QSC for AdS_3

with

Andrea, Bogdan, Alessandro

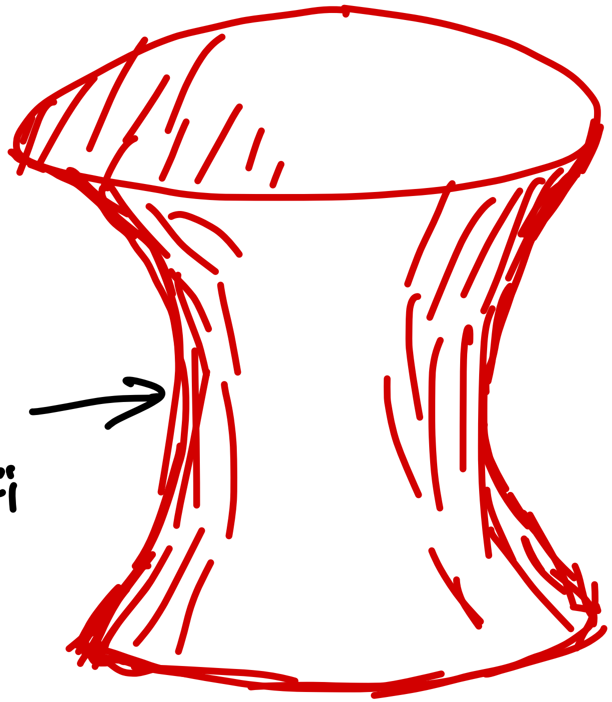
Crossing symmetry

hep-th

2109.05500

Cavaglia NG Stefanski

Torrielli



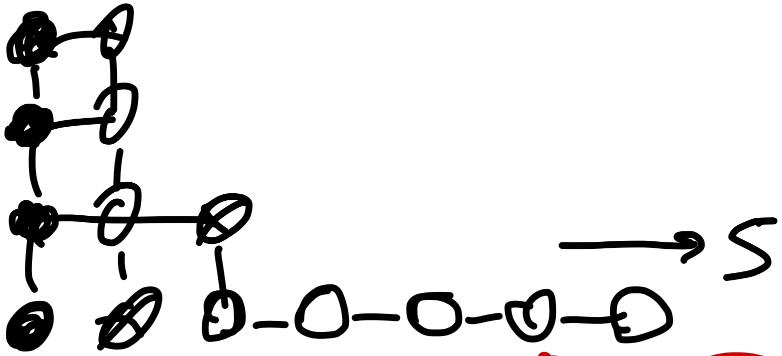
2109.06164

Ekhammar Volin

math-ph

AdS_5 and AdS_4

common feature:



Y-system

$$\ln Y_{1s} = \sum_t K_{s,t} * \ln(1+Y_{1,t})$$

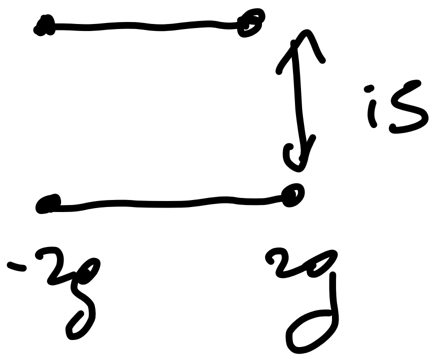
$Y \to T$:

$$1+Y_{1s} = \frac{T^+_{1s}\, T^-_{1s}}{T_{1s+1}\, T_{1s-1}}$$

$K_s * \ln T_{11}$

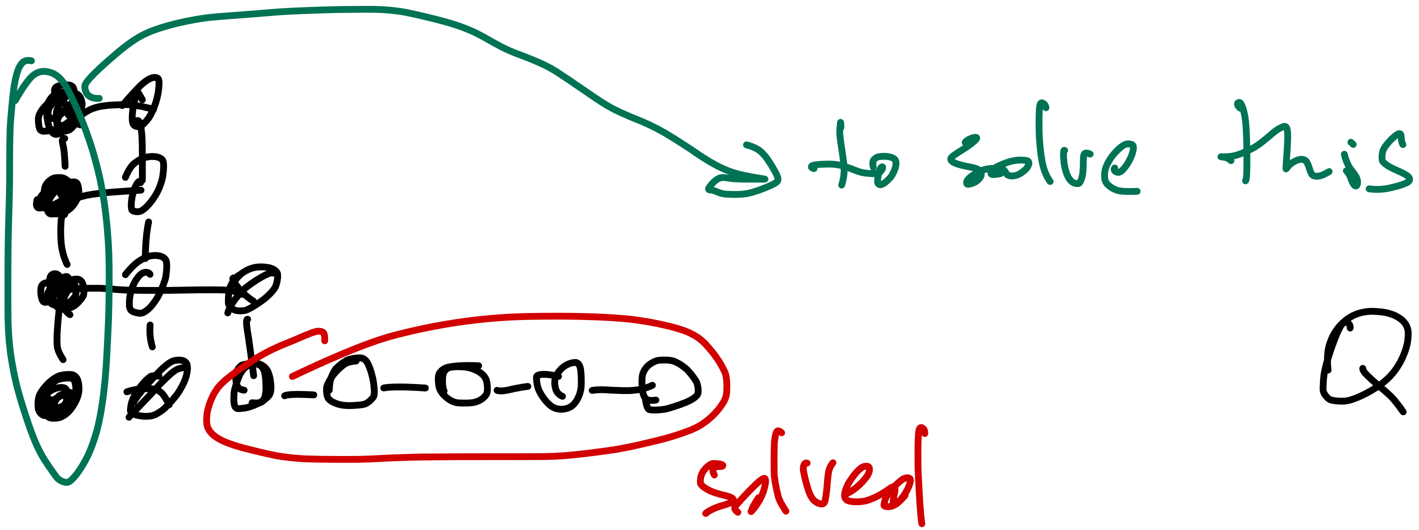
$T \to Q$

T_{1s}



P_a

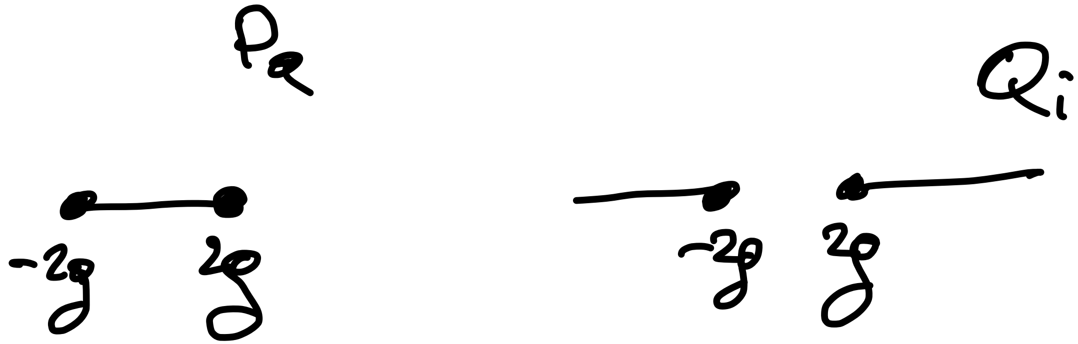
$$T_{1s} = P_1^{[+s]} P_2^{[-s]} - P_2^{[-s]} P_2^{[+s]}$$



Q_i

AdS_3

QSC = QQ-relations + analyticity



Hints from classics:

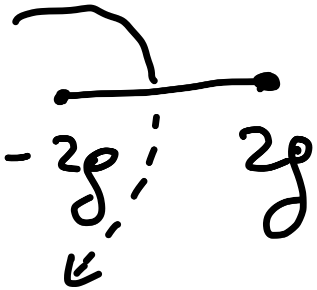
$J(u) \leftarrow$ flat $\Rightarrow$ $\Omega = P\exp\left(\int J_0\right)$

compact   non-compact

eigenvalues of $\Omega \longrightarrow p \,,\, q$

$$P_a \simeq e^{\int^u p_a du} \qquad Q_i \simeq e^{\int^u q_i du}$$

q_1



q_2

← dynamical cuts
= condensate
of Bethe roots

$$\Downarrow$$

$$Q_1^{\gamma} = Q_2$$

Asymptotics

$$\begin{pmatrix} p_1^A \\ p_2^A \\ p_1^S \\ p_2^S \end{pmatrix} \simeq \frac{1}{2hx} \begin{pmatrix} -\Delta - S - \hat{B} \\ +\Delta + S - \hat{B} \\ -J - K - \hat{B} \\ +J + K - \hat{B} \end{pmatrix} = \frac{1}{2hx} \begin{pmatrix} -\gamma - 2K_1 - L \\ +\gamma + 2K_3 + L \\ -2K_1 + 2K_2 - L \\ -2K_2 + 2K_3 + L \end{pmatrix},$$

$\Rightarrow Q \sim u^{\Delta}$

$$\begin{pmatrix} p_{\dot{1}}^A \\ p_{\dot{2}}^A \\ p_{\dot{1}}^S \\ p_{\dot{2}}^S \end{pmatrix} \simeq \frac{1}{2hx} \begin{pmatrix} +\Delta - S - \check{B} \\ -\Delta + S - \check{B} \\ +J - K - \check{B} \\ -J + K - \check{B} \end{pmatrix} = \frac{1}{2hx} \begin{pmatrix} +\gamma - 2K_{\dot{1}} + 2K_{\dot{2}} + L \\ -\gamma - 2K_{\dot{2}} + 2K_{\dot{3}} - L \\ -2K_{\dot{1}} + L \\ 2K_{\dot{3}} - L \end{pmatrix},$$

$$p_a^A\left(\frac{1}{x}\right) = p_{\dot{a}}^A(x), \quad a = 1, 2,$$

no dot

dot

Q-system

$$Q_{aA|I}Q_{A|Ii} = Q^+_{aA|Ii}Q^-_{A|I} - Q^-_{aA|Ii}Q^+_{A|I},$$
$$Q_{12|I}Q_{\emptyset|I} = Q^+_{1|I}Q^-_{2|I} - Q^-_{1|I}Q^+_{2|I},$$
$$Q_{A|12}Q_{A|\emptyset} = Q^+_{A|1}Q^-_{A|2} - Q^-_{A|1}Q^+_{A|2},$$

$$\mathbf{Q}_k \equiv Q_{\emptyset|k}, \quad \mathbf{P}_a \equiv Q_{a|\emptyset}, \quad \mathbf{Q}^k \equiv \epsilon^{kl}Q_{12|l}, \quad \mathbf{P}^a \equiv \epsilon^{ab}Q_{b|12},$$

$$Q^{a|i} \equiv \epsilon^{ab}\epsilon^{ij}Q_{b|j}.$$

in our case:

$$Q^+_{a|i} - Q^-_{a|i} = P_a Q_i \qquad Q^{a|i+} - Q^{a|i-} = -P^a Q^i$$

same function

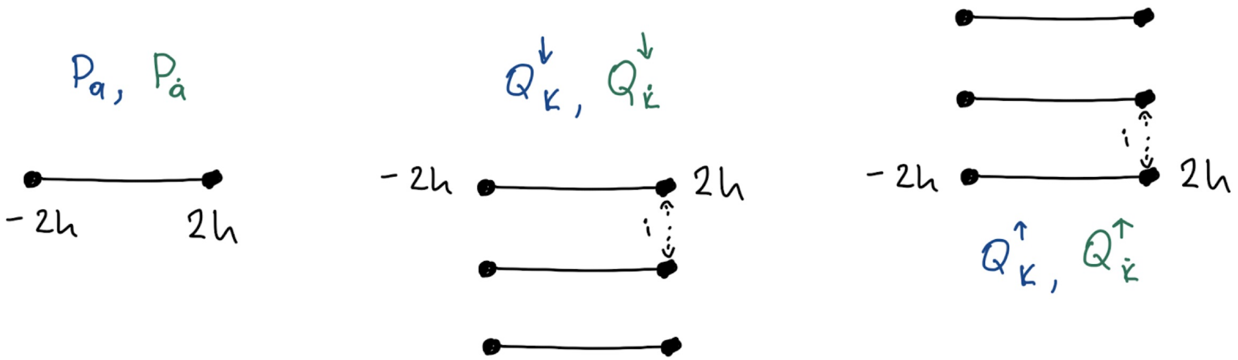
$$\mathbf{Q}^1\mathbf{P}^1 = -\mathbf{Q}_2\mathbf{P}_2 \;,\; \mathbf{Q}^1\mathbf{P}^2 = +\mathbf{Q}_2\mathbf{P}_1 \;,\; \mathbf{Q}^2\mathbf{P}^1 = +\mathbf{Q}_1\mathbf{P}_2 \;,\; \mathbf{Q}^2\mathbf{P}^2 = +\mathbf{Q}_1\mathbf{P}_1,$$

$$\frac{\mathbf{Q}^1}{\mathbf{Q}_2} = -\frac{\mathbf{Q}^2}{\mathbf{Q}_1} = -\frac{\mathbf{P}_2}{\mathbf{P}^1} = +\frac{\mathbf{P}_1}{\mathbf{P}^2} \equiv r.$$

Baxter equation

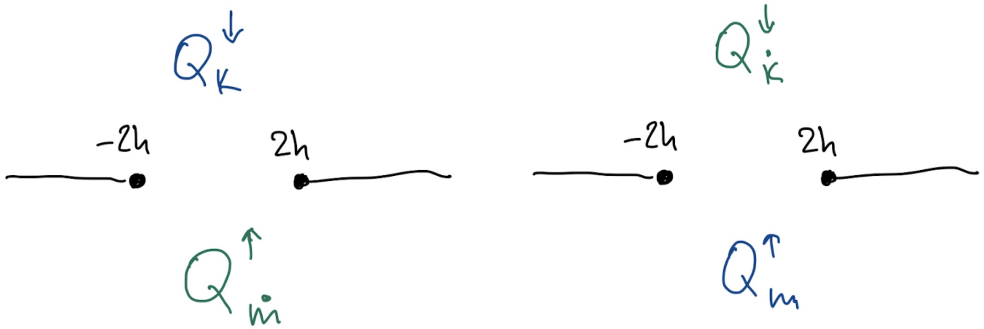
$$\mathbf{Q}_k^{++} D_1^- - \mathbf{Q}_k D_2 + \mathbf{Q}_k^{--} D_1^+ = 0 \,,$$

$$D_1 = \epsilon_{ab} \mathbf{P}^{a-} \mathbf{P}^{b+} \quad , \quad D_2 = \epsilon_{ab} \mathbf{P}^{a--} \mathbf{P}^{b++} - \mathbf{P}_c \mathbf{P}^{c--} \; \epsilon_{ab} \mathbf{P}^a \mathbf{P}^{b++}$$



$$\mathbf{Q}_k^{\uparrow} = \Omega_k{}^m \mathbf{Q}_m^{\downarrow} \quad , \quad \Omega_k{}^m(u+i) = \Omega_k{}^m(u) \;.$$

Gluing condition



following classical hints

$$\mathbf{Q}_k^{\downarrow}(u+i0) = G_k^{\ \dot n}\, \mathbf{Q}^{\uparrow}_{\dot n}(u-i0)$$

constant $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$

Non - sqrt

$$\mathbf{Q}_k^{\downarrow\gamma} = \omega_k{}^{\dot m}\mathbf{Q}_{\dot m}^{\downarrow} \qquad \omega_k{}^{\dot n} = G_k{}^{\dot m}\Omega_{\dot m}{}^{\dot n}$$

$$Q_{\dot k}^{\gamma^2} = \underbrace{\omega_{\dot k}{}^{\ell}\left(\delta_e^p - Q_e Q^p\right)\omega_p{}^{\dot n}}_{\mho} Q_{\dot n}$$

$$Q_{\dot k}^{\gamma^{2n}} = \mho^n Q_{\dot n}$$

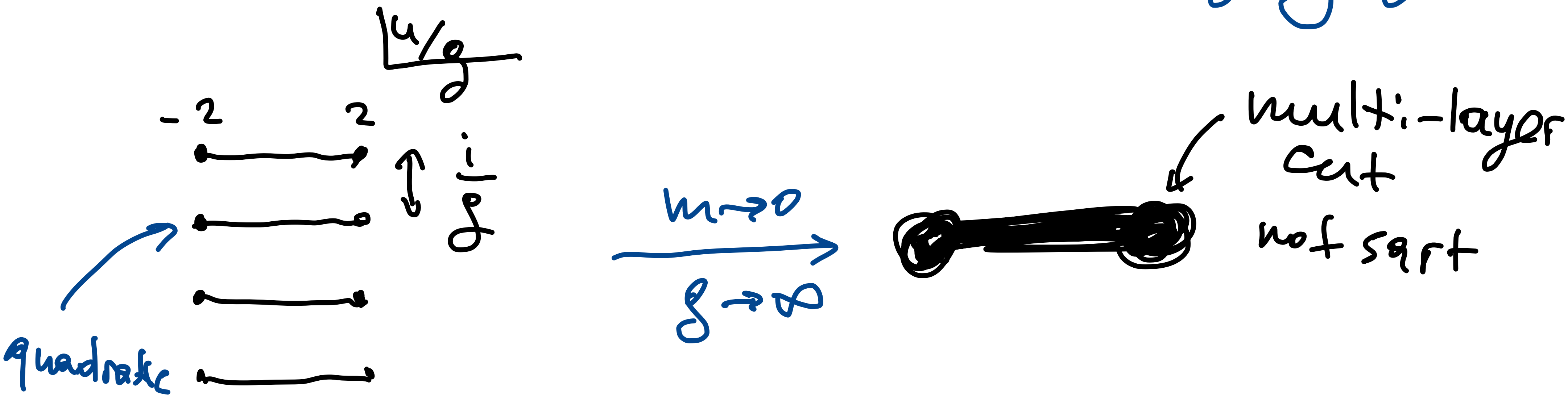
log vibe

Massless vs sqrt

mass

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i m}{g}$$

massless limit $m \to 0$ is roughly $g \to \infty$



Some things are sqrs

$$\frac{\mathbf{Q}^1}{\mathbf{Q}_2} = -\frac{\mathbf{Q}^2}{\mathbf{Q}_1} = -\frac{\mathbf{P}_2}{\mathbf{P}^1} = +\frac{\mathbf{P}_1}{\mathbf{P}^2} \equiv r.$$

$$r^{\gamma} = r^{\bar{\gamma}} = \dot{r}$$

r = rational function of $x(u)$

$$r = \frac{(x - y_{1,i})(1/x - y_{\dot{1},i})}{(x - y_{3,i})(1/x - y_{\dot{3},i})}$$

exact non ABA expression!

ABA Limit

$Q_1, Q_{\dot 1}$ are large$^{\text{exp}}$ $\quad Q_2, Q_{\dot 2}$ small$^{\text{exp}}$

assumption $\mu_1{}^{\dot 2}$ is sqrt in ABA

$$\mu^2{}_{\dot 1} = Q^-_{11}\,\omega_2{}^{\dot 1}\,Q^-_{\dot 1 1}$$

$\omega^{++} = \omega$

$\mu^{++} = \mu^{\gamma}$

Step 1 $\quad$ zeros of μ^+ are mom-carrying roots $\mathbb{Q}$

$$F_0 \equiv \frac{\mu\,\mathbb{Q}^+}{\mu^{++}\,\mathbb{Q}^-} = \frac{Q\,Q\,\mathbb{Q}^+}{Q\,Q\,\mathbb{Q}^-} \leftarrow \text{1cut} \qquad F\,F^{\gamma} \sim \frac{\mathbb{Q}^+}{\mathbb{Q}^-}$$

$$\mu_1{}^{\dot 2} \propto \mathbb{Q}^- f \bar f^{--}, \quad \omega^1{}_{\dot 2} \propto \frac{\bar f^{--}}{f}, \quad Q_{1|1}Q_{\dot 1|\dot 1} \propto \mathbb{Q}\,(f^+)^2, \qquad \frac{f^{++}}{f} = \frac{B_{(-)}}{B_{(+)}},$$

$$B_{(\pm)} = \prod\left(\frac{1}{x} - x_i^{\mp}\right)$$

$$B\tilde B = \mathbb{Q}$$

Fixing P, Q

takes care of asymptotics

arbitrary

contains zeros of P_1

$R = (x - x_i)$

$$\mathbf{P}_1 \propto x^{-L/2}\,\mathcal{A} \times R_{\tilde{1}} B_{\tilde{\dot{1}}}\, B_{2,(-)}, \qquad \mathbf{P}^2 \propto x^{-L/2}\,\mathcal{A} \times R_{\tilde{3}} B_{\tilde{\dot{3}}} B_{2,(-)},$$

$$\mathbf{Q}_1 \propto \frac{x^{L/2}}{\mathcal{A}'} \times R_1 B_{\dot{1}}\, f_2 \frac{f_{\dot{2}}}{B_{\dot{2},(+)}}, \qquad \mathbf{Q}^2 \propto \frac{x^{L/2}}{\mathcal{A}'} \times R_3 B_{\dot{3}}\, f_2 \frac{f_{\dot{2}}}{B_{\dot{2},(+)}}\,.$$

use that:

$$r \propto \frac{R_{\tilde{1}} B_{\tilde{\dot{1}}}}{R_{\tilde{3}} B_{\tilde{\dot{3}}}} \propto \frac{R_3 B_{\dot{3}}}{R_1 B_{\dot{1}}}\,,$$

Next:

fixed

$$Q^+_{11} - Q^-_{11} = Q_1 P_1 \implies A = A'$$

Dressing ?

$(\mathbf{P}_1)^\gamma \sim Q^+_{1|1}\omega^1_{\dot 2}\mathbf{Q}^{\dot 2}$

$$A^\gamma \dot{A} = \left(\frac{R_{2,(+)}}{R_{2,(-)}}\right) \bar{f}_2^{--} f_2^{++} \bar{f}_{\dot 2}^{--} f_{\dot 2}^{++}$$

$$\dot{A}^\gamma A = \left(\frac{R_{\dot 2,(+)}}{R_{\dot 2,(-)}}\right) \bar{f}_2^{--} f_2^{++} \bar{f}_{\dot 2}^{--} f_{\dot 2}^{++}$$

$$\frac{\mathcal{A}^\gamma}{\mathcal{A}^{\bar\gamma}} = \frac{R_{2,(+)}}{R_{2,(-)}}\frac{B_{\dot 2,(-)}}{B_{\dot 2,(+)}}\,, \quad \frac{\dot{\mathcal{A}}^\gamma}{\dot{\mathcal{A}}^{\bar\gamma}} = \frac{R_{\dot 2,(+)}}{R_{\dot 2,(-)}}\frac{B_{2,(-)}}{B_{2,(+)}}\,,$$

cannot be split

(But easy to solve)

ABA

$$\mathbf{P}_1 \propto x^{-L/2}\, R_{\tilde{1}} B_{\tilde{\dot{1}}} \sqrt{B_{2,(+)} B_{2,(-)}}\; \sigma_2^1\, \tilde{\sigma}_{\dot{2}}^1, \quad \mathbf{P}^2 \propto x^{-L/2}\, R_{\tilde{3}} B_{\tilde{\dot{3}}} \sqrt{B_{2,(+)} B_{2,(-)}}\; \sigma_2^1\, \tilde{\sigma}_{\dot{2}}^1,$$

$$\mathbf{Q}_1 \propto x^{L/2} R_1 B_{\dot{1}} \sqrt{\frac{B_{2,(-)}}{B_{2,(+)}}}\; \frac{f_2\, f_{\dot{2}}}{B_{\dot{2},(+)}\, \sigma_2^1\, \tilde{\sigma}_{\dot{2}}^1}, \quad \mathbf{Q}^2 \propto x^{L/2} R_3 B_{\dot{3}} \sqrt{\frac{B_{2,(-)}}{B_{2,(+)}}}\; \frac{f_2\, f_{\dot{2}}}{B_{\dot{2},(+)}\, \sigma_2^1\, \tilde{\sigma}_{\dot{2}}^1},$$

plugging into exact BE:

$$-1 = \frac{Q_{1|1}^{++} Q_{\emptyset|1}^{-} Q_{12|1}^{-}}{Q_{1|1}^{--} Q_{\emptyset|1}^{+} Q_{12|1}^{+}} = \frac{Q_{1|1}^{++} \mathbf{Q}_1^{-} \mathbf{Q}^{2-}}{Q_{1|1}^{--} \mathbf{Q}_1^{+} \mathbf{Q}^{2+}}$$

$$= \frac{(x^-)^L (\sigma_2^{1+} \tilde{\sigma}_{\dot{2}}^{1+})^2 \mathbb{Q}_2^{++} R_1^- B_{\dot{1}}^- R_3^- B_{\dot{3}}^- f_2^{[+3]} f_{\dot{2}}^{[+3]} f_2^- f_{\dot{2}}^- B_{2(-)}^- B_{2(+)}^+ [B_{\dot{2}(+)}^+]^2}{(x^+)^L (\sigma_2^{1-} \tilde{\sigma}_{\dot{2}}^{1-})^2 \mathbb{Q}_2^{--} R_1^+ B_{\dot{1}}^+ R_3^+ B_{\dot{3}}^+ f_2^+ \quad f_{\dot{2}}^+ \quad f_2^+ f_{\dot{2}}^+ B_{2(-)}^+ B_{2(+)}^- [B_{\dot{2}(+)}^-]^2},$$

simplifies to:

$$-1=\left(\frac{x^-}{x^+}\right)^L\times\frac{\mathbb{Q}_2^{++}}{\mathbb{Q}_2^{--}}\times(\sigma_2)^2\times\frac{R_1^-R_3^-}{R_1^+R_3^+}$$
$$\times\frac{B^+_{\dot{2},(-)}\,B^+_{\dot{2},(+)}}{B^-_{\dot{2},(-)}\,B^-_{\dot{2},(+)}}\times(\tilde{\sigma}_{\dot{2}})^2\times\frac{B_{\dot{1}}^-B_{\dot{3}}^-}{B_{\dot{1}}^+B_{\dot{3}}^+}\Bigg|_{u=u_{2,i}},\quad i=1,\dots,K_2.$$

↳ combination of BES, and

$$\chi^{\mathrm{HL}}(x,y)=\left(\int_{C^+}-\int_{C^-}\right)\frac{dw}{4\pi}\frac{1}{x-w}\Big[\log(y-w)-\log(y-\frac{1}{w})\Big].$$ ← HL

$$\chi^-(x,y)=\left(\int_{C^+}-\int_{C^-}\right)\frac{dw}{8\pi}\frac{1}{x-w}\log\Big[(y-w)\Big(1-\frac{1}{yw}\Big)\Big]-x\leftrightarrow y,$$ ← BOSST

aka

Bogdan-phase

Conclusions 1

- possible to construct QSC with $[psu(1,1|2)]^2$ symmetry
  but sqrt has to be dropped

- ABA limit reproduces correctly BOSS eqs with BOSST phases

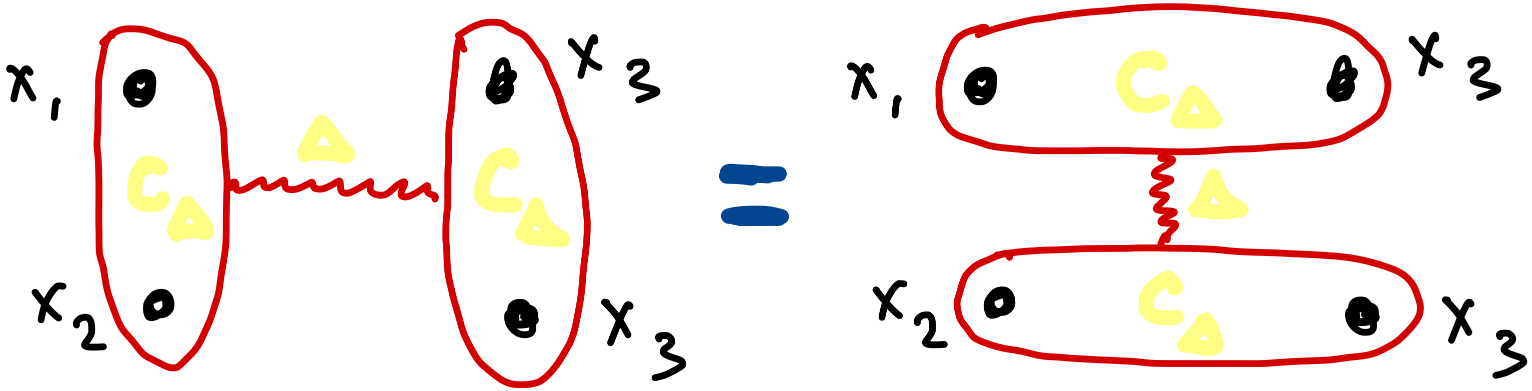
- more tests needed to detect massless modes
  BPS? numerics? weak coupling?

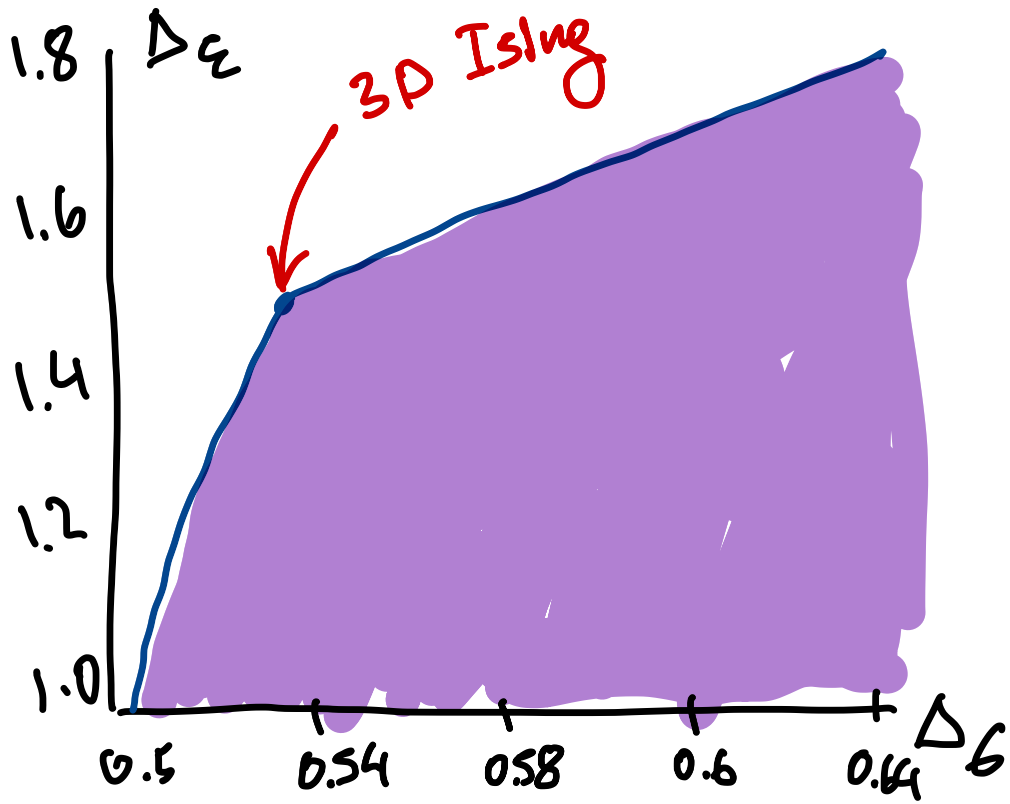
Bootstrability

in AdS/CFT

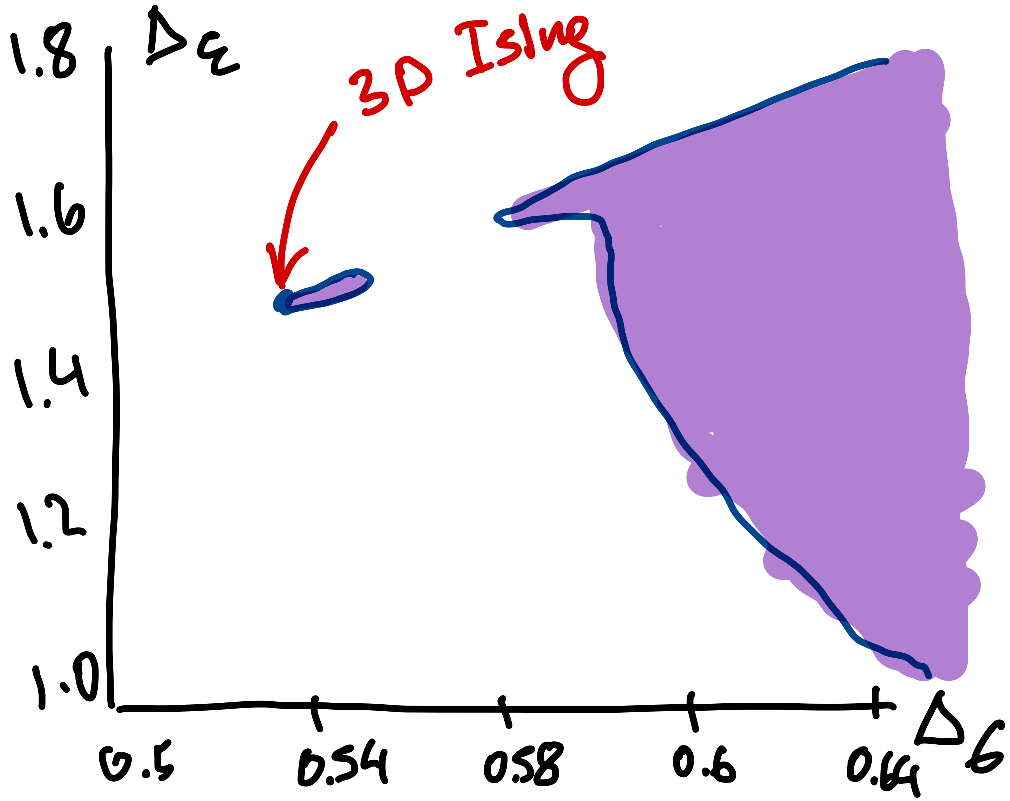
with Andrea, Julius, Miguel

Conformal Bootstrap





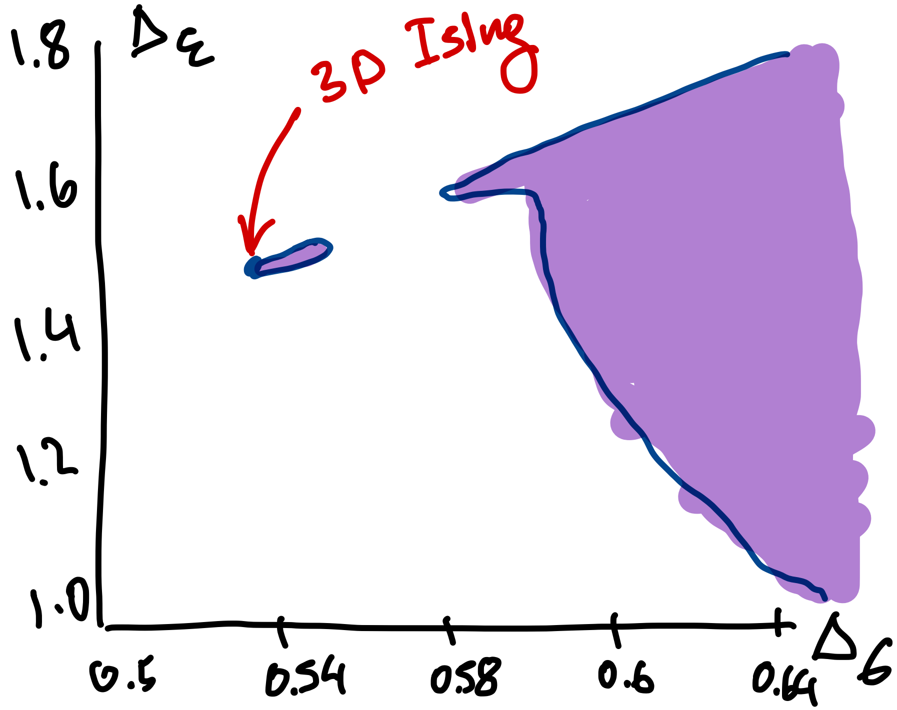
1 correlator



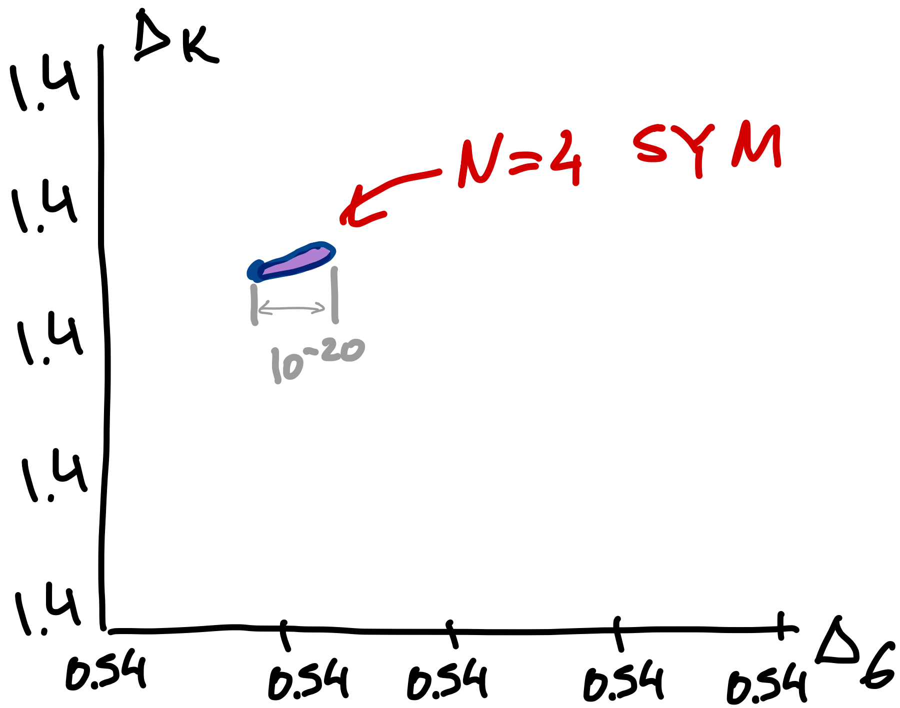
several correlators

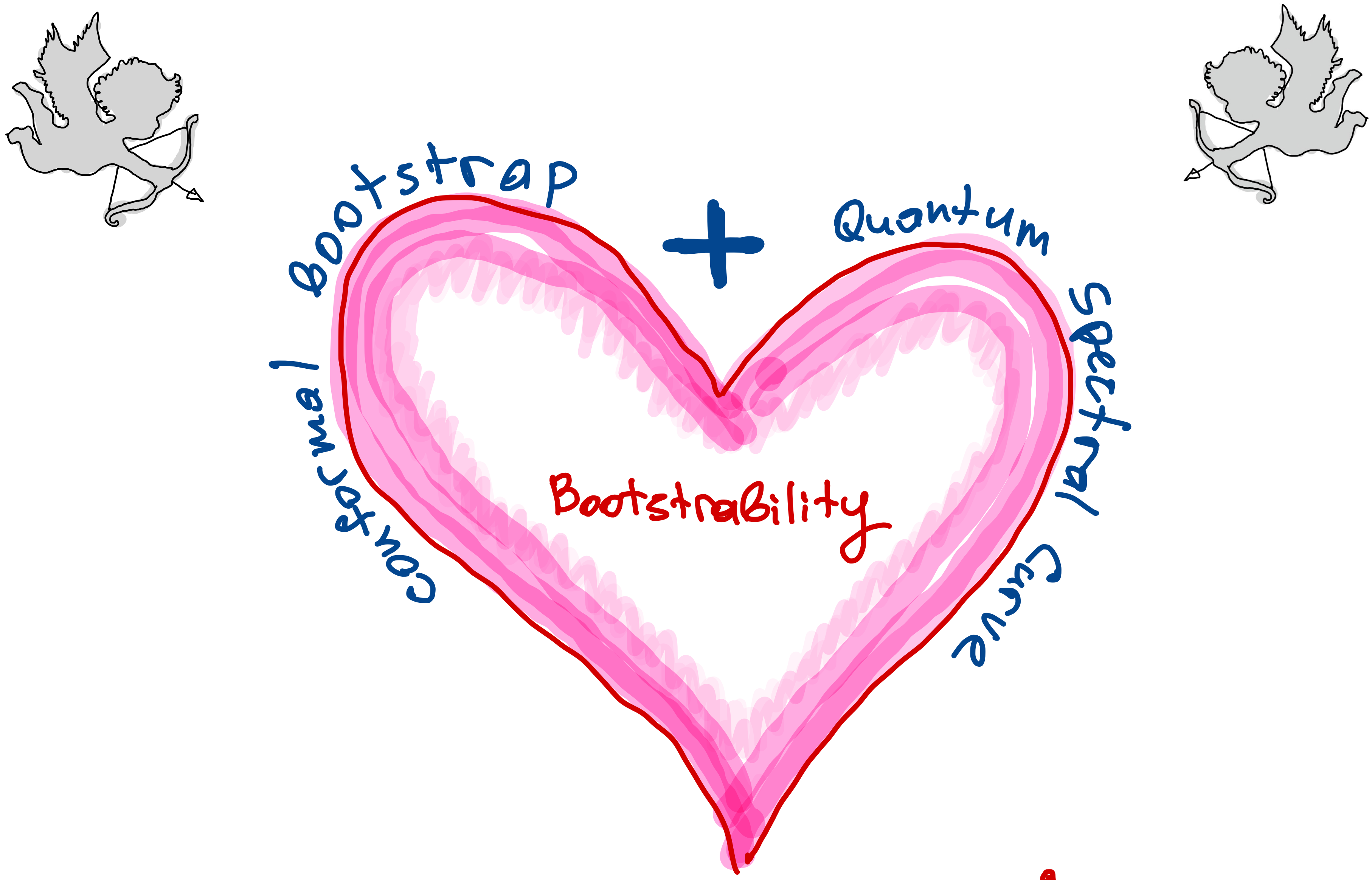
CB vs QSC

Conformal Bootstrap



QSC



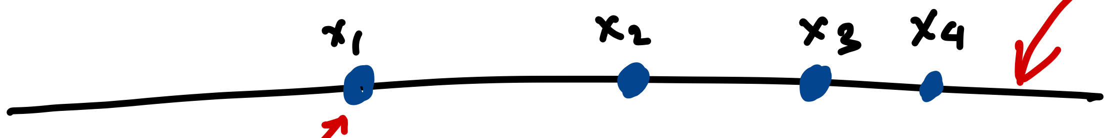


let them work together!

Set up

WL: $P\exp\int A_t + \varphi_{\parallel}$

1 of 6 scalars



$\mathcal{N}=4$ SYM operators:

e.g. $\varphi_{\parallel}$, $\varphi_i\varphi_i$, $\varphi_i \partial_t \varphi_i^2$

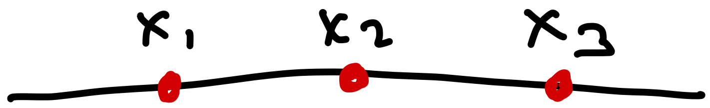
2pt:

$$\sim \frac{1}{(x_2 - x_1)^{\Delta}}$$

→ integrability QSC

N.G. Fedor

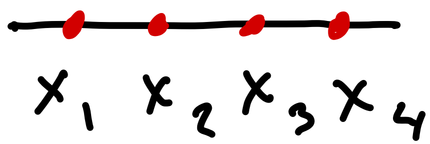
3pt:



$$\sim \frac{C}{x_{12}^{\Delta_{12}}\, x_{13}^{\Delta_{13}}\, x_{23}^{\Delta_{23}}}$$

→ hard due to wrapping

4pt:



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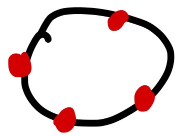

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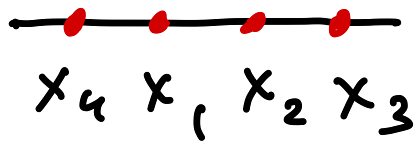


$$G(\chi) = G(1-\chi)$$

→ crossing

Symmetries

- 1D Conformal Symmetry

  $SL(2,\mathbb{R}) = \{D_t, P_t, K_t\}$

- R-symmetry : rotations in $\perp$ $\Phi_{\parallel}$ $\quad SO(5)$

- $SO(3)$ rotation in coordinate space $\perp$ WL

$$SL(2,\mathbb{R}) \times SO(3) \times SO(5) \subset OSP(2,2|4)$$

BPS-protected

$B_1 = \Phi^i \quad i = 1 \ldots 5 \quad \Delta = 1$

$B_2 = \Phi_\perp^{(i} \Phi_\perp^{i)} \quad \Delta = 2$

Non-protected

QSC

$O_1 = \Phi_{\parallel} \quad \Delta = 1 + 5g^2 + \ldots$

OPE for 4pt

$$\langle\langle \Phi_\perp(x_1)\, \Phi_\perp(x_2)\, \Phi_\perp(x_3)\, \Phi_\perp(x_4) \rangle\rangle = \frac{1}{x_{12}^2}\, \frac{1}{x_{34}^2}\, G(\chi)$$

protected

$$G(\chi) = \frac{3 W W''}{(W')^2}\, \chi^2 + \left(\frac{2}{\chi} - 1\right) f - (\chi^2 - \chi + 1)\, f'$$

$$W = \frac{2 I_1(\sqrt{\lambda})}{\sqrt{\lambda}} \qquad (W')^2 = 1 + C_{BPS}^2$$

- crossing $\quad (1-\chi)^2 f(\chi) + \chi^2 f(1-\chi) = 0$

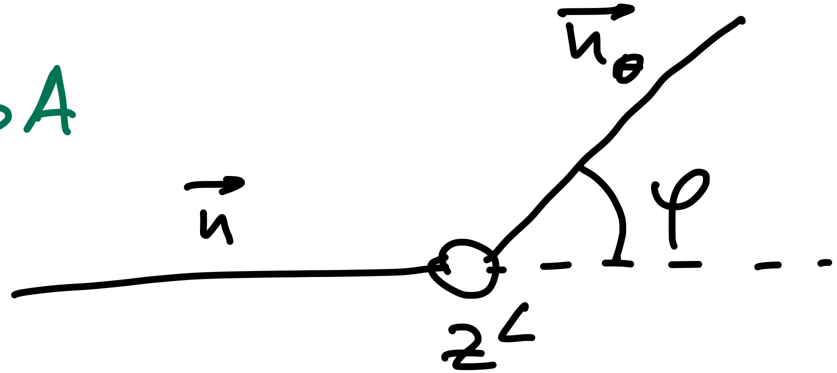
- OPE $\quad f(\chi) = \underbrace{\chi + C_{BPS}^2\, F_{BPS}(\chi)}_{\text{known}} + \sum_{\Delta} \underbrace{C_\Delta^2}_{\text{unknown}}\, \overbrace{F_\Delta(\chi)}^{\text{known}}$

$\{\Delta\}$ can be found from QSC!

singlets of $SO(5) \times SO(3)$

Spectrum

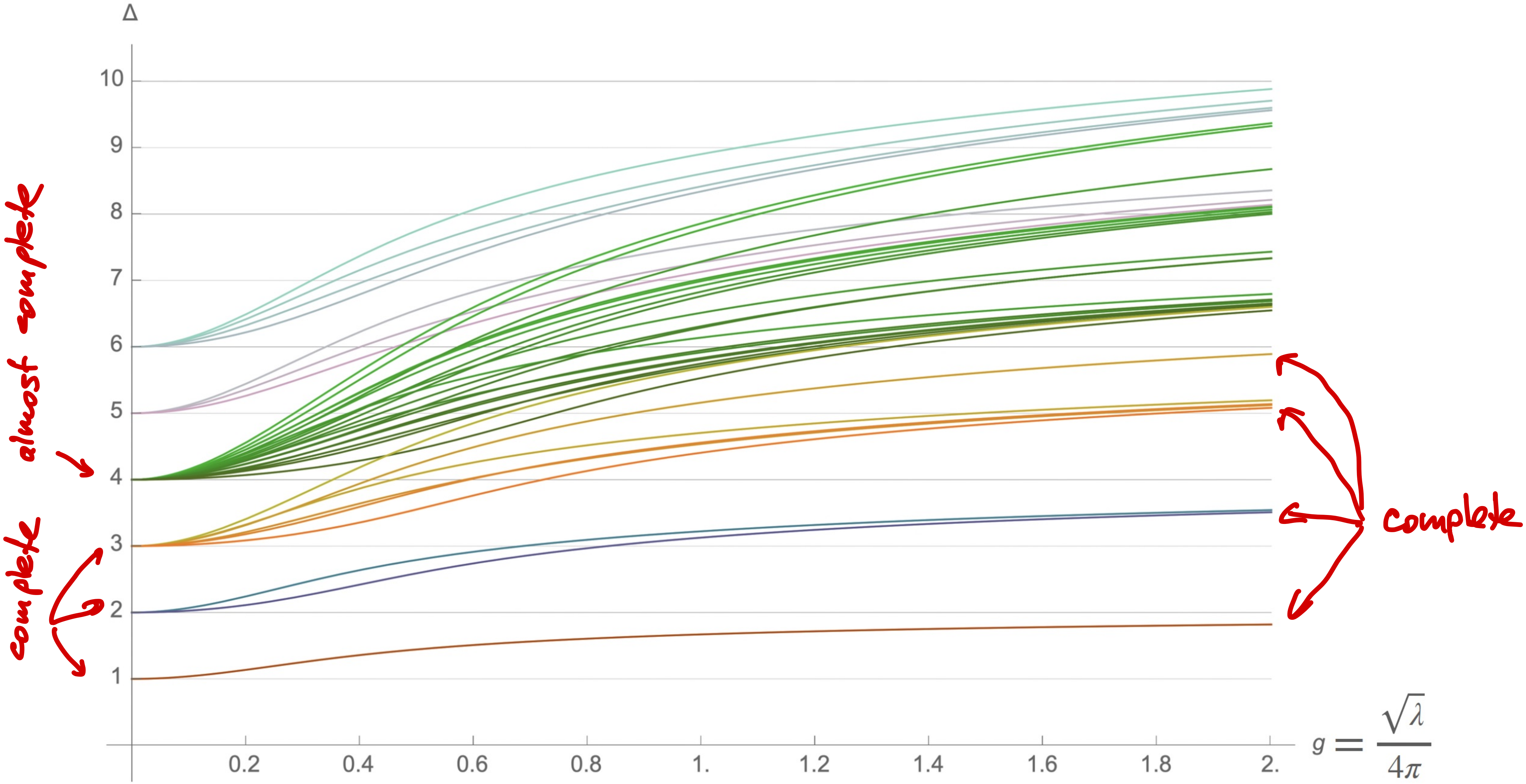
- Drukker Kawamoto '06 : open spin chains
- Drukker '12, Correa Maldacena Sever '12 : BTBA
- NG, Fedor '15 : QSC



Further improvements:

- $\varphi \to 0$ $\theta \to 0$ limit is not totally trivial
  Grabner NG Julius '20
- improved numerical method
  to ensure good convergence at small g
- improved perturbative method
  to generate good starting points for numerics

spectrum



35 states with high precision

Crossing equation

$$\sum_n \underset{\text{want to deduce}}{C_n^2}\ \underbrace{G_{\Delta_n}(x)}_{\text{known first many}} = \underbrace{H(x)}_{\text{fixed function}} \qquad x \in [0,1]$$

∞ many equations for ∞ many quantities (but linear!)

truncation strategy I

$$\sum_{n=1}^{N} C_n^2 \, G_{\Delta_n}(x_i) = H(x_i) \qquad i = 1 \ldots N$$

- select N random x_i, truncate to N C_n's, solve linear eq.

used successfully by He, Jacobsen, Saleur, Picco, Ribault, Santachiara '16

Fails in our case

Truncation strategy II

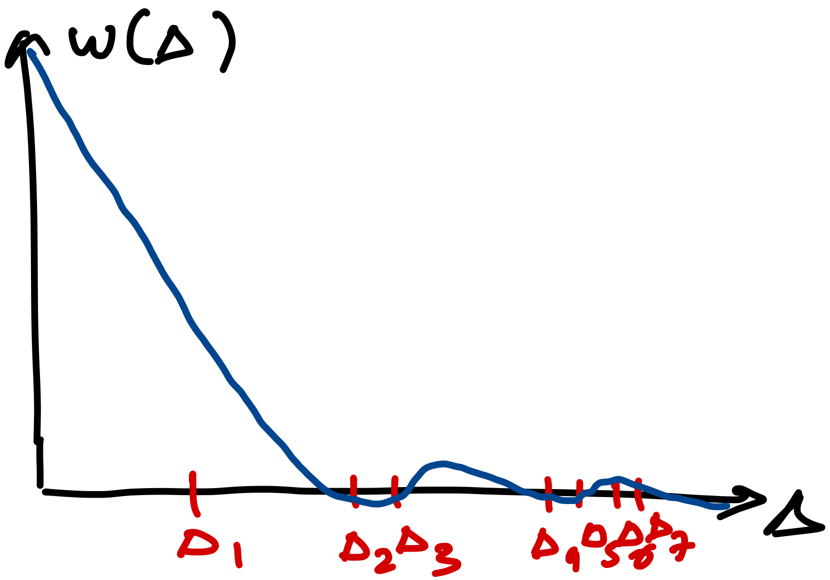
$$\sum_n c_n^2 G_{\Delta_n}(x) = H(x)$$

Define $\alpha_i[f(x)] \leftarrow$ functionals

$$\alpha_i[G_\Delta(x)] = \omega_i(\Delta)$$

→ decays well for large Δ
→ sufficiently different for different ith

does not need to be positive



$$c_1^2\, \omega(\Delta_1) \overset{\text{ideally}}{=} \alpha[H(x)]$$

works well, but hard to estimate error for c_1^2

Traditional method

[El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi '12]

- find 2 functionals $\alpha_\pm$ s.t.

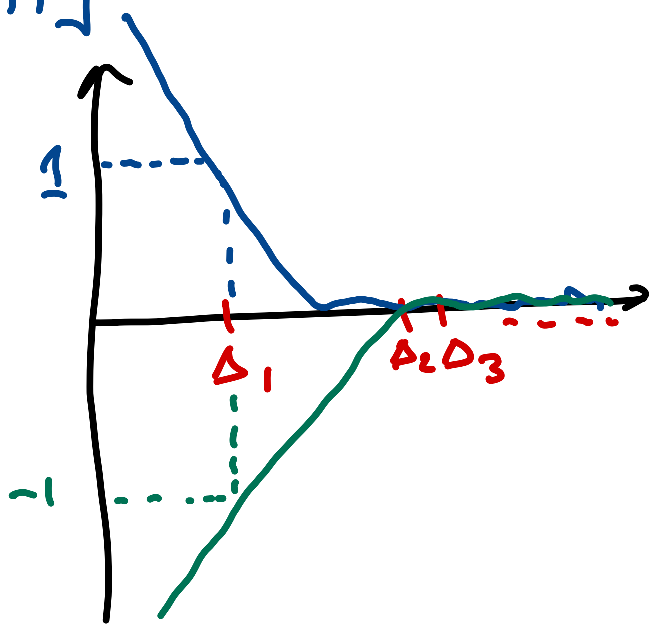
$$\alpha_\pm[G_{\Delta_1}] = \pm 1 \qquad \alpha_\pm[G_\Delta] > 0 \quad \Delta \geq \Delta_2$$

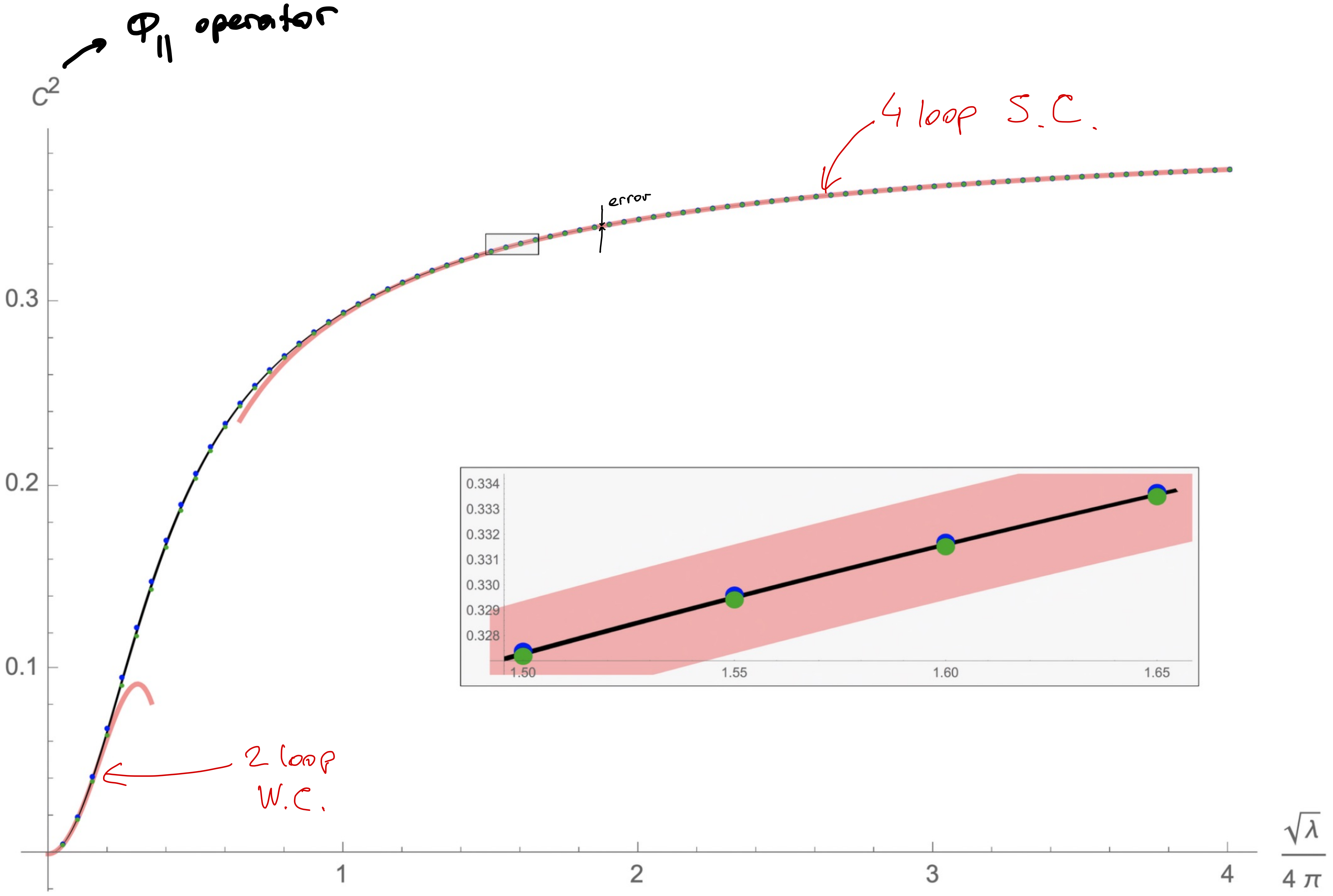
$\mp \alpha_\pm[H]$ is as big as possible

$$\pm C_1^2 + \underbrace{\sum_{n>1} C_n^2\, \omega_\pm(\Delta)}_{\geq 0 \text{ small}} = \pm \alpha_\pm[H]$$

$$\alpha_-[H] < C_1^2 < \alpha_+[H]$$

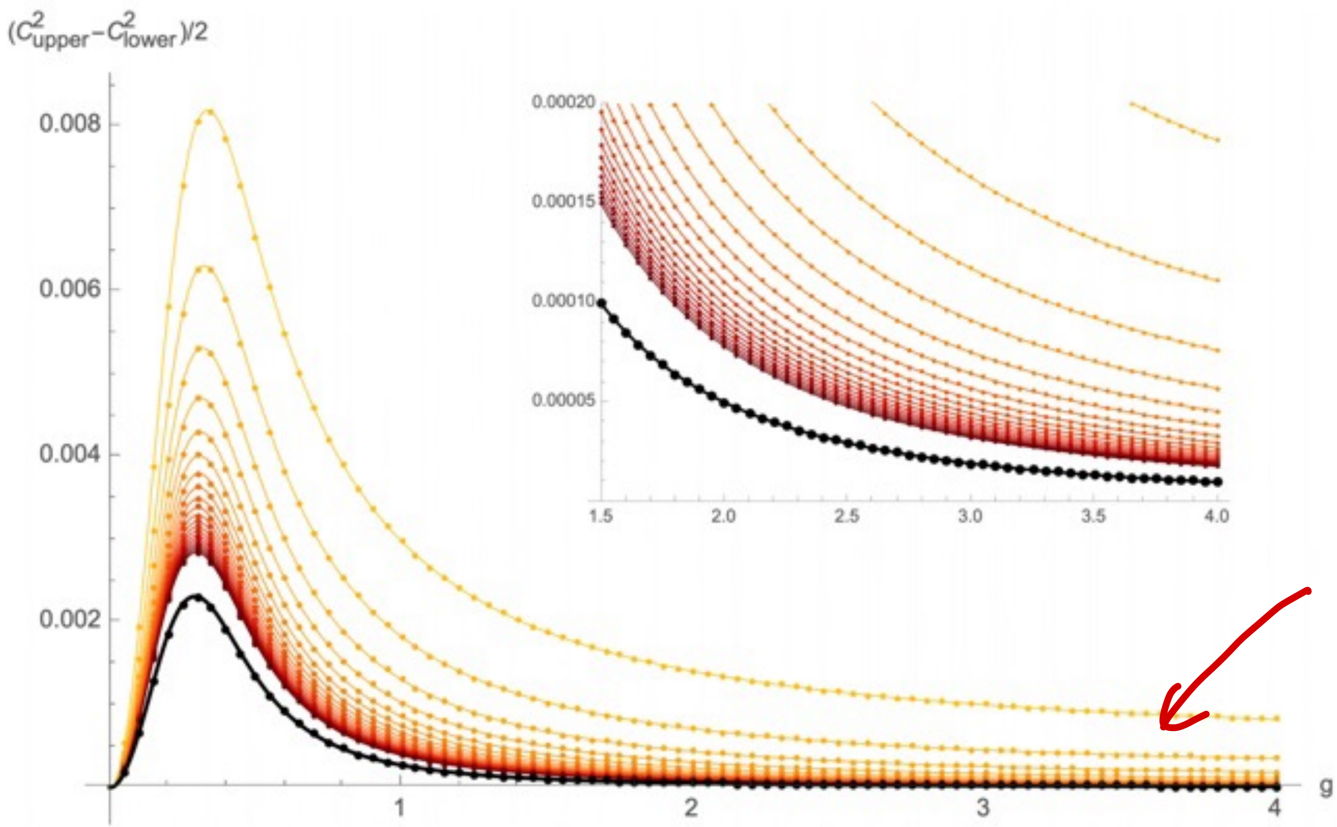
optimize $\alpha_\pm$ to close the gap





$\Phi_{||}$ operator
C^2
0.3
0.2
0.1
1
2
3
4
$\frac{\sqrt{\lambda}}{4\pi}$
4 loop S.C.
error
2 loop W.C.
0.334
0.333
0.332
0.331
0.330
0.329
0.328
1.50
1.55
1.60
1.65

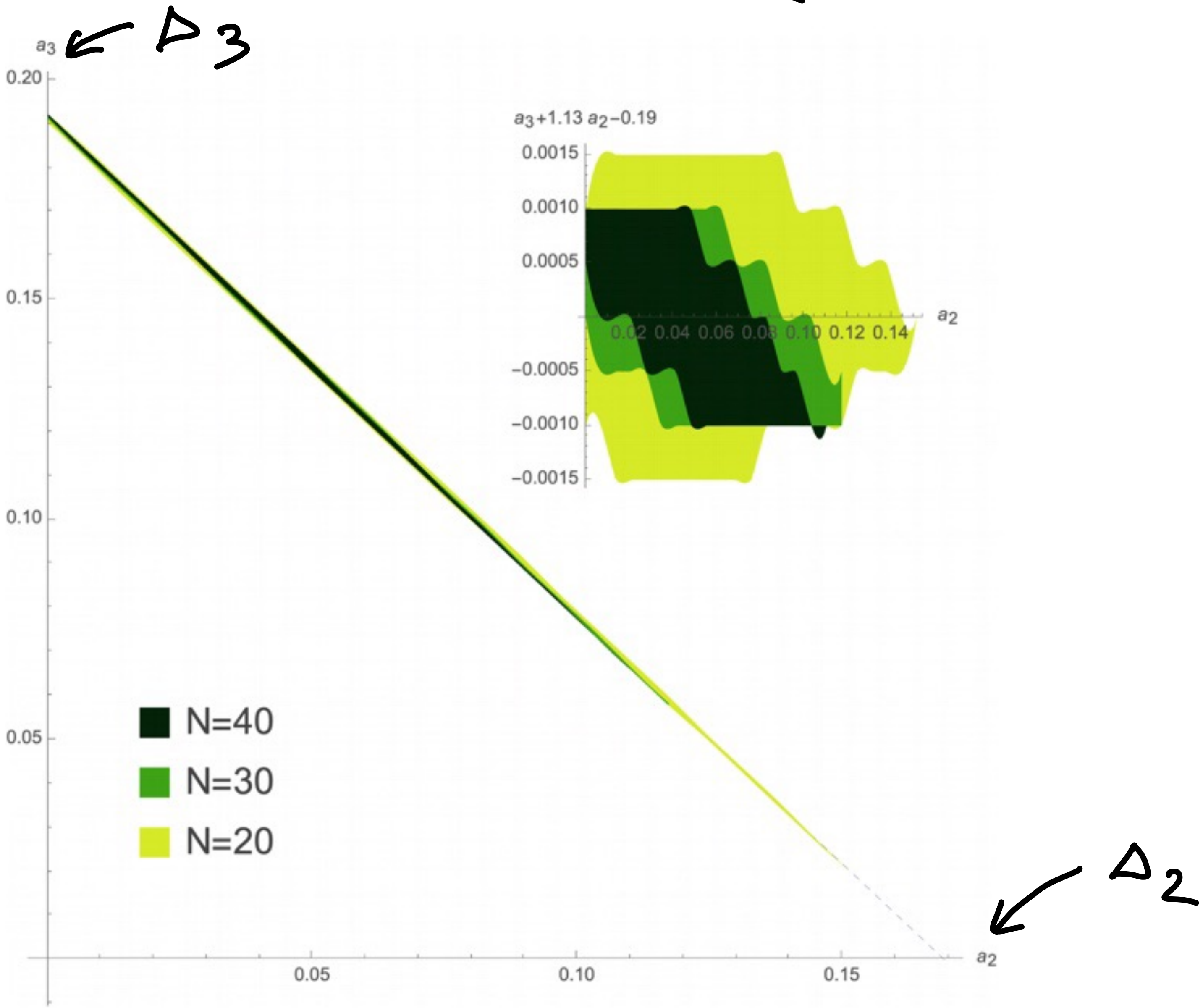
error:



5,7 ..., 45 derivatives
and extrapolation

Note: we only used 2 states (out of 35)
and only 1 correlator!

More states?



Δ3
a3
0.20
0.15
0.10
0.05
0.05
0.10
0.15
a2
Δ2
a3+1.13 a2−0.19
0.0015
0.0010
0.0005
−0.0005
−0.0010
−0.0015
0.02 0.04 0.06 0.08 0.10 0.12 0.14
a2
N=40
N=30
N=20

- Accurate OPE data from just 2 states
- Can we reduce the bound to zero with more states?
- Next – more correlators? More input from integrability?
- Analytically? see [Meneghelli Ferrero '18] at large g (no spectrum used, more correlators, perturbative input + HPL assumption)
- Work around double-traces problem for local ops?

in short: is QSC + Conformal bootstrap = solution of $\mathcal{N}=4$ SYM?