Integrable D-branes

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C. Kristjansen, D. Müller, K.Z., 2005.01392; 2011.12192; 2106.08116
G. Linardopoulos, K.Z., 2102.12381

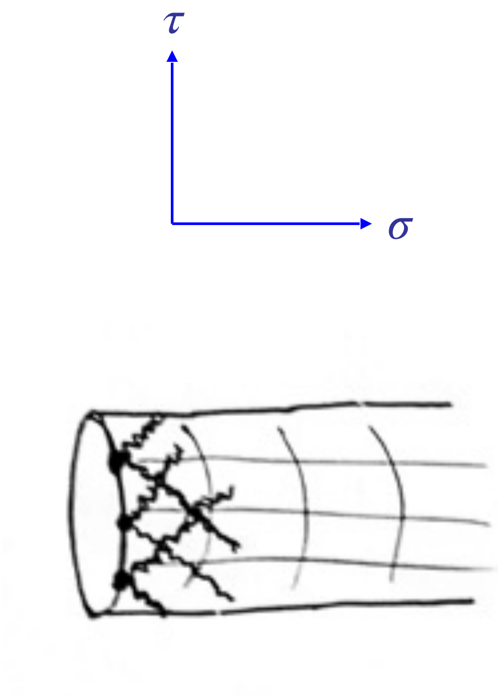
"Correlation functions and wavefunctions in solvable models," Saclay, 14.9.21

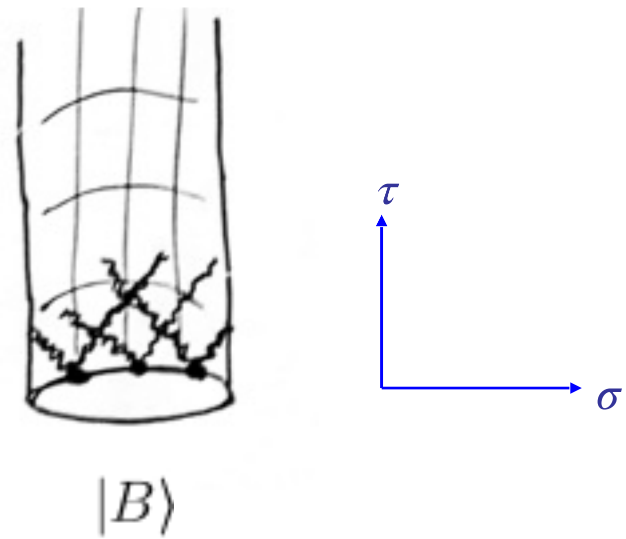
String states $\leftrightarrow$ Bethe eigenstates $\leftrightarrow$ operators in YM

$|\{u_j\}\rangle$ $\qquad$ $O_{\{u_j\}}(x)$

D-branes

- b.c.'s may (or may not) preserve integrability





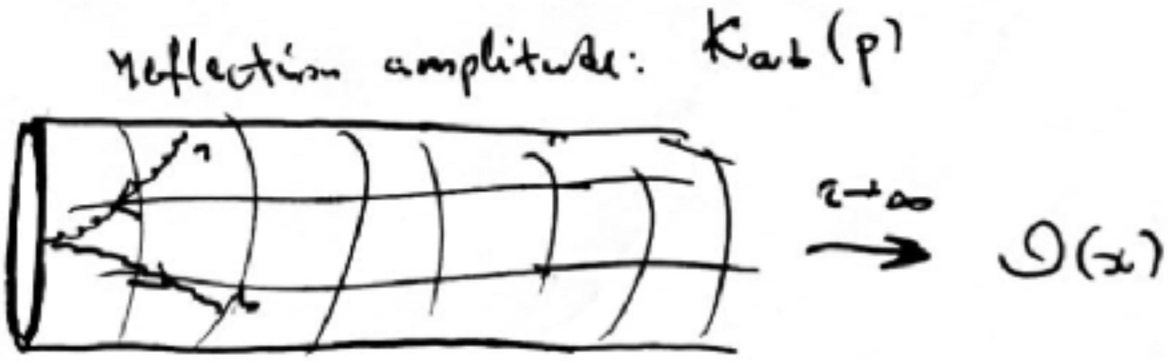
boundary cds

(open string)

boundary state

(closed string)

g-function



systematically
calculable
by TBA!

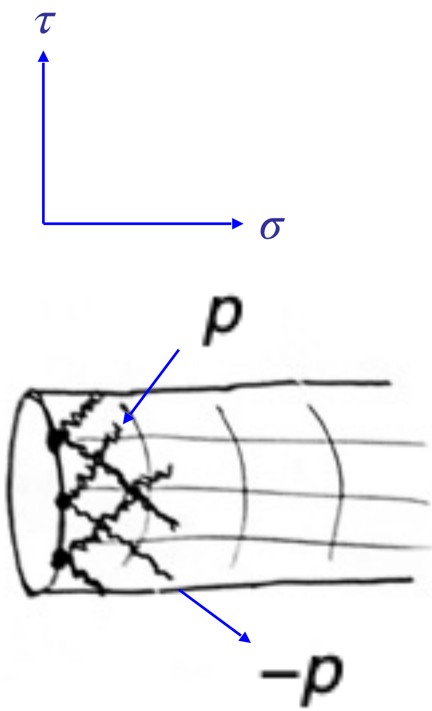
Integrable QFT on semi-infinite cylinder: the g-function

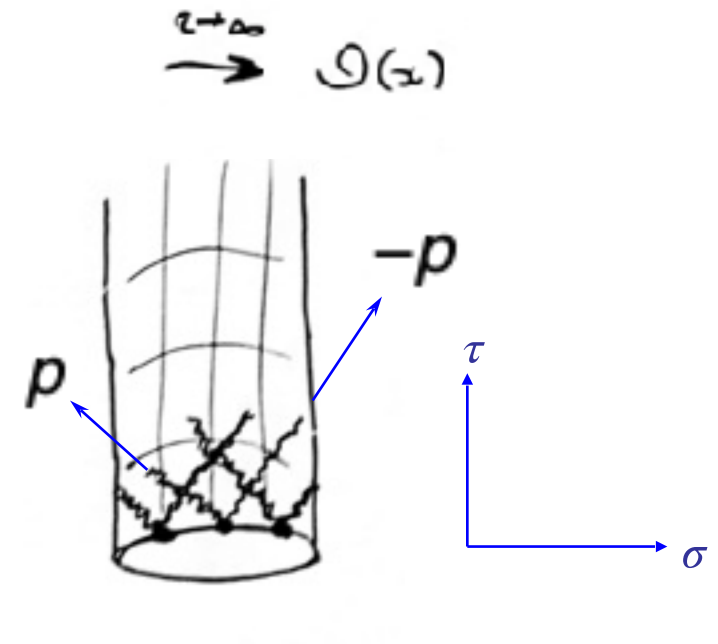
Affleck,Ludwig'91
Dorey,Fioravanti,Rim,Tateo'04
Pozsgay'10

In AdS/CFT:

$$\boxed{\text{g} - \text{function} = \langle O(x) \rangle}$$

Jiang,Komatsu,Vescovi'19
Komatsu,Wang'20





Boundary state is integrable if

$$|B\rangle = \sum C^{B}_{\{p_j,-p_j\}} \, |\{p_j,-p_j\}\rangle \qquad \Longleftrightarrow \qquad Q_{2n+1}\,|B\rangle = 0$$

Ghoshal, Zamolodchikov'93
Piroli, Pozsgay, Vernier'17

$$\langle \mathcal{O}(x) \rangle = \text{const} \, \frac{\langle B|\mathbf{p}\rangle}{\langle \mathbf{p}|\mathbf{p}\rangle^{\frac{1}{2}}}$$

AdS$_5$ sigma-model

AdS$_5$ = SO(4,2)/SO(4,1)

Coset representative: $g = \mathrm{e}^{iP_\mu x^\mu} z^D$

Moving-frame current:

so(4,1) → J_0; the rest → J_2

$$J = g^{-1} dg = J_0 + J_2$$

$$S = \int \mathrm{tr}\, J_2 \wedge * J_2 = \int \frac{dx^\mu \wedge * dx_\mu + dz \wedge * dz}{z^2}$$

Lax connection: moving frame

Lax current:

$$L(\mathrm{x}) = J + A(\mathrm{x})$$

$$A(\mathrm{x}) = 2\,\frac{J_2 - \mathrm{x} * J_2}{\mathrm{x}^2 - 1}$$

$$dL + L \wedge L = 0$$ $\Leftrightarrow$ Eqs of motion

Lax connection: fixed frame

Fixed frame (gauge inv. current):

$$j = g J_2 g^{-1}$$

$$j = \frac{1}{2z^2}\left[2(z\,dz + x\,dx)\;(D - xP) + \left(z^2 + x^2\right) P\,dx + K\,dx + L_{\mu\nu} x^{\mu} dx^{\nu}\right]$$

Gauge inv. Lax current:

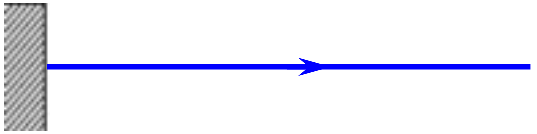
$$a(x) = 2\frac{j - x * j}{x^2 - 1} \qquad da + a \wedge a = 0$$

Integrable boundary conditions

Sklyanin'87

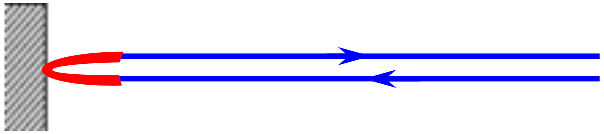
Monodromy matrix:

$$\mathcal{M}(\tau, \mathrm{x}) = g(0,\tau)\,\mathrm{P}\exp\int_0^\infty ds\, L_\sigma(s,\tau;\mathrm{x})$$



Double-row transfer matrix:

$$T(x) = M^{t}(-x)\,\mathbb{U}\,M(x)$$



reflection matrix

at string's endpoint

$$\frac{dT(x)}{d\boxtimes} = 0 \quad \Longleftrightarrow \quad a^t(-x)\,\mathbb{U} - \mathbb{U}\,a(x) = 0$$

charge-conjugation matrix: $\gamma_\mu^t = K^{-1}\gamma_\mu K$

Transposition bracket: $\langle A, B\rangle_\pm = K A^t K^{-1} B \pm B A$

$$\langle j_\boxtimes, \hat{\mathbb{U}}\rangle_+ = 0 \,, \qquad \langle j_\sigma, \hat{\mathbb{U}}\rangle_- = 0$$

$\hat{\mathbb{U}} = K\mathbb{U}$

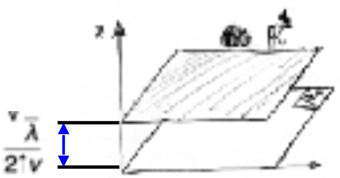
Solutions in $AdS_5 x S^5$ classified

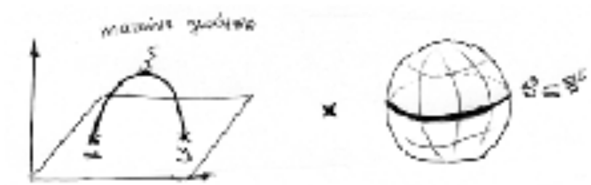
Dekel,Oz'11

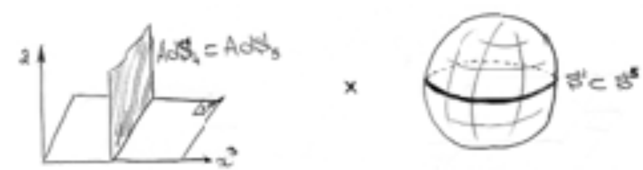
Integrable D-branes: examples

Dekel,Oz'11

Coulomb branch	Giant graviton	Domain wall
D3	D3	D5







Coulomb branch (D3):

$$\langle \Phi_i \rangle = \begin{bmatrix} v_i & \\ & \end{bmatrix}$$

$SU(N) \to SU(N-1) \times U(1)$

Giant graviton (D3):

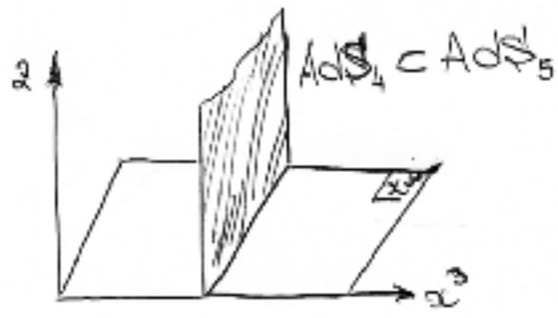
$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(y)\rangle$$

$$\mathcal{O} = \det Z$$

Jiang,Komatsu,Vescovi'19

Domain wall (D5): ✓

Domain wall



Boundary conditions:

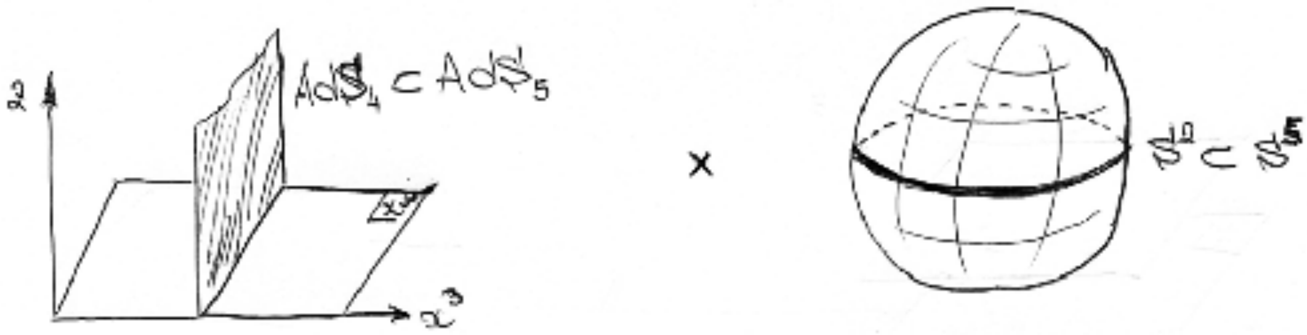
$x^3 = 0$ (Dirichlet)

$\acute{z} = 0, \quad \acute{x}^i = 0 \quad (i = 0, 1, 2)$ (Neumann)

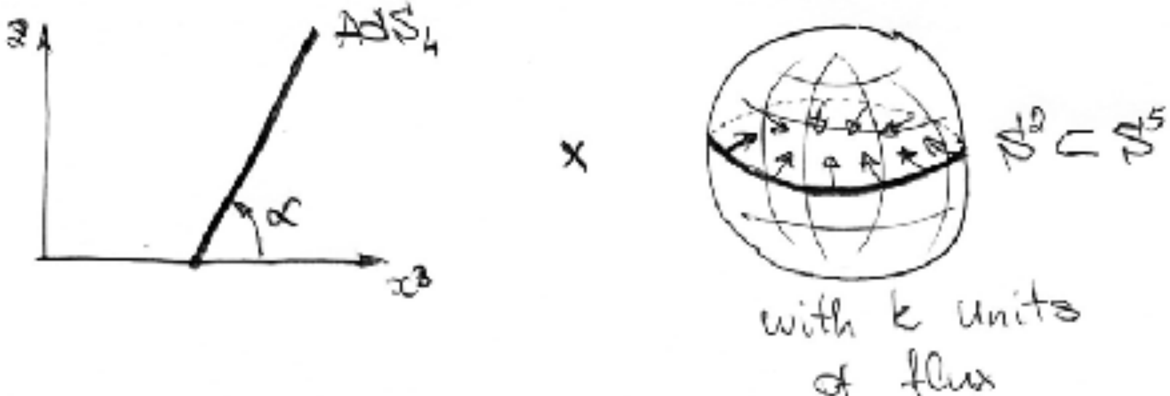
Refelction matrix:

$$\hat{\mathbb{U}} = n^\mu \gamma_\mu = \gamma_3$$

Karch-Randall brane



1-parameter family:



$$\tan \alpha = \sqrt{\frac{k}{\lambda}}$$

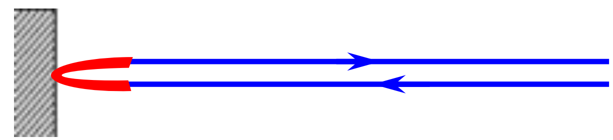
$$\oint_{S^2} F = k$$

Karch,Randall'01

Dynamical reflection matrices

Corrigan,Sheng'96
Aniceto,Bajnok,Gombor,Kim,Palla'17
Linardopoulos,Z.'21

$$T(x) = M^{t}(-x)\,\mathbb{U}(x^{m}(0,\tau);x)\,M(x)$$

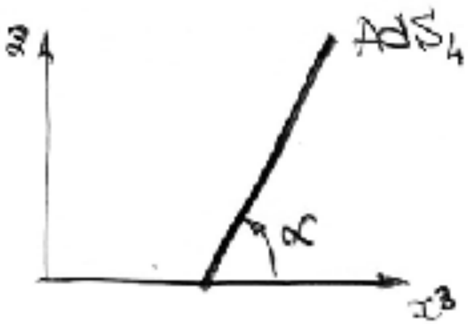


Integrability condition:

$$\frac{dT(x)}{d\tau} = 0$$

$$\dot{\mathbb{U}} = \frac{2}{x^2-1}\Big[\,(j_\tau^t\,\mathbb{U} + \mathbb{U}\,j_\tau) + x\,(j_\sigma^t\,\mathbb{U} - \mathbb{U}\,j_\sigma)\,\Big]$$

Reflection off Karch-Randall brane



$$\hat{\mathbb{U}} = \gamma_3 + \frac{2\cot \hookleftarrow}{x^2+1}\,\frac{x^\mu\gamma_\mu - \Uparrow_+ - (x^2+z^2)\Uparrow_-}{z}$$

in the defining rep of SU(2,2)

Linardopoulos,Z.'21

dynamical reflection

⇕

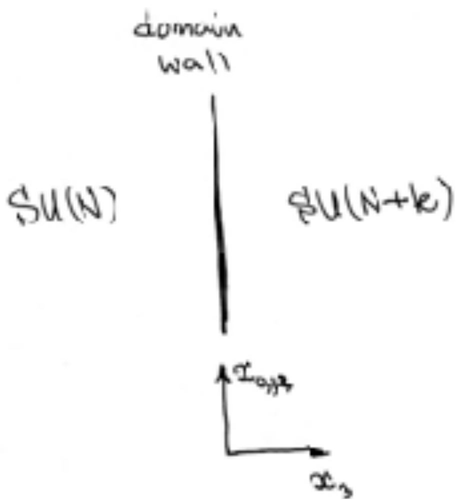
∞ many bound states in quantum theory

$$\Uparrow_\pm = \frac{1\pm\gamma_4}{2}$$

Komatsu,Wang'20

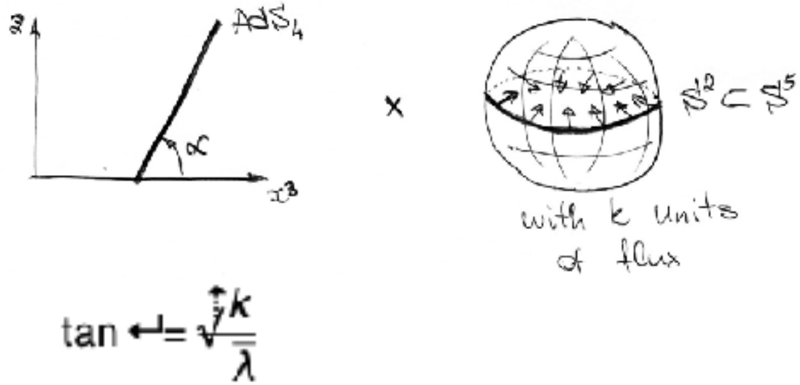
AdS/dCFT

defect CFT



⟺

Karch-Randal brane



$$\oint_{S^2} F = k$$

Nahm's equations

Energy of static domain wall:

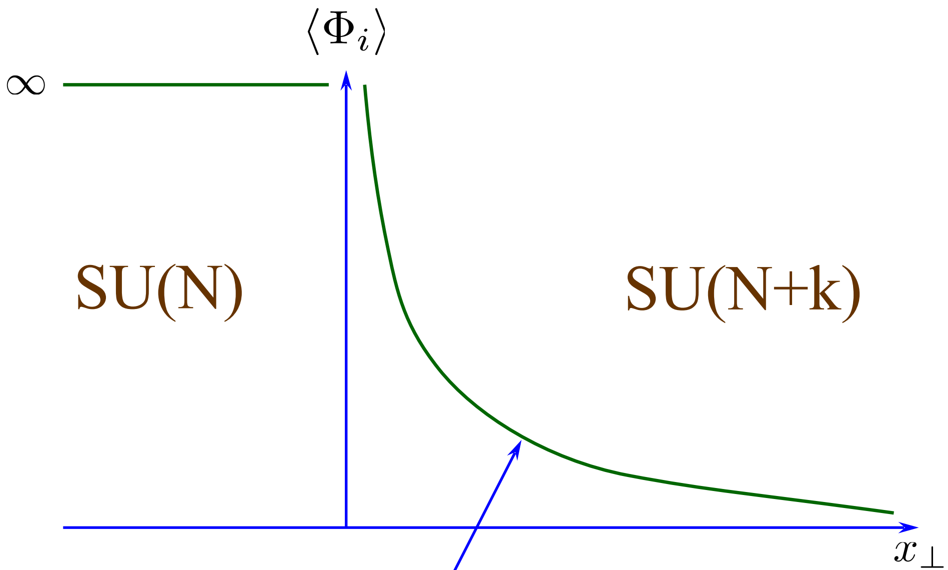
$$E = \int dx \, \mathrm{tr} \left\{ \left(\frac{d\Phi_i}{dx} \right)^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 \right\}$$

$$= \int dx \, \mathrm{tr} \left(\frac{d\Phi_i}{dx} - \frac{i}{2} ''_{ijk} [\Phi_j, \Phi_k] \right)^2 + \mathrm{tot.d.}$$

Nahm's equations:

$$\frac{d\Phi_i}{dx} = \frac{i}{2} ''_{ijk} [\Phi_j, \Phi_k] \quad \Longrightarrow \quad \frac{d^2\Phi_i}{dx^2} = 2[\Phi_j, [\Phi_j, \Phi_i]]$$

Domain walls in N=4 SYM



$$\Phi_i^{\rm cl} = \frac{1}{x_\perp} \begin{pmatrix} t_i & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(columns: k, N; rows: k, N)}$$

Constable,Myers,Tafjord'99

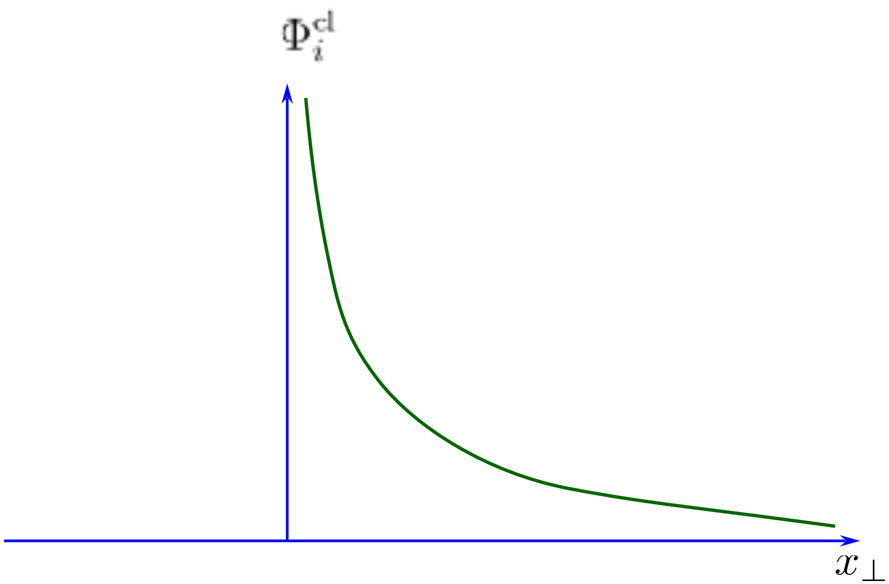
$$[t_i, t_j] = i\varepsilon_{ijk} t_k$$ ☛ k-dim. rep. of SU(2)

$m^2_{\rm Higgs} \sim \frac{1}{x_\perp^2}$ massive modes stay on one side

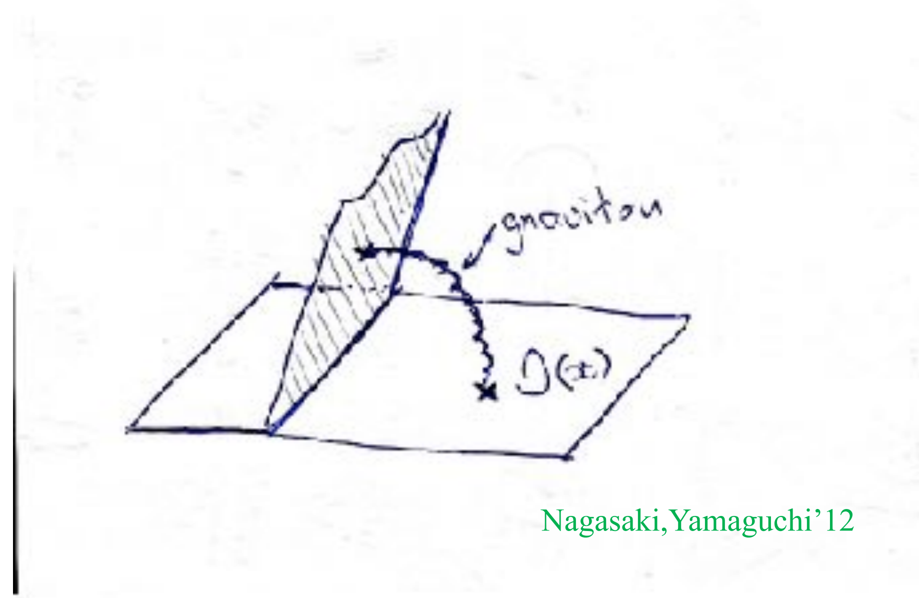
Buhl-Mortensen,de Leeuw,Ipsen,Kristjansen,Wilhelm'16

1pt functions

Weak coupling:



Strong coupling:

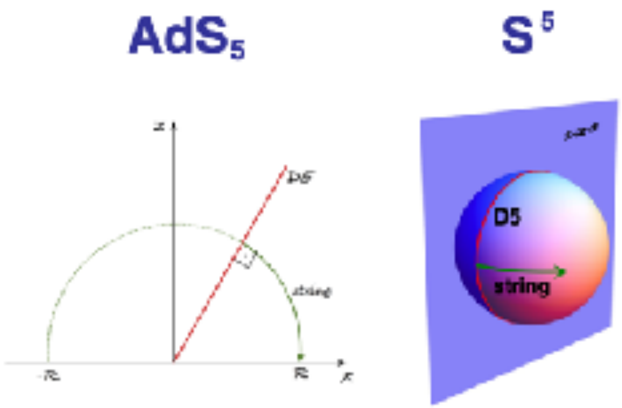


Nagasaki,Yamaguchi'12

$$\langle O(x)\rangle = {}^{i_1\ldots i_l}\ \mathrm{tr}\,\Phi^{\mathrm{cl}}_{i_1}\ldots\Phi^{\mathrm{cl}}_{i_L} + \text{quant. corr.}$$

Geodesic approximation

Buhl-Mortensen,de Leeuw,Kristjansen,Zarembo’15



Method of images:

$$
\begin{aligned}
' &= i\,!\,⊠ \\
x &= R \tanh !\,(⊠ + ⊠)\,, \\
z &= \frac{R}{\cosh !\,(⊠ + ⊠)}\,.
\end{aligned}
$$

$$! = \sqrt{\frac{L}{\lambda}} \qquad \sinh !\,⊠ = \sqrt{\frac{k}{\lambda}} \equiv \kappa$$

Evaluating string action:

$$\left\langle \operatorname{tr} Z^L \right\rangle = \left(\frac{\kappa + \sqrt{\kappa^2+1}}{\sqrt{2}}\right)^L \frac{1}{R^L} \stackrel{\kappa\to\infty}{\simeq} \left(\frac{\sqrt{2}\,\kappa}{R}\right)^L$$

Dobashi,Shimada,Yoneya’02

Tsuji’06

Janik,Surowka,Wereszczynski’10

Strong-coupling prediction:

$$\left\langle \mathrm{tr}\, Z^L \right\rangle = \left(\frac{\kappa + \sqrt{\kappa^2+1}}{\sqrt{2}} \right)^L \frac{1}{R^L} \overset{\kappa\to\infty}{\simeq} \left(\frac{\sqrt{2}\,\kappa}{R} \right)^L$$

agree

$$\kappa = \frac{\pi k}{\sqrt{\lambda}}$$

Weak coupling:

$$\left\langle \mathrm{tr}\, Z^L \right\rangle = \frac{1}{R^L} \left(\frac{8\pi^2}{\lambda} \right)^{\frac{L}{2}} \sum_{a=-\frac{k-1}{2}}^{\frac{k-1}{2}} a^L \overset{k\to\infty}{\simeq} \left(\frac{\sqrt{2}\pi k}{\sqrt{\lambda}\, R} \right)^L$$

Exact 1pt function for protected ops

Komatsu,Wang'20

From localization:

$$\langle \mathrm{tr}\, Z^L \rangle = \frac{2^{-L} L^{-\frac{1}{2}}}{x_{\perp}^{L}} \left[\left(\frac{16\pi^2}{\lambda} \right)^{\frac{L}{2}} \sum_{a=-\frac{k-1}{2}}^{\frac{k-1}{2}} x_a^L - k\delta_{L,2} - \left(\frac{\lambda}{16\pi^2} \right)^{\frac{L}{2}} \sum_{b \in \mathbb{Z} + \frac{k-1}{2}}^{\prime} \frac{1}{x_b^L} \right]$$

$$2x_a = a + \sqrt{a^2 + \frac{\lambda}{4\pi^2}}$$ Zhukovsky variable

$\langle B|0\rangle$ is not protected and receives quantum corrections!

Matrix product state

$$\mathcal{O} = \Psi^{i_1 \ldots i_L} \, \mathrm{tr}\, \Phi_{i_1} \ldots \Phi_{i_L}$$

$$\Phi_i \to \langle \Phi_i \rangle = \frac{t_i}{x_\perp}$$

$$\langle O(x) \rangle = \frac{1}{x_\perp^L} \quad {}^{i_1 \ldots i_L} \, \mathrm{tr}\, t_{i_1} \ldots t_{i_L}$$

$$\langle \mathcal{O}(x) \rangle = \frac{1}{x_\perp^L} \left(\frac{2\pi}{\sqrt{\lambda}} \right)^L \frac{1}{L^{\frac{1}{2}}} \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}$$

Matrix product state: $\mathrm{MPS}_{i_1 \ldots i_L} = \mathrm{tr}\, t_{i_1} \ldots t_{i_L}$

de Leeuw, Kristjansen, Z.’15

an integrable boundary state!

Overlap formulae

$$\frac{\langle B|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle^{\frac{1}{2}}} = \sum_{\alpha=1}^{d_B} \sqrt{\prod_j f_{B,\alpha}(u_j)\ \mathrm{Sdet}\, G}$$

$G_{jk} = \frac{\partial}{\partial u_k} \ln \mathrm{BAE}_j$ - Gaudin matrix

d_B - dim of twisted Yangian rep

$f_{B,\alpha}(u)$ - eigenvalues of double-row transfer matrix

de Leeuw, Gombor, Kristjansen, Linardopoulos, Pozsgay'19

Paired states

Z_2 invariance (consequence of integrability):

$$\langle B|\{u_j\}\rangle \neq 0 \qquad \Longleftrightarrow \qquad \{-u_j\} = \{u_j\}$$

$$|\mathbf{u}\rangle = \left|\{u_j, -u_j\}_{j=1\dots\frac{M}{2}}\right\rangle$$

Gaudin superdeterminant

$$e^{i\chi_j} \equiv \left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^L \prod_k \frac{u_j - u_k + i}{u_j - u_k - i} = -1 \qquad G_{jk} = \frac{\partial \chi_j}{\partial u_k}$$

For a paired state:

$$G = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad \begin{matrix} \text{columns: } \{u_j\}\ \{-u_j\} \\ \text{rows: } \{u_j\},\ \{-u_j\} \end{matrix}$$

Z_2 parity ($u_j \leftrightarrow -u_j$):

$$\boxtimes = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$$

Superdeterminant:

$$\text{Sdet}\, G = e^{\mathrm{tr}\, \boxtimes \ln G}$$

Explicit form:

$$\mathrm{Sdet}\, G = \frac{\det G^+}{\det G^-}$$

$$G^{\pm}_{jk} = \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_n K^+_{jn}\right)\delta_{jk} + K^{\pm}_{jk}$$

$$K^{\pm}_{jk} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}$$

$$\frac{\langle \mathrm{MPS} | \mathrm{u} \rangle}{\langle \mathrm{u} | \mathrm{u} \rangle^{\frac{1}{2}}} = 2\sqrt{\frac{Q\left(\frac{i}{2}\right)}{Q(0)}\, \mathrm{SdetG}}$$

Brockmann, DeNardis, Wouters, Caux'14

$$Q(u) = \prod_{j=1}^{M} (u - u_j)$$

Higher representations

$$\mathrm{MPS}_{i_1 \dots i_L} = \mathrm{tr}\, t_{i_1} \dots t_{i_L}$$

k-dim. rep. of su(2)

$$\langle \mathrm{MPS}_k \, \mathrm{u} \rangle = S_k \, Q\left(\frac{ik}{2}\right) \sqrt{Q\left(\frac{i}{2}\right) Q(0) \ \mathrm{Sdet}\, G}$$

$$S_k = \sum_{a=-\frac{k-1}{2}}^{\frac{k-1}{2}} \frac{a^L}{Q\left(\frac{2a+1}{2}\, i\right) Q\left(\frac{2a-1}{2}\, i\right)}$$

Buhl-Mortensen, de Leeuw, Kristjansen, Z.'15

related to transfer matrix eigenvalue

Beyond SU(2)

$$\left(\frac{u_{aj}-\frac{iq_a}{2}}{u_{aj}+\frac{iq_a}{2}}\right)^L \prod_{bk} \frac{u_{aj}-u_{bk}+\frac{iM_{ab}}{2}}{u_{aj}-u_{bk}-\frac{iM_{ab}}{2}} = (-1)^{F_a+1}$$

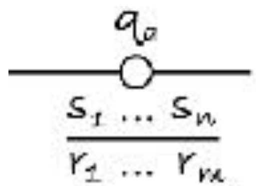
$$\frac{\langle B|\mathrm{u}\rangle}{\langle \mathrm{u}|\mathrm{u}\rangle^{\frac{1}{2}}} = \sqrt{\prod_a \frac{\prod_{i=1}^{n_a} Q_a\left(\frac{is_{ai}}{2}\right)}{\prod_{j=1}^{m_a} Q_a\left(\frac{ir_{aj}}{2}\right)} \, \mathrm{Sdet}\, G}$$

$$\mathrm{Sdet}\, G = \frac{\det G^+}{\det G^-}$$

$$G^{\pm}_{aj,bk} = \left(\frac{Lq_a}{u_{aj}^2+\frac{q_a^2}{4}} - \sum_{cl} K^+_{aj,cl} - \frac{1}{2}\sum_{\hookleftarrow} K^+_{aj,a\leftarrow 0}\right)\delta_{ab}\delta_{jk} + K^{\pm}_{aj,bk}$$

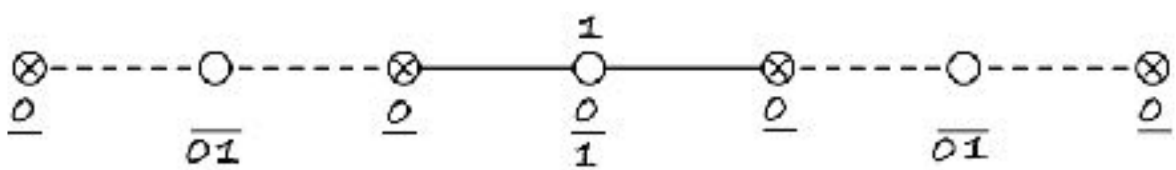
$$K^{\pm}_{aj,bk} = \frac{M_{ab}}{(u_{aj}-u_{bk})^2+\frac{M_{ab}^2}{4}} \pm \frac{M_{ab}}{(u_{aj}+u_{bk})^2+\frac{M_{ab}^2}{4}}$$

Graphic notations:



$:\quad \dfrac{Q_a(is_1/2)\ \dots\ Q_a(is_n/2)}{Q_a(ir_1/2)\ \dots\ Q_a(ir_m/2)}$

Ex:

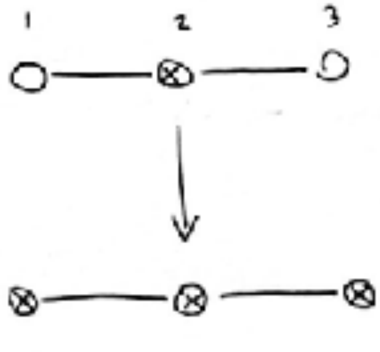


describes 1pt function in D3-D5 dCFT:

$$\frac{\langle \mathrm{D3D5}|u\rangle}{\langle u|u\rangle^{\frac{1}{2}}} = \sqrt{\frac{Q_1(0)\,Q_3(0)\,Q_4(0)\,Q_5(0)\,Q_7(0)}{Q_2(0)\,Q_2\left(\frac{i}{2}\right)Q_4\left(\frac{i}{2}\right)Q_6(0)\,Q_6\left(\frac{i}{2}\right)}\ \mathrm{Sdet}\, G}$$

Gombor,Bajnok'20

Fermionic duality

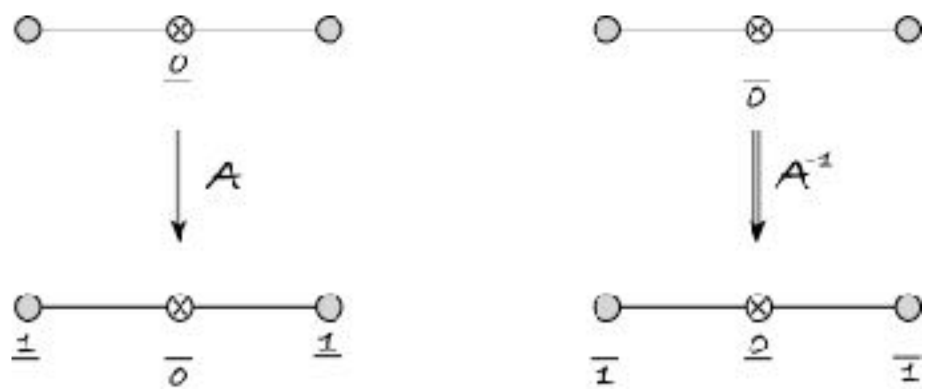


$$Q_1^+ Q_3^- - Q_1^- Q_3^+ = i(M_1 - M_3)\, \tilde{Q}_2 Q_2$$

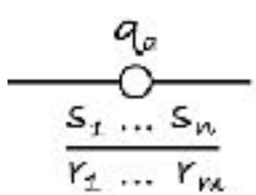
$$\mathrm{Sdet}\, \widetilde{G} = \text{const}\, \frac{Q_2(0)\widetilde{Q}_2(0)}{Q_1\left(\frac{i}{2}\right) Q\left(\frac{i}{2}\right)}\, \mathrm{Sdet}\, G$$

Kristjansen,Müller,Z.'20

Transformation rules for overlap formulas

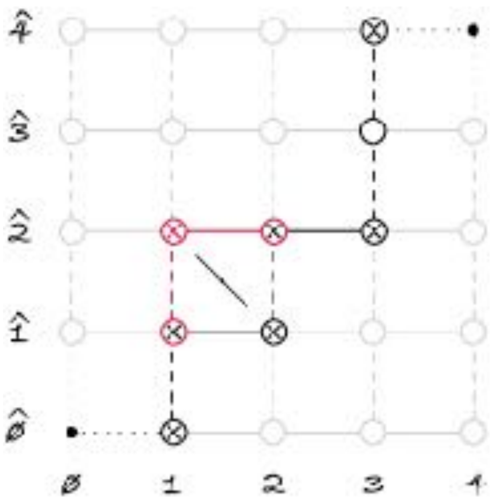


In the graphic notations:

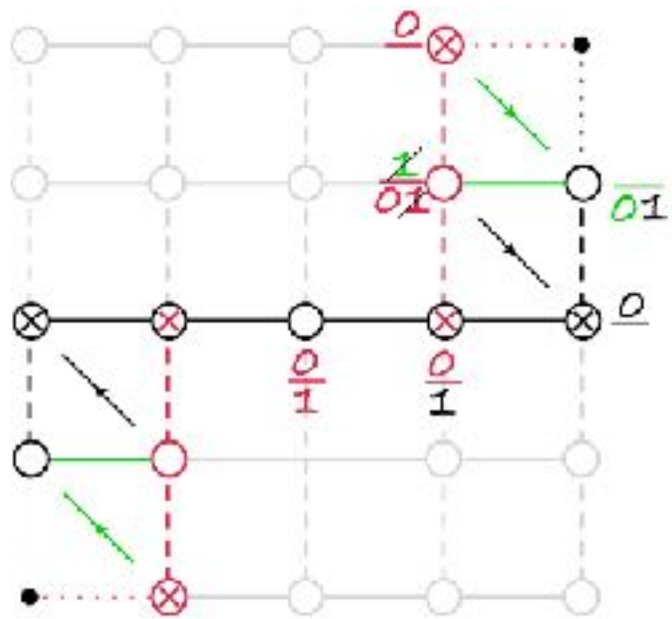


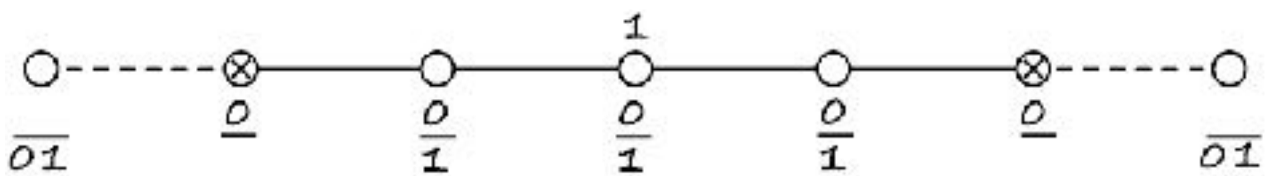
$$:\quad \frac{Q_a(is_1/2)\dots Q_a(is_n/2)}{Q_a(ir_1/2)\dots Q_a(ir_m/2)}$$

Fermionic duality for PSU(2,2|4):

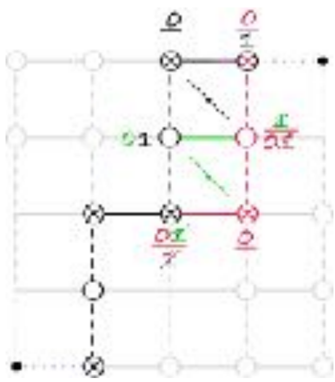


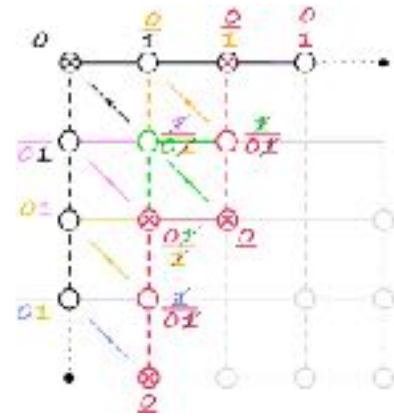
Ex: from alternating to Beauty:

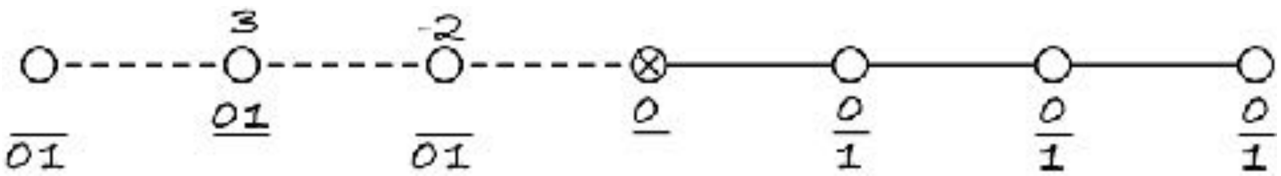




from Beauty to the Beast:







useful for gluonic operators:

$$\mathrm{tr}\, F^{+}_{\mu_1\nu_1}\ldots F^{+}_{\mu_L\nu_L}$$

Kristjansen,Müller,Z.’20

Gluon sector

$$f_i = F_{\mu\nu}\eta_i^{\mu\nu}$$

’t Hooft symbol

$$\mathcal{O} = \Psi^{i_1 \dots i_L} \operatorname{tr} f_{i_1} \dots f_{i_L}$$

Ferretti,Heise,Z.’04
Beisert, Ferretti,Heise,Z.’04

Mixing matrix:

permutation trace

$$H = \sum_{l=1}^{L} (1 - P_{l,l+1} + 2K_{l,l+1})$$

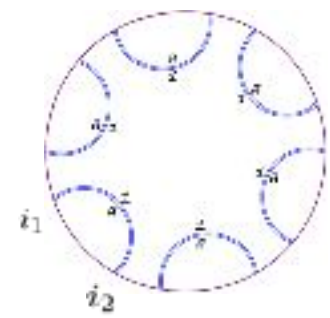
- SU(2) spin-1 (Zamolodchikov-Fateev) model

Zamolodchikov,Fateev’80
Kulish,Reshetikhin,Sklyanin’81
Reshetikhin’85

Overlap from dualities:

$$\frac{\langle B|\mathbf{u}\rangle}{\langle \mathbf{u}|\mathbf{u}\rangle} = \sqrt{\frac{1}{Q(0)Q(\frac{i}{2})}\,\frac{\det G^{+}}{\det G^{-}}}$$

Agrees with explicit calculation, which gives:



$$B_{i_1 \ldots i_L} = \delta_{i_1 i_2} \ldots \delta_{i_{L-1} i_L}$$

Same formula valid for any spin and $\langle B| = (\mathrm{Proj}_{\mathrm{singlet}})^{\otimes \frac{L}{2}}$

$$G^{\pm}_{jk} = K^{\pm}_{jk} + \delta_{jk}\left(\frac{2SL}{u_j^2 + S^2} - \sum_{l} K^{+}_{jl}\right)$$

$$K^{\pm}_{jk} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}$$

Questions

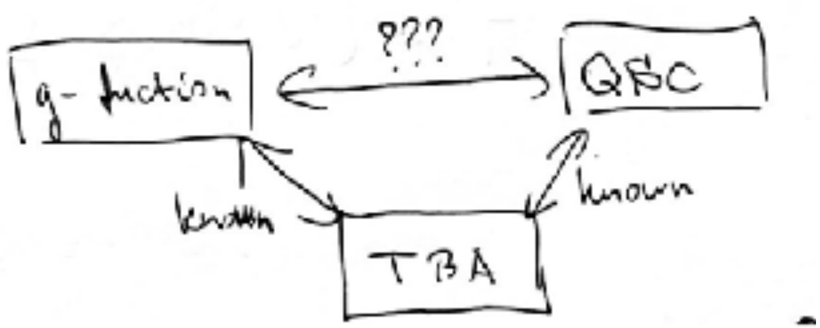
- Exact g-function of D3-D5 dCFT?

  can be tested using localization result for <O_{CPO}>

  Komatsu,Wang'20

- Relation to Quantum Spectral Curve?

  Caetano,Komatsu'20



- 1pt functions for other integrable D-branes?
  E.g. on Coulomb branch