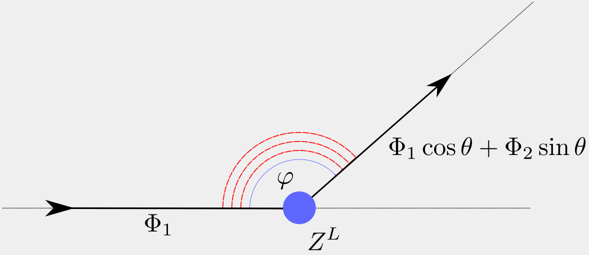
Provides an example of the iterative structures typical of the fishnet model in the theory of pure $\mathcal{N}=4$ SYM.



$$W = -\frac{1}{N}\,\mathrm{tr}\,\mathrm{P}\exp\int_0^\infty dt\,4\pi g\left(iA\cdot x'(t) + \Phi_1|x'(t)|\right)\times Z(0)^L \times$$
$$\mathrm{P}\exp\int_0^\infty ds\,4\pi g\left[iA\cdot x'(s) + (\Phi_1\cos\theta + \Phi_2\sin\theta)|x'(s)|\right]\ ,$$



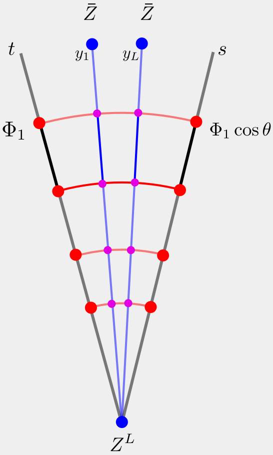
We are able to map the scaling dimension of this observable to the spectrum of an integrable spin chain with open boundary conditions (Gromov, Julius, NP 2020), the open Fishchain.



We take $g \to 0$, $\theta \to i\infty$ with $\hat{g} \equiv \frac{g}{2}\exp\left(-i\,\frac{\theta}{2}\right)$ kept fixed (Erickson, Semenoff and Zarembo, 2000; Correa, Henn, Maldacena and Sever 2012)

- Gluons and fermions decouple
- Φ_2 drops out

CFT Wavefunction



$$\psi = \langle \mathrm{tr}(W\,\bar{Z}(y_1)\ldots\bar{Z}(y_L))\rangle \tag{8}$$

Graph-destroying Operator and 4D Hamiltonian:

$$\hat{H}\,\psi = (\hat{B}^{-1} - 1)\psi = 0\ ,\ \hat{B}^{-1} \equiv \frac{(-1)^L}{(4\hat{g}^2)^{L+1}} \frac{\prod_{i=0}^{L}(y_i - y_{i+1})^2}{|y'_0||y'_{L+1}|}\,\partial_t \partial_s \prod_{j=1}^{L} \Box_{y_j}$$

Embed in 6D and define the coupling $\xi \equiv (2\,i)^{\frac{1}{J+1}}\hat{g}$:

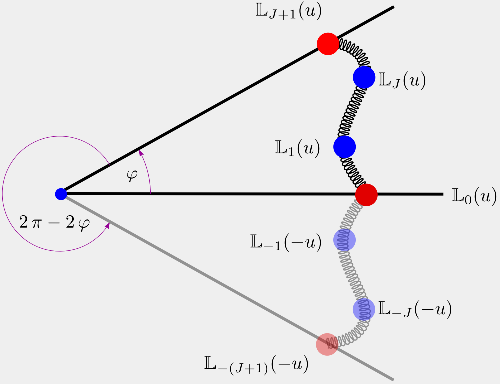
$$L \;=\; \xi\left(\frac{\dot{X}_0^2}{4} + \sum_{i=1}^{L}\frac{\dot{X}_i^2}{2} + \frac{\dot{X}_{L+1}^2}{4} - \sum_{i=0}^{L}\frac{1}{2}\log\frac{-X_i.X_{i+1}}{2e}\right)\ , \qquad (9)$$

Integrability

$$L = \xi \left(\frac{\dot{X}_0^2}{4} + \sum_{i=1}^{L} \frac{\dot{X}_i^2}{2} + \frac{\dot{X}_{L+1}^2}{4} - \sum_{i=0}^{L} \frac{1}{2} \log \frac{-X_i . X_{i+1}}{2e} \right) , \quad (10)$$

- Not solvable via Bethe Ansatz
- The boundary sites X_0 and X_{L+1} have 1 dof each, so they are dynamical: uncommon for integrable systems with boundaries!

Lax Representation



$$\mathbb{T}(u) = \operatorname{tr} \mathbb{L}_{-L}(u).\mathbb{L}_{-1}(u).\mathbb{L}_0(u).\mathbb{L}_1(u) \cdots \mathbb{L}_L(u).\mathbb{L}_{L+1}(u).G^4.G^4$$

$$\mathbb{T}(u) = \mathrm{tr}\,\mathbb{L}_{-L}(u).\mathbb{L}_{-1}(u).\mathbb{L}_0(u).\mathbb{L}_1(u)\cdots\mathbb{L}_L(u).\mathbb{L}_{L+1}(u).G^4.G^4$$

$$\mathbb{L}_i = u\,\mathbb{I}_{4x4} + \frac{i}{2}q_i^{MN}\Sigma_{MN}\,, \qquad \mathbb{L}_{-i}(u) = C^4.\mathbb{L}_i^t(-u).C^4$$

Where:

$$\hat{q}_j^{MN} = -\frac{i}{\xi}\left(X_j^N\frac{\partial}{\partial X_{j,M}} - X_j^M\frac{\partial}{\partial X_{j,N}}\right)$$

$$\hat{q}_0^{NM} \equiv -i\frac{2}{\xi}(Y_0^M\dot{Y}_0^N - Y_0^N\dot{Y}_0^M)\partial_t\,,\;\; Y_0 = \{\cosh t, -\sinh t, 1, 0, 0, 0\}$$

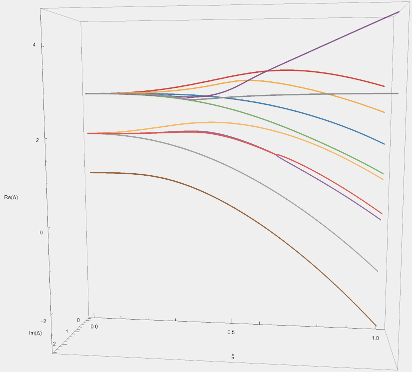
$$\hat{q}_{L+1}^{NM} \equiv -i\frac{2}{\xi}(Y_{L+1}^M\dot{Y}_{L+1}^N - Y_{L+1}^N\dot{Y}_{L+1}^M)\partial_s\,,\;\; Y_{L+1} = \{\cosh s, -\sinh s, 1, 0, 0, 0\}$$

The Baxter Equation

The transfer matrix satisfies the Baxter equation:

$$\begin{aligned}&\mathbb{T}^{\mathbf{1}}(v+i)Q(v+2i)+\mathbb{T}^{\mathbf{4}}(v+i/2)Q(v+i)\\&\quad+\mathbb{T}^{\mathbf{6}}(v)Q(v)+\mathbb{T}^{\bar{\mathbf{4}}}(v-i/2)Q(v-i)+\mathbb{T}^{\bar{\mathbf{1}}}(v-i)Q(v-2i)=0\ .\end{aligned}$$

The Q-functions contain information on the anomalous dimension of the observable, so we can use this equation + a quantisation condition to retrieve Δ numerically.



Verified the weak coupling result of (**Correa et al. 2012**):

$$\Delta = L + \hat{g}^{2L+2}\frac{(-1)^L 2^{4L+3}\pi^{2L+1}\csc(\varphi)B_{2L+1}\left(\frac{\varphi}{2\pi}\right)}{\Gamma(2L+2)} + \mathcal{O}(\hat{g}^{4L+4}) \,,$$

Outlook and Conclusions

- The fishchain can be used to compute with high numerical precision Scaling dimensions of operators with nontrivial Δ in fishnet theory
- We are able to use the same techniques to compute the scaling dimension of some observables in the full $\mathcal{N} = 4$ SYM theory

Thank You!

Explicitly Local form of L

Lagrangian is subject to the constraints

$$|\dot{X}_0| = |\dot{X}_{J+1}|$$

and

$$2\prod_{i=0}^{J}(-X_i \cdot X_{i+1})^{-\frac{1}{J+1}} = 1$$

Notice that $2|\dot{X}_0||\dot{X}_{J+1}| = \dot{X}_0^2 + \dot{X}_{J+1}^2 - (|\dot{X}_0| - |\dot{X}_{J+1}|)^2$ and replace $|\dot{X}_0||\dot{X}_{J+1}| \to \frac{\dot{X}_0^2}{2} + \frac{\dot{X}_{J+1}^2}{2}$

Define $y = 2\prod_{i=0}^{J}(-X_i \cdot X_{i+1})^{-\frac{1}{J+1}} \simeq 1$ on constraints

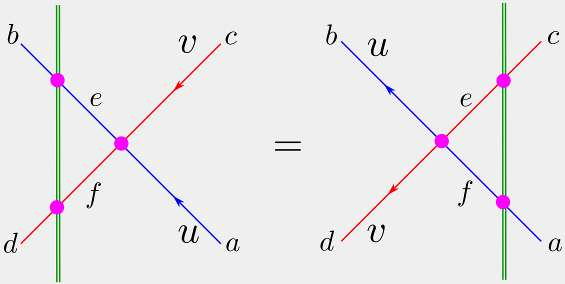
We have $y = e^{\log y} = 1 + \log y + \mathcal{O}(\log^2 y)$ so that

$$L \;=\; \xi\left(\frac{\dot{X}_0^2}{4} + \sum_{i=1}^{J}\frac{\dot{X}_i^2}{2} + \frac{\dot{X}_{J+1}^2}{4} - \sum_{i=0}^{J}\frac{1}{2}\log\frac{-X_i.X_{i+1}}{2e}\right) \;,$$

Yang-Baxter

$$\hat{\mathbb{L}}_i{}^a{}_b(u) = u\,\delta^a_b + \frac{i}{2}\hat{q}_i^{MN}\Sigma_{MN}{}^a{}_b$$

$$\hat{q}_j^{MN} = -\frac{i}{\xi}\left(X_j^N\frac{\partial}{\partial X_{j,M}} - X_j^M\frac{\partial}{\partial X_{j,N}}\right)$$



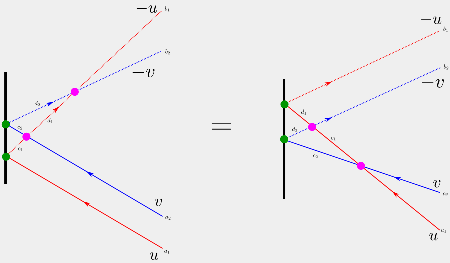
$$R^{b\,c}_{a\,d}(u) = \mathbb{I}_{(\mathrm{aux}\times\mathrm{aux})} + \frac{i}{\hat{g}\,u}\mathbb{P} \;= \delta^b_a\,\delta^c_d + \frac{i}{\hat{g}\,u}\delta^b_d\,\delta^c_a$$

Boundary Yang-Baxter

$$\hat{\mathbb{K}}(u) = C \cdot \hat{\mathbb{L}}_0(u - \tfrac{i}{2\xi})$$

$$\hat{\mathbb{L}}_0{}^a{}_b(u) = u\,\delta^a_b + \frac{i}{2}\hat{q}_0^{MN}\Sigma_{MN}{}^a{}_b$$

$$\hat{q}_0^{NM} \equiv -i\frac{2}{\xi}(Y_0^M\dot{Y}_0^N - Y_0^N\dot{Y}_0^M)\partial_t\ ,\ \ Y_0 = \{\cosh t, -\sinh t, 1, 0, 0, 0\}$$



Proof of $\mathbb{T}^{\mathbf{6}}(0) = 4\hat{B}^{-2}$

$$f(y_m^1,\ldots,y_m^4) \to \frac{1}{X_m^{-1}+X_m^0} f\left(\frac{X_m^1}{X_m^{-1}+X_m^0},\ldots,\frac{X_m^4}{X_m^{-1}+X_m^0}\right)$$

$$\hat{\mathbb{L}}_i^{\mathbf{6}\ MN}(0) = \frac{\hat{q}_i^{2\ MN}}{2} - \frac{i}{\xi}\hat{q}_i^{MN} - \frac{\eta^{MN}}{8}\,\mathrm{tr}\,\hat{q}_i^2 + \frac{\eta^{MN}}{4\xi^2} = \frac{:\hat{q}_i^2:^{MN}}{2} = \frac{1}{2\xi^2}X_i^M X_i^N \partial^2_{X_i^K}$$

$$\begin{aligned} u\,\xi\,\hat{\mathbb{K}}^{\mathbf{6}\ MN}(u)\Big|_{u=0} &= -\frac{i}{2}(\hat{q}_0^2\quad)^{NM} = -\frac{2i}{\xi^2}Y_0^M\hat{\partial}_t Y_0^N\hat{\partial}_t \\ u\xi\left(G^{\mathbf{6}}\hat{\bar{\mathbb{K}}}^{\mathbf{6}}(u)(G^{\mathbf{6}})^{-1}\right)^{MN}\Big|_{u=0} &= +\frac{i}{2}(\hat{q}_{J+1}^2)^{MN} = +\frac{2i}{\xi^2}Y_{J+1}^N\hat{\partial}_s Y_{J+1}^M\hat{\partial}_s \end{aligned}$$

$$\begin{aligned} Y_0.X_1X_1.X_2\ldots X_J.Y_{L+1}\partial_s\partial_t\prod_{i=1}^{L}\Box_i^{(6)} &= \\ \left(-\tfrac{1}{2}\right)^{L+1}\frac{\partial_s\partial_t}{|y_0'||y_{L+1}'|}\prod_{i=0}^{L}(y_i-y_{i+1})^2\prod_{i=1}^{L}\Box_i^{(4)} &= \left(\tfrac{1}{2}\right)^{L+1}(4\hat{g}^2)^{L+1}\hat{B}^{-1} \end{aligned}$$

Details of Numerical Solution - Asymptotics

$$
\begin{aligned}
q_1 &= e^{+\varphi v} v^{+\Delta+S-J}\left(1+\frac{c_{1,1}}{v}+\dots\right)\ ,\\
q_2 &= e^{-\varphi v} v^{+\Delta-S-J}\left(1+\frac{c_{2,1}}{v}+\dots\right)\ ,\\
q_3 &= e^{+\varphi v} v^{-\Delta-S-J}\left(1+\frac{c_{3,1}}{v}+\dots\right)\ ,\\
q_4 &= e^{-\varphi v} v^{-\Delta+S-J}\left(1+\frac{c_{4,1}}{v}+\dots\right)\ .
\end{aligned}
$$

- Baxter equation reduces to a linear problem asymptotically — gives very good approximation for large $|\mathrm{Im}\, v|$
- Use Baxter equation itself to find $q(v)$ in terms of $q(v+in),\ n=1,\dots,4$ — gradually decrease $|\mathrm{Im}\, v|$
- Two options: start from $+i\infty$ or from $-i\infty$ — find 4 analytic solutions in the upper-half plane: $q_i^{\downarrow}$, 4 analytic in the lower-half plane: $q_i^{\uparrow}$
- Can have only four independent solutions — $q_i^{\uparrow}$ and $q_i^{\downarrow}$ are related by a linear transformation.

Details of Numerical Solution - Quantisation Condition

$$q_i^{\uparrow}(v) = \Omega_i^{\,j}(v) q_j^{\downarrow}(v), \quad \Omega_i^{\,j}(v+i) = \Omega_i^{\,j}(v) \;,$$

Quantisation condition can be obtained by comparing with the QSC for cusp (**Gromov:2015dfa**)

$$\omega_{ik} = \Omega_i^{\,j} \Gamma_{jk}$$

$$\Gamma_{jk} = \begin{pmatrix} 0 & \gamma_1 \sinh(2\pi v) & 0 & \gamma_3 \\ \gamma_2 \sinh(2\pi v) & 0 & \gamma_4 & 0 \\ 0 & \gamma_5 & 0 & 0 \\ \gamma_6 & 0 & 0 & 0 \end{pmatrix}$$

Input from QSC: ω anti-symmetric