Exact Matrix Elements of the Field Operator in the Thermodynamic Limit of the Lieb-Liniger Model

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Correlation Functions and Wave Functions in Solveable models

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- We take the example of the Lieb-Liniger model.
- For brevity, it's an algebraic R-matrix, with real Bethe roots only.

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- Each overlap may be written as a determinant related to the Slavnov determinant,
- namely the Matsuo-Kostov determinant:

$$\langle\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}\rangle = e^{\frac{\imath L}{2}\sum_j \theta_j-\tilde{\theta}_j}\det(\mathbb{1}-K),\quad K_{ij}=\frac{E_i}{u_i-u_j+\imath c}$$

$$E_i\equiv e^{-\imath L u_i}\frac{\prod_k(u_i-u_k+\imath c)}{\prod_{k\neq i}(u_i-u_k)},\qquad \boldsymbol{u}\equiv\boldsymbol{\theta}\cup\tilde{\boldsymbol{\theta}}$$

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- We need to invert $1 - e^{-Ny}\mathcal{K}$, for $\mathcal{K}$.
  (We have yet to discuss $\mathcal{K}$)

The Operator K

- $$K_{ij} = e^{-\imath L u_i} \frac{Q_{\boldsymbol{u}}(u_i + \imath c)}{Q'_{\boldsymbol{u}}(u_i)} \frac{1}{u_i - u_j + \imath c}$$

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- $e^{-\imath L x} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)} \psi(x+\imath c)$ is a candidate for $\mathcal{K}\psi$.
- More precisely:

$$(\mathcal{K}\psi)(y) = \oint_{\boldsymbol{u}} \frac{e^{-\imath L x}}{y-x} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)} \psi(x+\imath c)$$

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The Kostov-Matsuo operator is just $1+\mathcal{K}$ with:

$$\mathcal{K} = \mathcal{P} e^{\Phi} e^{\imath c \partial}$$

with

$$e^{\Phi(x)} = e^{-\imath Lx} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)}, \quad (\mathcal{P}f)(x) = \oint \frac{f(y)}{x-y}$$

Taking the Trace of the Resolvent

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The trace is given by:

$$\mathrm{tr}(\mathbb{1} - e^{-Ny}K)^{-1} = \sum_j \operatorname*{Res}_{\lambda \to u_j}(R_y(\lambda, u_j)) =$$

$$= \oint \oint \frac{R_y(\lambda, w)}{\lambda - w} \frac{dz}{2\pi\imath} \frac{dw}{2\pi\imath},$$

where the w integral surrounds the λ integral.

Solution through Saddle Point

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- To invert the matrix $\mathbb{1} - e^{-Ny}K$, solve:

$$R_y(z,w) - \oint \frac{e^{N(\Phi(z')-y)} R_y(z' + \imath c, w)}{z - z'} = \frac{1}{z - w}$$

with $w \in \boldsymbol{u}$ and $e^{N\Phi(x)} = e^{-\imath Lx} \frac{Q_{\boldsymbol{u}}(x+\imath c)}{Q_{\boldsymbol{u}}(x)}$

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- Previously we considered $c \sim \frac{1}{N}$. Another limit, $c \sim N$, is interesting. Now interested in $c \sim 1$.

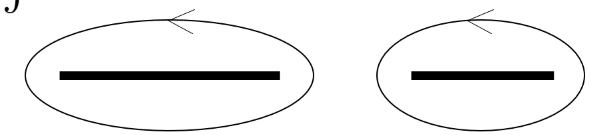
Fourier Transformed Integral Equation

The integral equation can be solved in the large N by first applying a Fourier transform.

$$\frac{1}{z-w} = R_y(z,w) - \oint \frac{e^{\Phi(z')-Ny} R_y(z'+\imath c, w)}{z-z'}$$

$$\oint e^{-N\imath Pz} R_y(z,w) = e^{NS(P,w)}$$

$$\oint e^{N(\Phi(z)-\imath Pz)} = e^{N\tilde{\varphi}(P)}$$



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For $P > 0$ this becomes $(e^{N\Phi(z)} \to e^{\imath N \tilde{\varphi}(P)})$:

$$e^{-\imath NPw} = e^{\imath NS(P,w)} - \int_0^\infty e^{\imath N[S(Q,w)+\imath cQ+\tilde{\varphi}(P-Q)]} dQ$$

Ansatz for Solving the Integral Equation

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$$e^{\imath NS(P,w)} = \begin{cases} -e^{-\imath N(\Phi(w)+(w+\imath c)P)} & P < -\frac{1}{c}\mathrm{Im}(\Phi(w)) \\ e^{-\imath NPw} & -\frac{1}{c}\mathrm{Im}(\Phi(w)) < P \end{cases}$$

Substituting the Ansatz

$$e^{-\imath NPw} \stackrel{?}{=} -\int_0^\infty e^{\imath N[S(Q,w)+\imath cQ+\tilde{\varphi}(P-Q)]}dQ$$

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Which is valid due to a saddle point justification.

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- One can then compute for example $\langle \mathrm{in}|\psi|\mathrm{out}\rangle$.
- In order to compare with known results we take small c (but not $c \sim N^{-1}$) and compare with semiclassics.
- The comparison is not 100% conclusive since semiclassics demands $c \sim N^{-1}$ and we have $N^{-1} \ll c \ll 1$. Nevertheless, the similarity between the two results leaves in my mind little doubt.

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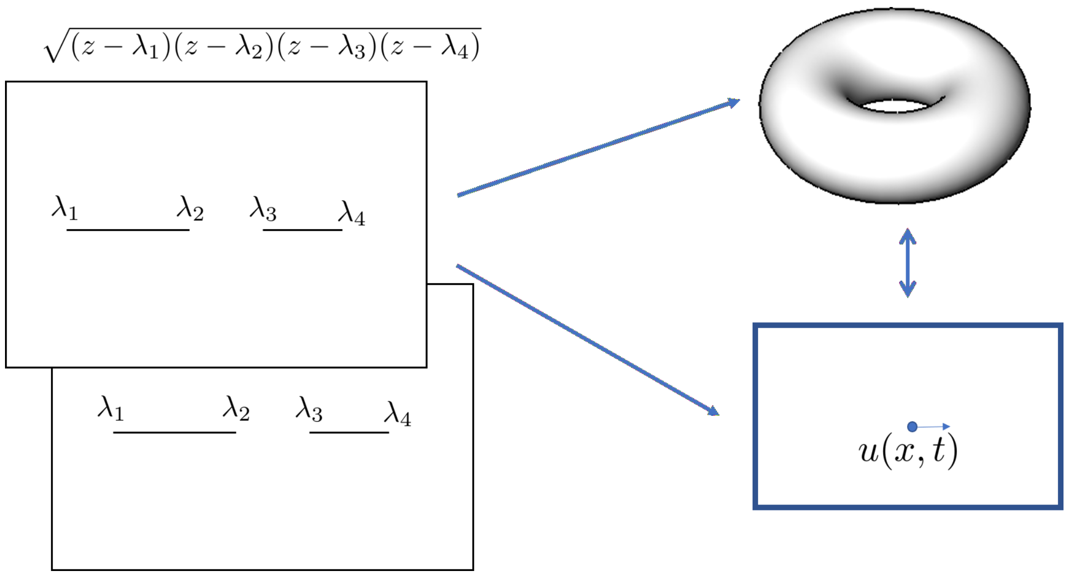
- Classical inverse scattering deals with a spectral surface of hyperelliptic type.
- The multi-gap spectral weight on the upper sheet of an operator from inverse scattering theory corresponds to density of Bethe roots.

The Semiclassics Field

Consider an elliptic Riemann surface



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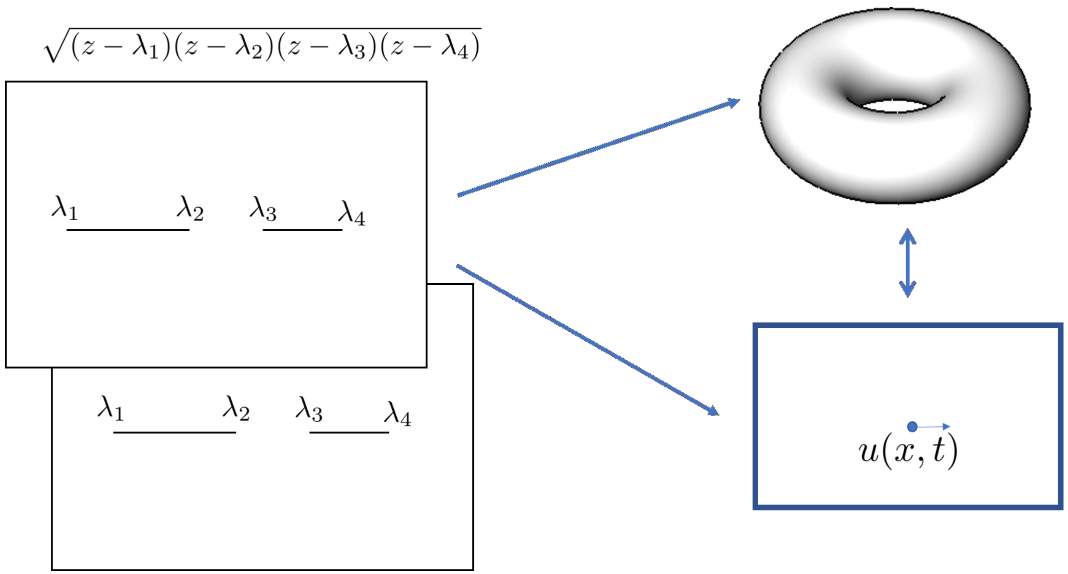

$$\psi(x,t) = e^{\frac{4\eta u_a}{\pi} u(x,t)} \frac{\sigma(u(x,t)-u_a)}{\sigma(u(x,t)+u_a)}, \qquad u(x,t) = k(x-vt)$$

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$$\langle \mathrm{out}|\psi(0)|\mathrm{in}\rangle = C \oint e^{\frac{4\eta u_a}{\pi} u(z)} \frac{\sigma(u(z)-u_a)}{\sigma(u(z)+u_a)} dz$$

Comparison Between the Classical and Quantum Results

- Classical average over space-time (Flaschka, McLaughlin, Forest):

$$\langle\psi(x)\rangle = \oint e^{\frac{4\eta u_a}{\pi} u(z)} \frac{\sigma(u(z)-u_\infty)}{\sigma(u(z)+u_\infty)} \frac{dz}{\sqrt{\prod(z-\lambda_i)}}$$

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- The measure of integration is different but the object integrated over is the same

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- We approach the semiclassical limit (but do not actually take it).
- The comparison is suggestive that the approach is correct.