Flags, Fusion, [ODE/IM] and Bethe Algebra

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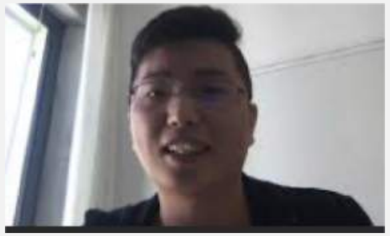
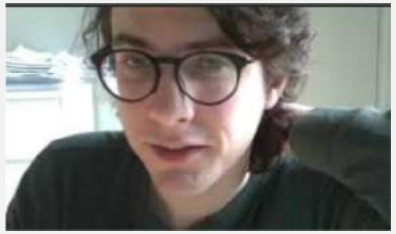
talk at IPhT, 14/09/21

Paul Ryan



Hongfei Shu



Simon Ekhammar

2002.12341 2008.10597 2104.04539, 21xx.xxxxx

GL(M|N) Q-systems in spin chains

[Tsuboi'10]

...Tsuboi in Paris...

....

Fusion of Grassmannians for GL(M|N) Q-systems

[Kazakov,Leurent,D.V'15]

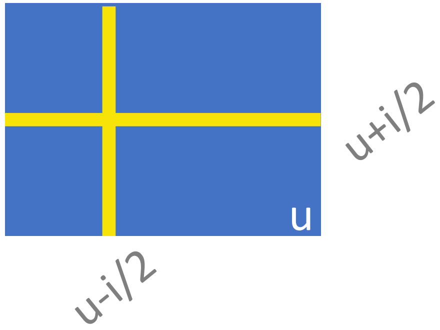
QQ-relations (and more) for simple Lie algebra in ODE/IM

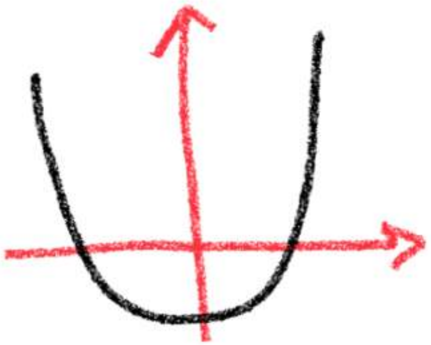
[Sun'12]

[Masoero, Raimondo,Valeri`15]

...Hongfei Shu in Stockholm...

Fused Flags



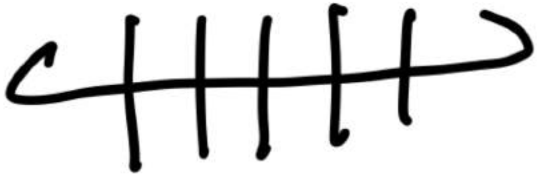
$y = x^2 - 1$ 

$\mathbb{C}[x,y]/(y-x^2+1)$

Bethe equations

$$\ldots = \prod \frac{u-u'+i}{u-u'-i}$$

Bethe algebra

 $=$

$= T(u),\ Q(u),\ \ldots$

Separation of Variables

$$\langle 0| \, T^1(\theta_1)\, T^0(\theta_2) \dots T^1(\theta_L) = \langle 10 \dots 1|$$

1,0 - powers

[Maillet, Niccoli'18]

$$\langle 100 \dots 1|\Psi\rangle = \prod_{\ell=1}^{L} \tau^{\delta_\ell}(\theta_\ell)$$

Bonus:

$$T(u) = \frac{\pi(u-\theta_\ell)\, Q(u-i) + \pi(u-\theta_\ell+i)\, Q(u+i)}{Q(u)}$$

$$\langle 100 \dots 1|\Psi\rangle = \prod_{\ell=1}^{L} \tau^{\delta_\ell}(\theta_\ell) = \# \prod_{\ell=1}^{L} Q(x_\ell) \qquad x_\ell = \theta_\ell + i\delta_\ell$$

Mathematical perspective:

- $\mathcal{B}$ – Bethe algebra. It is a maximal commutative algebra acting on spin chain. Spin chain is its <u>regular</u> representation.

$$b_a \cdot b_b = \sum f_{ab}^c \, b_c \, ;$$

$$\langle 0 | b_a = \langle b_a |$$

$$\langle b_a | b_b = \sum f_{ab}^c \langle b_c |$$

- Spin chain, as a vector space, is identified with $\mathcal{B}$

- Finding good basis in spin chain $\equiv$ Finding good basis in $\mathcal{B}$

Separation of Variables

$$\langle 0|\, T^1(\theta_1)\, T^0(\theta_2) \ldots T^1(\theta_L) = \langle 10\ldots 1|$$

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$$\langle 100\ldots 1|\Psi\rangle = \prod_{\ell=1}^{L} \tau^{\delta_\ell}(\theta_\ell) = \# \prod_{\ell=1}^{L} Q(x_\ell) \qquad x_\ell = \theta_\ell + i\delta_\ell$$

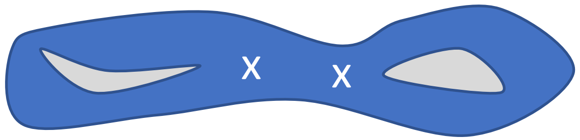
Good basis for GL(N) spin chains

$$\langle 100..1|\Psi\rangle = \prod_{\ell=1}^{L} \tau^{\delta_\ell}(\theta_\ell) = \# \prod_{\ell=1}^{L} Q(x_\ell) \qquad x_\ell = \theta_\ell + i\delta_\ell$$

$$\boldsymbol{\Psi(x) = \prod_{\alpha=1}^{L} \prod_{k=1}^{N-A} \det\ Q_i(x_{kj}^{\alpha})}$$ [Ryan,D.V.'18-20]

Why good?

- Diagonalises B_{good} who is a quantisation of classical B controlling dynamical divisor of classical spectral curve



$$\det(\lambda - M(u)) = 0$$

- $N = 2$: B of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ **[Sklyanin'89]**
- $N = 3$: **[Sklyanin'92]**
- **Classical B, any N: [Scott'94], [Gekhtman'95]**
- quantum B any N: **[Smirnov'01]**

[Gromov, Levkovich-Maslyuk, Sizov '16]

- Q's solve Baxter equation (quantum spectral curve)
- Can do functional SoV

$$\boldsymbol{\det\left(1 + M(u) e^{-\hbar\partial_u}\right) Q_i = 0}$$

(papers by subsets of **{Cavaglia,Gromov, Levkovich-Maslyuk,Ryan,D.V} 17-21**)

$\mathfrak{g} \longrightarrow Y(\mathfrak{g}) \longrightarrow$  $\ni$ 

$= T_\pi(u)$

rational
spin chain

$[T_\pi(u), T_{\pi'}(u')] = 0 \to$ form Bethe algebra

$\mathfrak{g} \longrightarrow Y(\mathfrak{g}) \longrightarrow$  $\ni$  $= T_\pi(u)$

rational spin chain

$[T_\pi(u), T_{\pi'}(u')] = 0 \rightarrow$ form **Bethe algebra**

Question of this talk:

What is a **good** description of Bethe algebra if symmetry is not GL(N)?

- No quantisation of spectral curve
- No B_{good}
- No Baxter equation
- No functional SoV

$\mathfrak{g} \longrightarrow Y(\mathfrak{g}) \longrightarrow$  $\ni$  $= T_\pi(u)$

rational spin chain

$[T_\pi(u), T_{\pi'}(u')] = 0 \rightarrow$ form **Bethe algebra**

Question of this talk:

What is a **GREEN** description of Bethe algebra if symmetry is not GL(N)?

- No quantisation of spectral curve
- No B_{good}
- No Baxter equation
- No functional SoV

Disclaimer:

- $\mathfrak{g} = \mathfrak{sl}_n$ - answer was known before
- $\mathfrak{g}$ – simply laced - formulas are correct
- $\mathfrak{g}$ – non-simply laced (= related to twisted affine) - formulas are morally correct, but we omit important technical details
- We won't do supersymmetry

The answer is a universal algebraic structure

- XXX (Yangian, twisted Yangian)
- XXZ (quantum [twisted] affine)
- Elliptic (?)
- AdS/CFT
- TBA
- ODE/IM

…..

- q-characters
- [finite difference] opers
- [equivariant quantum] cohomology rings
- Bethe/gauge

…..

- 4dCS

``Geometric and Representation-Theoretic Aspects of Quantum Integrability”

August 29-October 21, 2022 @Simons Center

[organisers: P.Koroteev, E.Pomoni, B.Vicedo, D.Volin, A.Zeitlin]

$$\mathfrak{g} = \mathfrak{sl}_N \; ; \; N = r+1$$

$$\text{source} = -\prod_{b=1}^{r} \prod_{j=1}^{M_b} \frac{u_{a,i} - u_{b,j} + \frac{\hbar}{2} C_{a,b}}{u_{a,i} - u_{b,j} - \frac{\hbar}{2} C_{a,b}}$$

$$Q_a = \text{source} \times \underbrace{\prod_{i=1}^{M_a} (u - u_{a,i})}_{q_a}$$

$$-1 = \prod_{b=1}^{r} \frac{Q_b^{[+C_{a,b}]}}{Q_b^{[-C_{a,b}]}} \Bigg|_{u = u_{a,i}}$$

Notation conventions:

$$f^{\pm}(u) := f(u \pm \tfrac{\hbar}{2}) \; ; \; f^{[n]}(u) := f(u + n\tfrac{\hbar}{2})$$

$$\mathfrak{g} = \mathfrak{sl}_N \; ; \; N = r+1$$

Baxter equation

$$\left(\sum_{a=0}^{N} (-1)^a \, T_{a}(u) \, e^{-a\hbar \partial_u} \right) Q^i = 0$$

$$\Downarrow$$

$$W(Q^1, \dots, Q^N) = 1$$

Wronskian Bethe equations

$$Q^i = \text{source} \times q^i$$

$$\text{source} = -\prod_{b=1}^{r} \prod_{j=1}^{M_b} \frac{u_{a,i} - u_{b,j} + \frac{\hbar}{2} C_{a,b}}{u_{a,i} - u_{b,j} - \frac{\hbar}{2} C_{a,b}}$$

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$$-1 = \prod_{b=1}^{r} \left. \frac{Q_b^{[+C_{a,b}]}}{Q_b^{[-C_{a,b}]}} \right|_{u = u_{a,i}}$$

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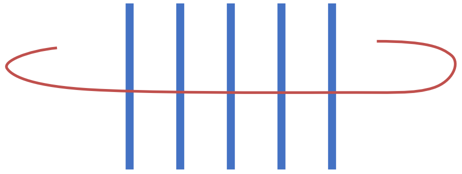
Vector (defining) representation of sl(N) $\rightarrow$ $\begin{pmatrix} Q^1 \\ \vdots \\ Q^N \end{pmatrix}$

$$Q_a = W(Q^1, Q^2, \dots, Q^a)$$

$$\left(\sum_{a=0}^{N} (-1)^a\, T_{[1^a]}(u)\, e^{-a i \partial_u}\right) Q^i = 0$$

sl(N) here is the symmetry (Galois group) of Baxter equation



! sl(N) is not the symmetry of spin chain !

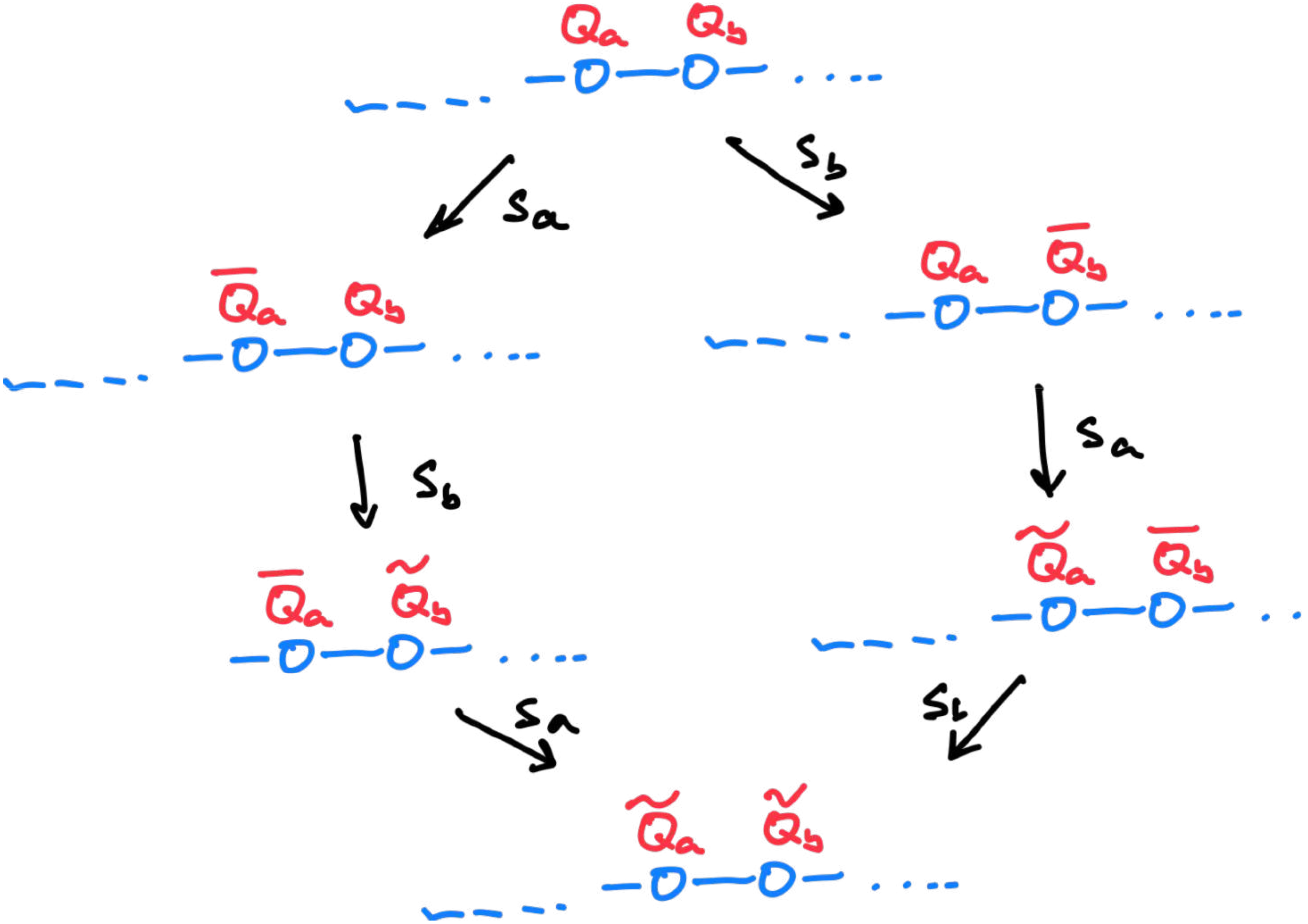
$$W(Q_a, \bar{Q}_a) = \prod_{b \sim a} Q_b$$

$$\frac{Q_a^{[2]}}{Q_a^{[-2]}} \prod_{b \sim a} \frac{Q_b^+}{Q_b^-} \Bigg|_{\text{zeros of } Q_a} = -1$$

$\downarrow$

Beyond equator (Pronko Stroganov)
Bosonic duality (Gromov Vieira) $\Big\}$ su(2)
Reproduction (Mukhin et al) – any g

method of constant variation (Lagrange, Euler)



Q_a Q_b
s_a
s_b
$\bar{Q}_a$ Q_b
Q_a $\bar{Q}_b$
s_b
s_a
$\bar{Q}_a$ $\tilde{Q}_b$
$\tilde{Q}_a$ $\bar{Q}_b$
s_a
s_b
$\tilde{Q}_a$ $\check{Q}_b$

$$s_a^2 = 1$$

$$(s_a s_b)^3 = 1$$

Group of duality transformations:

$$S_a^2 = 1$$

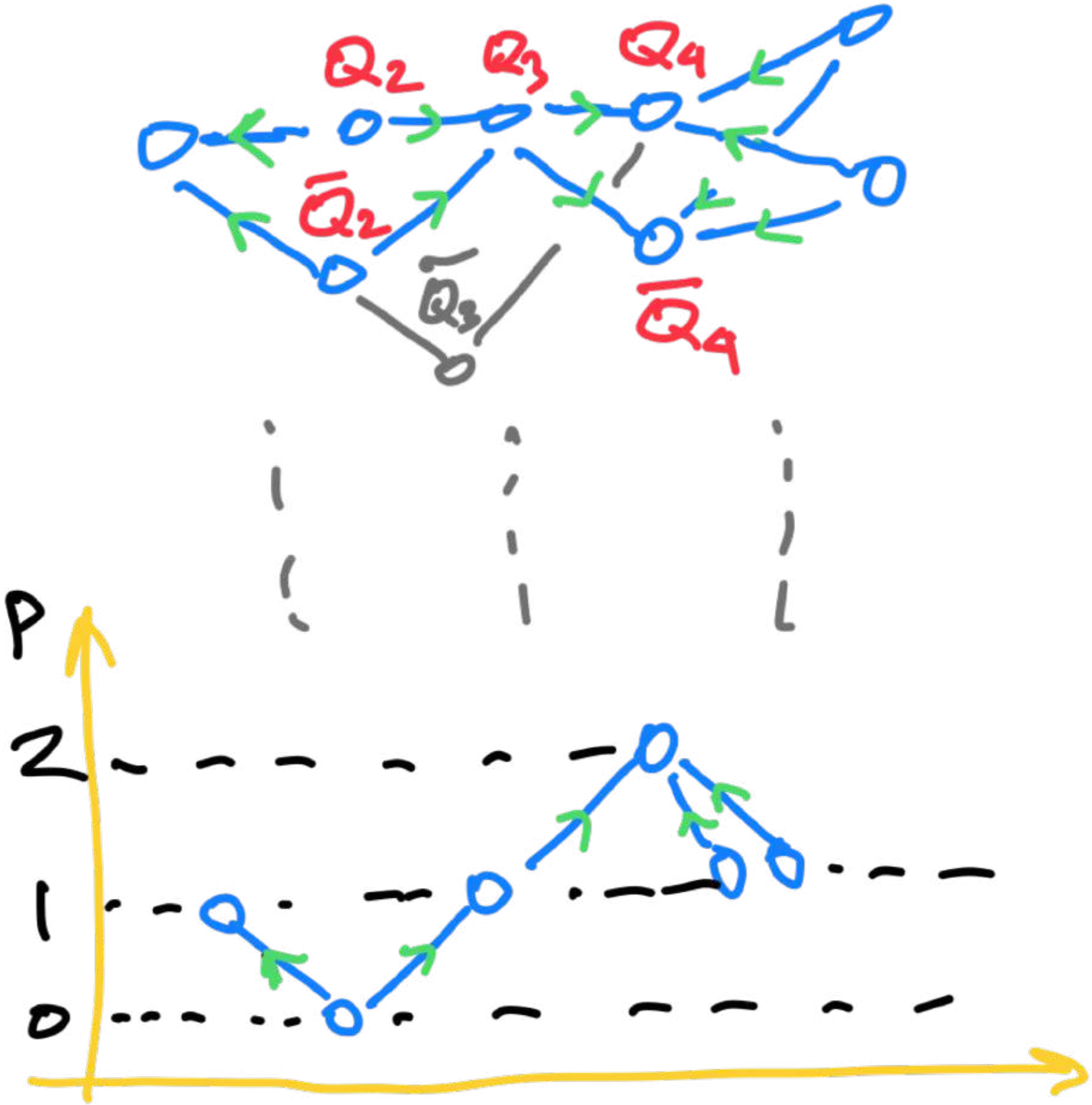
$$(S_a S_b)^3 = 1, \quad a \sim b$$

$$(S_a S_b)^2 = 1, \quad a \not\sim b$$

It is Weyl group of your algebra?

tuple $(Q_1, \dots, Q_r)$ transforms forming an orbit under Weyl group action.

Hasse diagrams



Hasse diagram

$\updownarrow$

oriented graph
without loops

Orientations

$\updownarrow$

Ways to do
nested Bethe ansatz

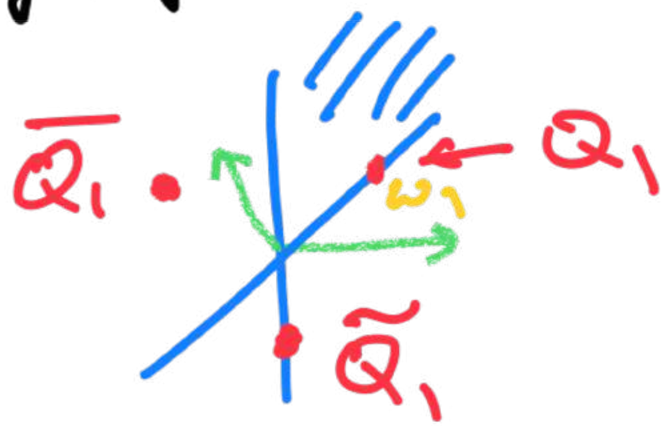
$\updownarrow$

choice of Coxeter
element

Q_a transforming under Weyl group $\longleftrightarrow$ highest weight vector in a'th fundamental representation transforming under Weyl group

$Q_a \sim u^{M_a}$ $\qquad$ $Q_a^{(\sigma)} \sim u^{M_a^{(\sigma)}}$

SU(3):



$$M_a = (\omega_a, \lambda_{max} - \lambda)$$

$$M_a^{(\sigma)} = (\omega_a, \lambda_{max} + \rho) - (\omega_a^{(\sigma)}, \lambda + \rho)$$

$Q^{\sigma(1)}_{(a)}$ ← Weyl orbits

← highest weight components of

$Q^{\sigma(1)}_{(a)} \longleftrightarrow \sigma \cdot \omega_a$

$Q^{1}_{(a)} = Q_a$

Issue: solution of QQ not unique, can take linear combinations

$$W(Q_a, \bar{Q}_a) = \prod_{b \sim a} Q_b$$

$$0 = p^2 + 2(V-E) = \left(p + \sqrt{2(E-V)}\right)\left(p - \sqrt{2(E-V)}\right)$$

$\downarrow$ quantisation

$$0 = \left(-\hbar^2 \frac{d^2}{du^2} + 2(V-E)\right)\Psi$$

$$p = -\frac{\hbar}{i}\frac{\Psi'}{\Psi}$$

Riccati equation:

$$p^2 + 2(V-E) = i\hbar p'$$

$$0 = \det(\lambda - M(u)) = \lambda^N + \sum_{a=1}^{N} (-1)^a \chi_{\square\}a}(M)\, \lambda^{N-a} = \prod_{i=1}^{N} (\lambda - \lambda_i)$$

S_N

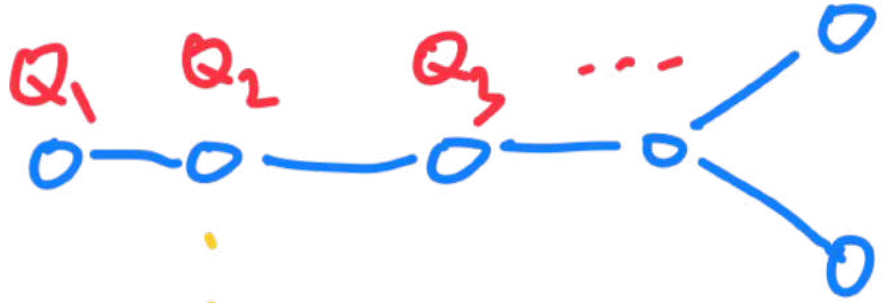
$$0 = \det\left(1 - M e^{-\hbar\partial_u}\right) Q = \left(\sum_{a=0}^{N} (-1)^a T_{\square\}a}\, e^{-a\hbar\partial_u}\right) Q = \prod_{i=1}^{N}\left(1 - \Lambda_i e^{-\hbar\partial_u}\right) Q$$

$s\ell_N$

(PGL_N)

$M_{ij} =$ 

$$\Lambda \sim \frac{Q^{++}}{Q} = 1 + \hbar\frac{Q'}{Q} + \dots$$



← highest weight components of

$Q^{\sigma(1)}_{(a)} \longleftrightarrow \sigma\cdot\omega_a$

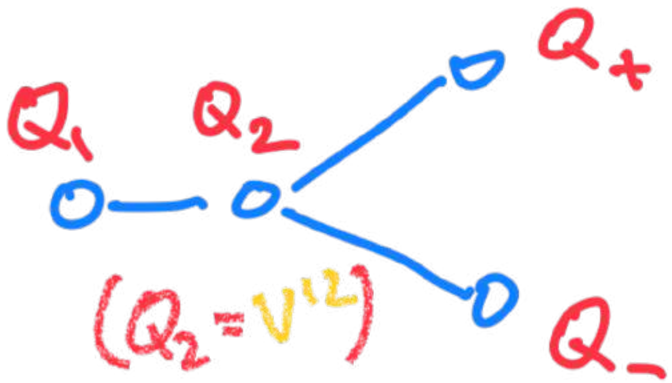
$Q^{\sigma(1)}_{(a)}$ ← Weyl orbits

$Q^{1}_{(a)} = Q_a$

$Q_{(a)}$ ← Full representation $\quad Q_{(a)} = \sum_{i=1}^{\dim L_a} Q^{i}_{(a)}\, e_i$

Extended Q-system

SO(8):



$(Q_2 = V^{12})$

$(Q_1 =)$ V^1
V^2
V^3
V^4
V^{-4}
V^{-3}
V^{-2}
V^{-1}

$\downarrow$

V^{ij}

$i+j \neq 0$

$$\Psi = \begin{pmatrix} \psi_+^{\alpha} \\ \psi_-^{\dot{\alpha}} \end{pmatrix}$$

$\downarrow$

V^i $\quad$ V^{ij} $\quad$ $\Psi = \begin{pmatrix} \psi_+^{\alpha} \\ \psi_-^{\dot{\alpha}} \end{pmatrix}$

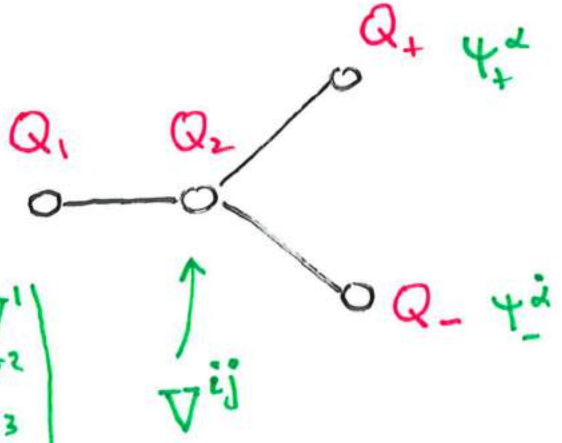
General picture

Q-functions are organised in irreps of ${}^{L}\mathfrak{g} \leftarrow \mathcal{W} \rightarrow \mathfrak{g} \rightarrow Y(\mathfrak{g})$ (or $U_q(\mathfrak{g}^{(r)})$)

Weyl group

covariant description of Bethe algebra for

SO(8)



$$Q_1 = \begin{pmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \\ V^{-4} \\ \vdots \\ V^{-1} \end{pmatrix}$$

$Q_2 = V^{12}$

$Q^{1}_{(a)} = Q_a$

a'th fund representation

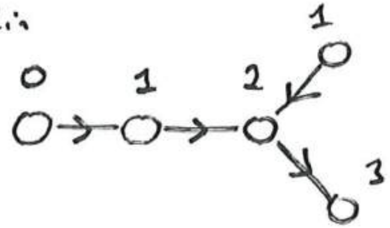
- Derivation of relations between Q-functions is heavily based on ODE/IM and in particular on [Sun`12], [Masoero,Raimondo,Valeri`15]

- All relations between Q-functions are summarised into/follow from the **fused flag** property

- Fused flag is equivalent to (G,q)-oper in [Frenkel, Koroteev, Sage, Zeitlin'20], sl_N case of this equivalence is [Kazakov, Leurent, D.V.'16] vs [Koroteev,Zeitlin'18]
- Overall, sl_N equations were known under many disguises well before, see e.g. [Krichever, Lipan, Wiegman, Zabrodin'97],[Tsuboi'09]

Fused flag

Orient Dynkin graph:



p_a = # at a'th node

Main result:

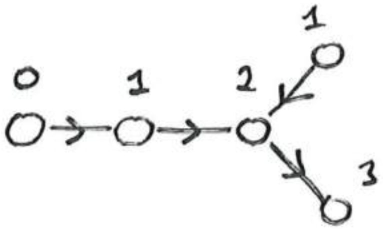
$\underline{\forall p, \exists G \text{ s.t.}}$: $Q_{(a)}(u + \frac{\hbar}{2} p_a) \propto G[p](u) \cdot |HWS\rangle_a$

(for almost all u)

Orbit of $\bigotimes_a |HWS\rangle_a$ under G-action is $\underbrace{G/B}_{\text{flag manifold}}$

$\forall p \leftarrow$ fusion property (speciality of Q-system)

Fused flag

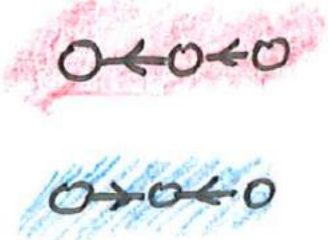


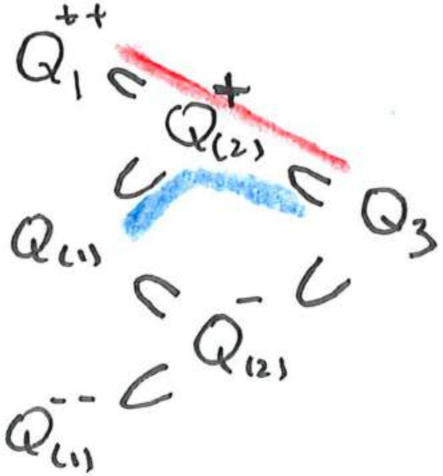
p_a = # at a'th node

$$Q_{(a)}\left(u+\tfrac{\hbar}{2}p_a\right) \propto G[p](u)\cdot |HWS\rangle_a$$

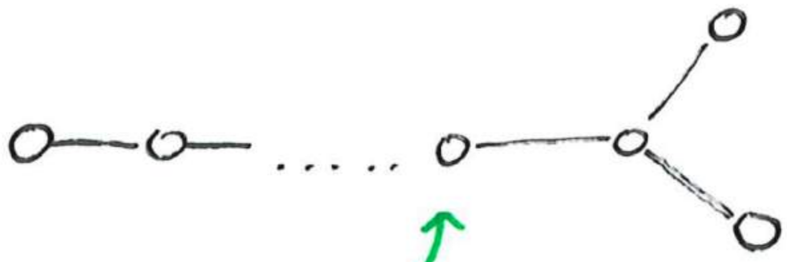
(for almost all u)

$$\{Q_{(1)}, Q_{(2)}, Q_{(3)}\} = G\{e_1, e_1\wedge e_2, e_1\wedge e_2\wedge e_3\}$$





Solution of T-system

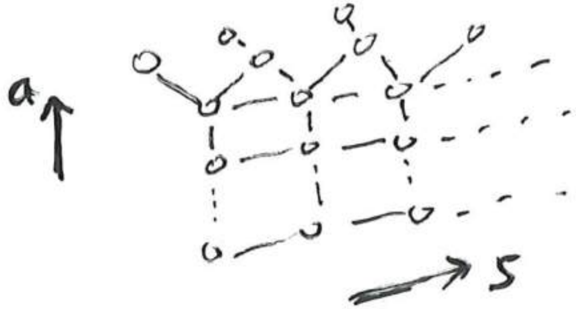


$Q_{(a)}$

vector in a'th fundamental representation

$$Q^{1}_{(a)} = Q_a$$

Application: $$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + \prod_{b\sim a} T_{b,s}$$

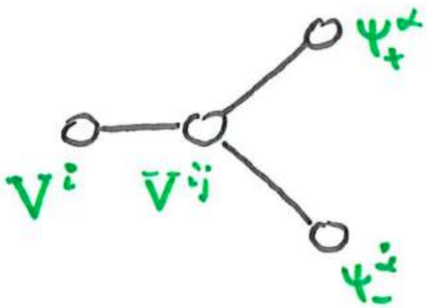


$$T_{a,s} = \left\langle Q^{[s+\frac{h}{2}]}_{(a)},\; Q^{[-s-\frac{h}{2}]}_{(a^*)} \right\rangle$$

h - Coxeter number

SO(8) in detail

(cf. [Ferrando, Frassek, Kazakov `20])



$\rightarrow$ self-dual plane

$\rightarrow$ anti-self-dual plane

SO(8) Flag: $\phi \subset \mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \begin{matrix} \subset \mathbb{C}^4_+ \\ \subset \mathbb{C}^4_- \end{matrix} \subset \mathbb{C}^8$

planes are isotropic (null): $G \cdot \left[e_1 \subset e_1 \wedge e_2 \subset e_1 \wedge e_2 \wedge e_3 \begin{matrix} \subset e_{123} \wedge e_4 \\ \subset e_{123} \wedge e_4 \end{matrix} \right]$

Fusion:

$V_{(1)}^{++} \subset V_{(2)}^{+} \subset V_{(3)}$

$\bar{V}_{(1)} \subset V_{(2)}^{+}$, $\bar{V}_{(1)} \subset V_{(2)}^{-}$

$V_{(1)}^{--} \subset V_{(2)}^{-} \subset V_{(3)}$

$V^{ij} = w(v^i, v^j)$

$V^{ijk} := w(v^i, v^j, v^k) = \bar{\Psi}_+ \gamma^{ijk} \psi_-$

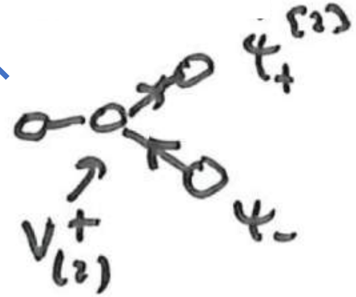
Example of fused Fierz relation

Fusion continued

	$V_{(1)}^{[-2]}$	$V_{(1)}$	$V_{(1)}^{[2]}$	$V_{(2)}^{-}$	$V_{(2)}^{+}$	$V_{(3)}$	Ψ_+	Ψ_-
1	x			x		x	x	x
2		x		x	x	x	x	x
3			x		x	x	x	x
4							x	x
-4								
-3								
-2								
-1								

$$\bar{V}^{ijk} = \bar{\Psi}_+ \gamma^{ijk} \Psi_-$$

$$V^{ij} = \bar{\Psi}_{\pm}^{[1]} \gamma^{ij} \Psi_{\pm}^{[-1]}$$



$$= \langle Q_{\underset{\pm}{\overset{\shortparallel}{(a)}}}^{[h/2]}, Q_{\underset{\pm}{\overset{\shortparallel}{(a^+)}}}^{[-h/2]} \rangle$$

Fusion continued

	$V_{(1)}^{[-2]}$	$V_{(1)}$	$V_{(1)}^{[2]}$	$V_{(2)}^{-}$	$V_{(2)}^{+}$	$V_{(3)}$	Ψ_+	Ψ_-	$\Psi_+^{[2]}$	$\Psi_+^{[-2]}$	$\Psi_-^{[2]}$	$\Psi_+^{[4]}$
1	x			x		x	x	x	x	x x	x	
2		x		x	x	x	x	x	x		x	x
3			x		x	x	x	x			x	x
4							x	x	x	x		
-4										x		
-3												x
-2												
-1									x		x	x

$$\bar V^{ijk} = \bar\Psi_+ \gamma^{ijk} \Psi_-$$

$$V^{ij} = \bar\Psi_\pm^{[1]} \gamma^{ij} \Psi_\pm^{[-1]}$$

$$V^{i} = \bar\Psi_\pm^{[2]} \gamma^{i} \Psi_\mp^{[-2]}$$

$$1 = \bar\Psi_\pm^{[3]} \cdot \Psi_\pm^{[-3]} = \langle Q_{(a)}^{[h/2]}, Q_{(a^+)}^{[-h/2]} \rangle, \quad a = \pm,\ a^+ = \pm$$

$$T_{\pm,s} = \langle \bar\Psi_\pm^{[3+s]}, \Psi_\pm^{[-3-s]} \rangle$$

$$\bar\Psi_\pm \gamma^{ij} \Psi_\pm = 0$$

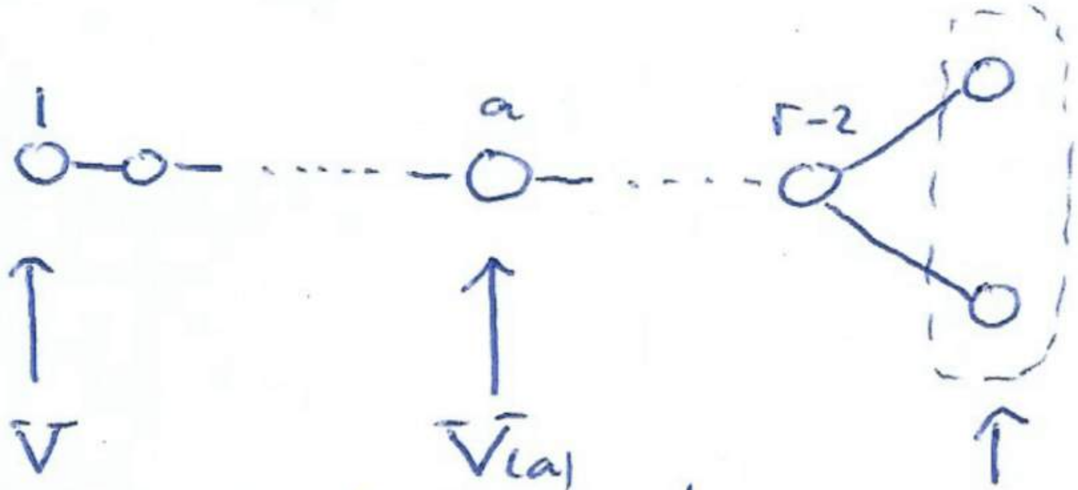
$$\bar\Psi_\pm \gamma^{i} \Psi_\mp = 0$$

$$\bar\Psi_\pm \Psi_\pm = 0$$

Pure spinors

Summary:

D_r:
(SO_{2r})



null vector V

$V_{(a)}$
(null plane)

Ψ - pure [Dirac] spinor

$$w(v^{i_1},\dots,v^{i_a}) = V^{i_1\dots i_a} = \bar{\Psi}^{[r-a-1]}\gamma^{i_1\dots i_a}\psi^{[-(r-a-1)]}$$

(extended Q - system for D_r)

$SU(r)$ decomposition (for us gl_r decomp.)

of components of ψ is $2^r = 1 + r + \binom{r}{2} + \dots$

As representation of $gl_r \subset so_{2r}$:

$$\underbrace{\psi = \psi_0 + \psi_{(1)} + \psi_{(2)}} + \dots$$

enough to reconstruct the rest

[Cartan'1936]

$$\Psi_{(2n)} = \frac{1}{n!}\frac{\Psi_{(2)} \wedge \Psi_{(2)} \wedge \cdots \wedge \Psi_{(2)}}{\Psi_{\emptyset}{}^{n-1}},$$

$$\Psi_{(2n-1)} = \frac{1}{(n-1)!}\frac{\Psi_{(1)} \wedge \Psi_{(2)} \wedge \cdots \wedge \Psi_{(2)}}{\Psi_{\emptyset}{}^{n-1}}$$

Pure spinor Q-system for D_r

$$\boxed{\begin{aligned} W(\psi_1, \psi_2, \psi_3, \psi_4) &= \psi_0^+ \psi_0^- \\ W(\psi_a, \psi_b) &= W(\psi_{ab}, \psi_0) \end{aligned}}$$

$$V = (V^1, \ldots, V^4, V^{-4}, \ldots, V^{-1})$$

$$V_a := V^{-a}$$

Pure spinor Q-system for D_r

$$\boxed{\begin{aligned} W(\Psi_1, \Psi_2, \Psi_3, \Psi_4) &= \Psi_0^+ \Psi_0^- \\ W(\Psi_a, \Psi_b) &= W(\Psi_{ab}, \Psi_0) \end{aligned}}$$

$$V^a = P^a$$

$$V_a = \mu_{ab} P^b$$

$$P_a = \frac{\Psi_a}{\Psi_0} \; ; \; \mu_{ab} = \frac{\Psi_{ab}}{\Psi_0} \; ; \; \Phi = \frac{1}{\Psi_0}$$

$P\mu$-system for D_r

$$W(\mathbf{P}_1, \ldots, \mathbf{P}_r) = \Phi^{[r-1]} \Phi^{[-r+1]}$$

$$W(\mathbf{P}_a, \mathbf{P}_b) = \mu_{ab}^+ - \mu_{ab}^-$$

For comparison:

$$\boxed{\begin{array}{l} \underline{\text{QSC for } AdS_5/CFT_4:} \\ P_a \tilde{P}_b - P_b \tilde{P}_a = \mu_{ab}^{[2]} - \mu_{ab} \\ \tilde{P}_a = \mu_{ab} P^b \, , \; P_a P^a = 0 \end{array}}$$

Analytic Bethe Ansatz

For rational spin chains with periodic boundary conditions:

$$Q^{i}_{(a)} = \sigma_a \times q^{i}_{(a)}$$

dressing factor

Polynomial in u

$$\deg q^{i}_{(a)} = (\omega_a, \lambda_{\max} + \rho) - (\gamma_{(a),i}, \lambda + \rho)$$

Fundamental weight

Weight of "ferromagnetic vacuum"

Weyl vector

Weight of the i'th basis vector in a'th fundamental irrep

$$\frac{\prod_{b \wedge a} \sigma_b}{\sigma_a^+ \sigma_a^-} = P_a$$

Drinfeld polynomial

Analytic Bethe Ansatz

Polynomial in u

For rational spin chains with periodic boundary conditions:

$$Q^{i}_{(a)} = \sigma_a \times q^{i}_{(a)}$$

dressing factor

Solving this system using the ansatz:

$$\boxed{\begin{array}{l} W(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = \Psi_0^+ \Psi_0^- \\ W(\Psi_a, \Psi_b) = W(\Psi_{ab}, \Psi_0) \end{array}}$$

- Already very efficient method in practice (likely can be improved further)
- Abundant evidence for completeness

SO(8):

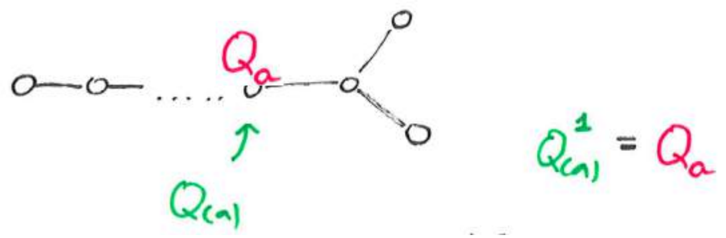
L	V	$\wedge^2 V$	Ψ_-	Ψ_+
2	3(0.9s)	10(11.5s)	3(2.0s)	3(2.4s)
3	7(2.0s)	68(95.0s)	7(5.0s)	7(11.0s)
4	26(11.2s)	631(1177s)	26(29.6s)	26(120s)
5	85(28.0s)	-	85(78s)	85(322s)
6	365(79s)	-	365(1435s)	365(2278s)
7	1456(1483s)	-	-	-

SO(10):

L	V	$\wedge^2 V$	$\wedge^3 V$	Ψ_-	Ψ_+
2	3(2.4s)	9(21.1s)	20(108s)	3(8s)	3(18.5s)
3	7(4.7s)	60(176s)	-	9(22.8s)	9(136s)
4	25(21.0s)	-	-	42(325s)	42(1571s)
5	82(215s)	-	-	-	-

Conclusions

- We propose to extend Q-functions to a set of Q-functions which is covariant w.r.t. to action of Langlands dual of the symmetry algebra



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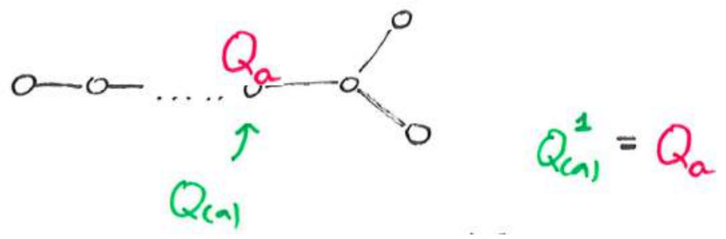


Bethe equations

$$-1=\prod_{b=1}^{r}\frac{Q_b^{[+c_{ab}]}}{Q_b^{[-c_{ab}]}}$$

[Ogievetsky,Wiegmann’86]

QQ system:

$$W\left(Q_a, Q^2_{(a)}\right)=\prod_{b\sim a} Q_b$$

[Pronko, Stroganov`98]
[Voros`99]
[Masoero, Raimondo, Valeri`15]
[Frenkel, Hernandez’16]

Q system on Weyl orbit:

$$W\left(Q^{s(1)}_{(a)}, Q^{s(2)}_{(a)}\right)=\pm_s\prod_{b\sim a} Q^{s(1)}_{(b)}$$

[Pronko, Stroganov`00][Bazhanov,Hibberd,Khoroshkin’01]
[Tsuboi’09]
[Mukhin,Varchenko’05]
[Masoero, Raimondo`18]
[Ferrando, Frassek, Kazakov `20]
[Koroteev, Zeitlin’21]

Extended Q system:

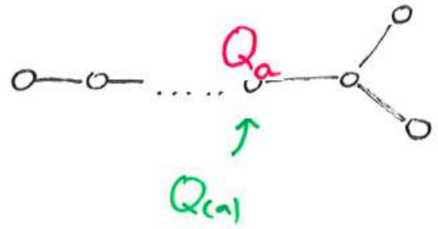
$$Q_{(a)}\left(u+\frac{\hbar}{2}p_a\right)=G[p](u)|HWS\rangle$$

Generalised Plücker coordinates (terminology of [Fomin,Zelevinsky’98])

Extended Plücker coordinates

Conclusions

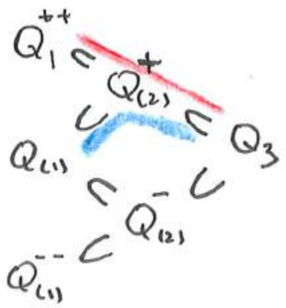
- We propose to extend Q-functions to a set of Q-functions which is covariant w.r.t. to action of Langlands dual of the symmetry algebra



$$Q^{1}_{(a)} = Q_a$$

- Fused flag condition

$$Q_{(a)}(u + \frac{\hbar}{2} p_a) = G[p](u)|HWS\rangle$$



- Can solve T-system:

$$T_{a,s} = \langle Q_{(a)}^{[s+\frac{\hbar}{2}]} , Q_{(a^*)}^{[-s-\frac{\hbar}{2}]} \rangle$$

- Efficient for computations

$$W(\mathbf{P}_1, \ldots, \mathbf{P}_r) = \Phi^{[r-1]}\Phi^{[-r+1]}$$
$$W(\mathbf{P}_a, \mathbf{P}_b) = \mu^{+}_{ab} - \mu^{-}_{ab}$$

- Universal algebraic system: XXX,XXZ,TBA,ODE/IM …, physics is determined by analytic input

Remark about non-simply laced case

We should compute Langlands dual not of Lie algebra but **of affine Lie algebra** [Masoero, Raimondo, Valeri`15] [Frenkel, Hernandez'16]

	Integrable model based on	Q-system is based on evaluation representations of
Periodic spin chains:	$U_q(\hat{X})$ or $\mathcal{Y}(X)$, $X = A_r, D_r, E_6, E_7, E_8$	$\hat{X}$
	$U_q(\hat{X})$ or $\mathcal{Y}(X)$, $X = B_r, C_r, F_4, G_2$	$A_{2r-1}^{(2)}$ $D_{r+1}^{(2)}$ $E_6^{(2)}$ $D_4^{(3)}$
Spin chains with boundary:	$U_q(X^{(2)})$ or $\mathcal{Y}(X,\sigma)$ $X = A_{2r}, A_{2r-1}, D_r, E_6$	$A_{2r}^{(2)}$ $\hat{B}_r$ $\hat{C}_{r+1}$ $\hat{F}_4$
	$U_q\left(D_4^{(3)}\right)$	$\hat{G}_2$

… Idea for fused flag is the same but quite a few subtleties, work in progress [S.Ekhammar, D.V.' math-ph/21xx.xxxxx]

Open directions

The answer is a

universal algebraic structure

- XXX (Yangian, twisted Yangian)
- XXZ (quantum [twisted] affine)
- Elliptic (?)
- AdS/CFT
- TBA
- ODE/IM

…..

- q-characters
- [finite difference] opers
- [equivariant quantum] cohomology rings
- Bethe/gauge

…..

- 4dCS

… Many open directions.

If the topic is interesting for your research:

- e-mail us and let us discuss
- apply for the workshop:

``Geometric and Representation-Theoretic Aspects of Quantum Integrability”

August 29-October 21, 2022 @Simons Center

[organisers: P.Koroteev, E.Pomoni, B.Vicedo, D.Volin, A.Zeitlin]

$$0 = p^2 + 2(V-E) = \left(p + \sqrt{2(E-V)}\right)\left(p - \sqrt{2(E-V)}\right)$$

$\downarrow$ quantisation

$$0 = \left(-\hbar^2 \frac{d^2}{du^2} + 2(V-E)\right)\Psi$$

$$p = -\frac{\hbar}{i}\frac{\Psi'}{\Psi}$$

Riccati equation:

$$p^2 + 2(V-E) = i\hbar p'$$

$$0 = \det(\lambda - M(u)) = \lambda^N + \sum_{a=1}^{N} (-1)^a \chi_{\{1\}^a}(M)\, \lambda^{N-a} = \prod_{i=1}^{N} (\lambda - \lambda_i)$$

S_N

$$0 = \det\left(1 - M e^{-\hbar\partial_u}\right) Q = \left(\sum_{a=0}^{N} (-1)^a T_{\{1\}^a} e^{-a\hbar\partial_u}\right) Q = \prod_{i=1}^{N} \left(1 - \Lambda_i e^{-\hbar\partial_u}\right) Q$$

SL_N
(PGL_N)

$$M_{ij} = \overset{i}{+\!\!+\!\!+\!\!+}\overset{j}{}$$

$$\Lambda \sim \frac{Q^{++}}{Q} = 1 + \hbar\frac{Q'}{Q} + \dots$$

$$W(Q^1, \ldots, Q^N) = 1$$

$$-1 = \prod_{b=1}^{r} \frac{Q_b^{[+C_{a,b}]}}{Q_b^{[-C_{a,b}]}} \Bigg|_{u=u_{a,i}}$$

- Can prove completeness

  [Mukhin, Tarasov, Varchenko’12] [Chernyak,Leurent,D.V.’20]

- Have efficient ways to solve them (GREEN!)

- Compact expressions for transfer matrices:

$$\det_{1\le i,j\le N} Q_i^{[2(\lambda_j+1-j)]} = T_\lambda = \sum_{\tau\in SSYT(\lambda)} \prod_{(a,s)\in\lambda} \Lambda_{\tau_{a,s}}^{[2ct(a,s)]}$$

$$\Lambda_a = \frac{Q_a^{++}}{Q_a}\frac{Q_{a-1}^{-}}{Q_{a-1}^{+}}$$

- Green philosophy is essential in AdS/CFT QSC